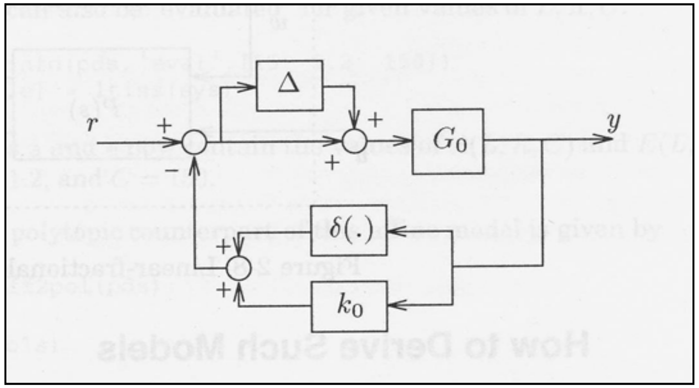
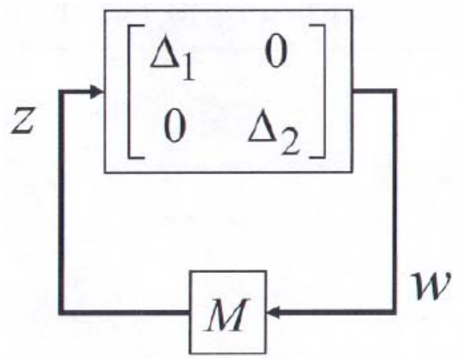
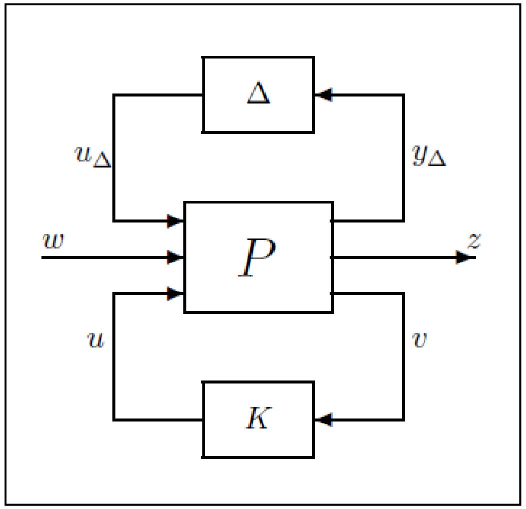






Structured Singular Value

$$\mu(M(j\omega)) < 1, \quad \forall\omega$$

Summary

1. **Uncertainty in a $\mathcal{H}_\infty$ Context**

2. **μ Analysis and Synthesis**

3. **D – K Iteration**

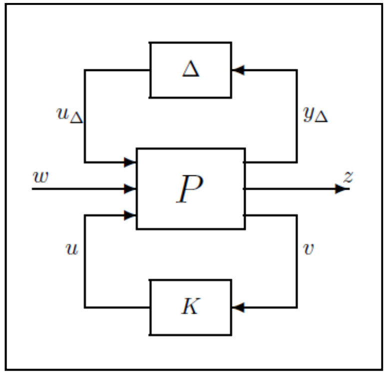
4. Examples

5. **Controller Reduction**

- To devise a structured, automated procedure for the design of a norm-based MIMO compensator capable of achieving nominal **as well robust stability and performance**.

Uncertainty in a $\mathcal{H}_\infty$ Context

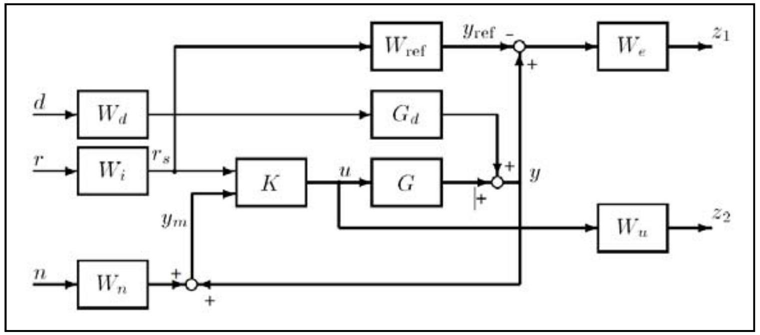
- The 3-block structure and $\mathcal{H}_\infty$ optimization can be used to formalize a robust control problem in the presence of unstructured as well as structured uncertainties.

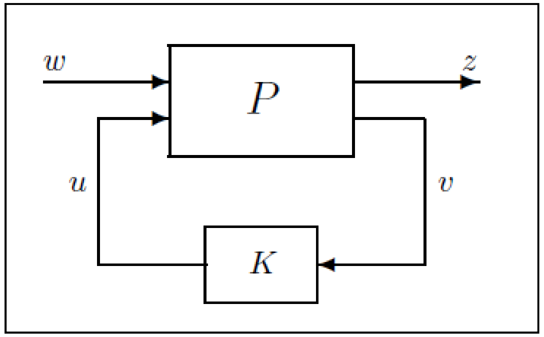


- The nominal system requirements are specified in terms of $\mathcal{H}_\infty$ - norm of closed loop transfer matrix and/or mixed sensitivity (γ – iteration)
- $\mathcal{H}_\infty$ - norm is used as bound on unstructured uncertainty (limits on maximum and minimum singular values of loop transfer matrix)
- Structured and unstructured uncertainties can be ‘easily’ combined in a single uncertainty block Δ, for compact algebraic manipulation
- Upper and Lower LFT can be applied to handle analysis and synthesis

Uncertainty in a $\mathcal{H}_\infty$ Context

$\mathcal{H}_\infty$ Control Sequence Review





$$G \triangleq \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]; G_d \triangleq \left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right]$$

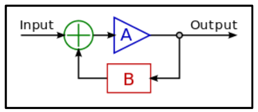
$$W_i \triangleq \left[\begin{array}{c|c} A_W^i & B_W^i \\ \hline C_W^i & D_W^i \end{array}\right]; i = \begin{bmatrix} n \\ i \\ d \\ ref \\ e \\ u \end{bmatrix}$$

$$P(s) = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right]$$

$$w = \begin{bmatrix} d \\ r \\ n \end{bmatrix} \qquad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$v = \begin{bmatrix} r_s \\ y_m \end{bmatrix} \qquad u = u$$

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x + D_{11} w + D_{12} u$$
$$v = C_2 x + D_{21} w + D_{22} u$$



Uncertainty in a $\mathcal{H}_\infty$ Context

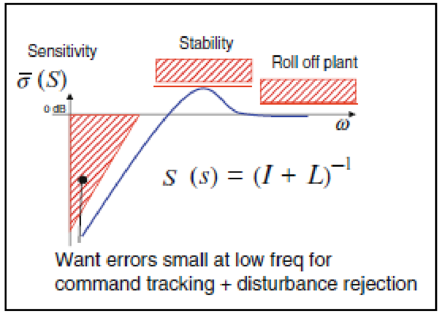
Classical Synthesis:

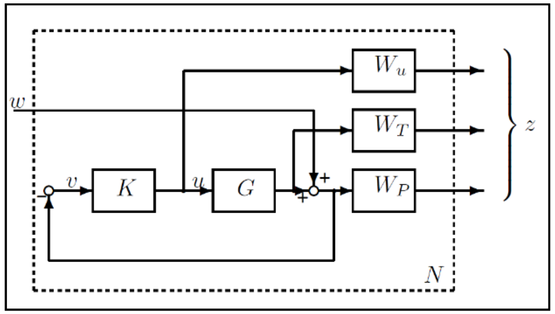
- Assuming specific structural conditions are satisfied on $P(s)$, we can find a stabilizing controller $K(s)$, such that the feedback $u(s) = K(s)v(s)$ minimizes

$$\left\| \mathrm{F}_l \left[P(s), K(s) \right] \right\|_\infty < \gamma$$

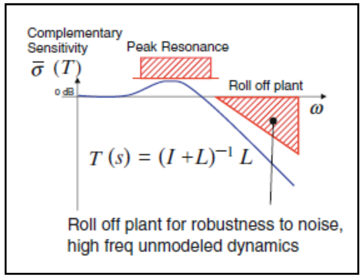
$$\mathrm{F}_l \left[P(s), K(s) \right] = T_{zw}(\mathrm{s}) = \left\{ P_{11} + P_{12} K(s) [I - P_{22} K(s)]^{-1} P_{21} \right\}$$

Mixed Sensitivity Synthesis:

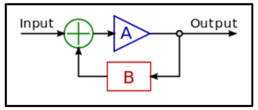




$$\min_K \|N(K)\|_\infty, \quad N = \begin{bmatrix} W_u K S \\ W_T T \\ W_P S \end{bmatrix}$$



Controller Structure and Algorithm from Bounded Real Lemma. See Skogestad-Postlethwaite Text, Sections 9.3 and 9.4



Uncertainty in a $\mathcal{H}_\infty$ Context

$\mathcal{H}_\infty$-(sub) optimal output-feedback problem

- Consider the system **(5.1)**. Suppose that the assumptions (A1) - (A8) hold.

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned} \qquad \textbf{(5.1)}$$

- Then there exists a controller $\boldsymbol{u} = K\boldsymbol{y}$, which achieves the $\mathcal{H}_\infty$-norm bound (or equivalently the inequality below):

$$J_\infty(K) < \gamma := \left\| T_{zw}(s) \right\|_\infty := \sup_{\omega \in \Re} \sigma_{\max}\left[T_{zw}(j\omega) \right] < \gamma$$

$$:= \left\| F_l(P,K) \right\|_\infty = \max_{\boldsymbol{w}(t) \neq 0} \frac{\left\| \boldsymbol{z}(t) \right\|_2}{\left\| \boldsymbol{w}(t) \right\|_2} < \gamma$$

- if and only if the following conditions are satisfied:

a) There exists a symmetric positive definite or semidefinite solution X to the Riccati equation

$$A^T X + XA - XB_2 (D_{12}^T D_{12})^{-1} B_2^T X + \gamma^{-2} X B_1 B_1^T X + C_1^T C_1 = 0$$

such that the matrix below in is stable (all eigenvalues with negative real part).

$$A - B_2 (D_{12}^T D_{12})^{-1} B_2^T X + \gamma^{-2} B_1 B_1^T X$$

b. There exists a symmetric positive definite or semidefinite solution Y to the Riccati equation:

$$AY + YA^T - YC_2^T(D_{21}D_{21}^T)^{-1}C_2Y + \gamma^{-2}YC_1^TC_1Y + B_1B_1^T = 0$$

such that the matrix:

$$A - YC_2^T(D_{21}D_{21}^T)^{-1}C_2 + \gamma^{-2}YC_1^TC_1$$

is stable.

c. $\rho(XY) = \max_i \left|\lambda_i(XY)\right| < \gamma^2.$ $\qquad \rho(A) = \max\{|\lambda_1|, \ldots, |\lambda_n|\}.$

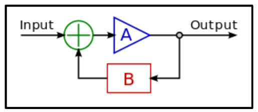
d. When these conditions are satisfied, such a controller is:

$$K_{sub}(s) = \left(\begin{array}{c|c} A_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right)$$

where:

$$A_\infty = A + \gamma^{-2}B_1B_1^TX + B_2(D_{12}^TD_{12})^{-1}F_\infty - Z_\infty L_\infty C_2$$

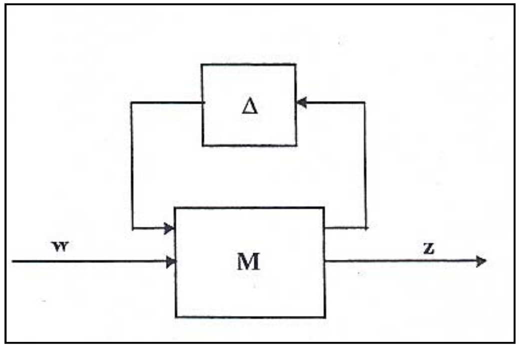
$$F_\infty = -(D_{12}^TD_{12})^{-1}B_2^TX$$
$$L_\infty = -YC_2^T(\tilde{D}_{21}\tilde{D}_{21}^T)^{-1}$$



$$K_{sub}(s) = \left(\begin{array}{c|c} A_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right) \Rightarrow K_{sub}(s) = -F_\infty(sI - A_\infty)^{-1}Z_\infty L_\infty$$

Uncertainty in a $\mathcal{H}_\infty$ Context

- The basic idea of representing uncertainty (structured and unstructured) is to isolate it in a separate Block, to be connected in feedback to the plant, via additional “fictitious” input – output pair.

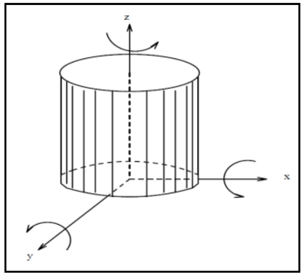


- **Unstructured Uncertainty**: unknown nature, modeled by a generic frequency dependent matrix
- **Structured Uncertainty**: specified nature, specific frequency dependent model, modeled by block diagonal real and complex elements

- **Example: (Zhou p. 232; Scherer p. 41)**

  - Consider a symmetric spinning body with torque inputs, T_1 and T_2, along two orthogonal transverse axes, x and y. Assume that the angular velocity of the spinning body with respect to the z axis is constant, Ω. Assume further that the inertia of the spinning body with respect to the x; y, and z axes are I_1, $I_2 = I_1$, and I_3, respectively. Denote by ω_1 and ω_2 the angular velocities of the body with respect to the x and y axes, respectively. Then the Euler's equation of the spinning body is given by

$$\begin{aligned} I_1\dot{\omega}_1 - \omega_2\Omega(I_1 - I_3) &= T_1 \\ I_1\dot{\omega}_2 - \omega_1\Omega(I_3 - I_1) &= T_2 \end{aligned} \qquad I_3\dot{\omega}_3 = 0$$



Define

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} := \begin{bmatrix} T_1/I_1 \\ T_2/I_1 \end{bmatrix}, \; a := (1 - I_3/I_1)\Omega.$$

Then the system dynamical equations can be written as

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Now suppose that the angular rates ω_1 and ω_2 are measured in scaled and rotated coordinates:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\cos\theta} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

where $\tan\theta := a$. (There is no specific physical meaning for the measurements of y_1 and y_1 but they are assumed here only for the convenience of discussion.) Then the transfer matrix for the spinning body can be computed as

Uncertainty in a $\mathcal{H}_\infty$ Context

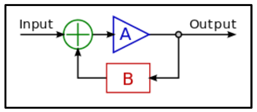
$$Y(s) = G(s)U(s) \qquad G(s) = \frac{1}{s^2+a^2}\begin{pmatrix} s-a^2 & a(s+1) \\ -a(s+1) & s-a^2 \end{pmatrix} \qquad G = \left[\begin{array}{cc|cc} 0 & a & 1 & 0 \\ -a & 0 & 0 & 1 \\ \hline 1 & a & 0 & 0 \\ -a & 1 & 0 & 0 \end{array}\right]$$

- Assume the model is valid within some actuator tolerance, taken into account by a parametric uncertainty on the nominal input matrix $B = I$.

$$\begin{pmatrix} 1+\delta_1 & 0 \\ 0 & 1+\delta_2 \end{pmatrix} \qquad \frac{1}{s^2+a^2}\begin{pmatrix} s-a^2 & a(s+1) \\ -a(s+1) & s-a^2 \end{pmatrix}\left(I + \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}\right)$$

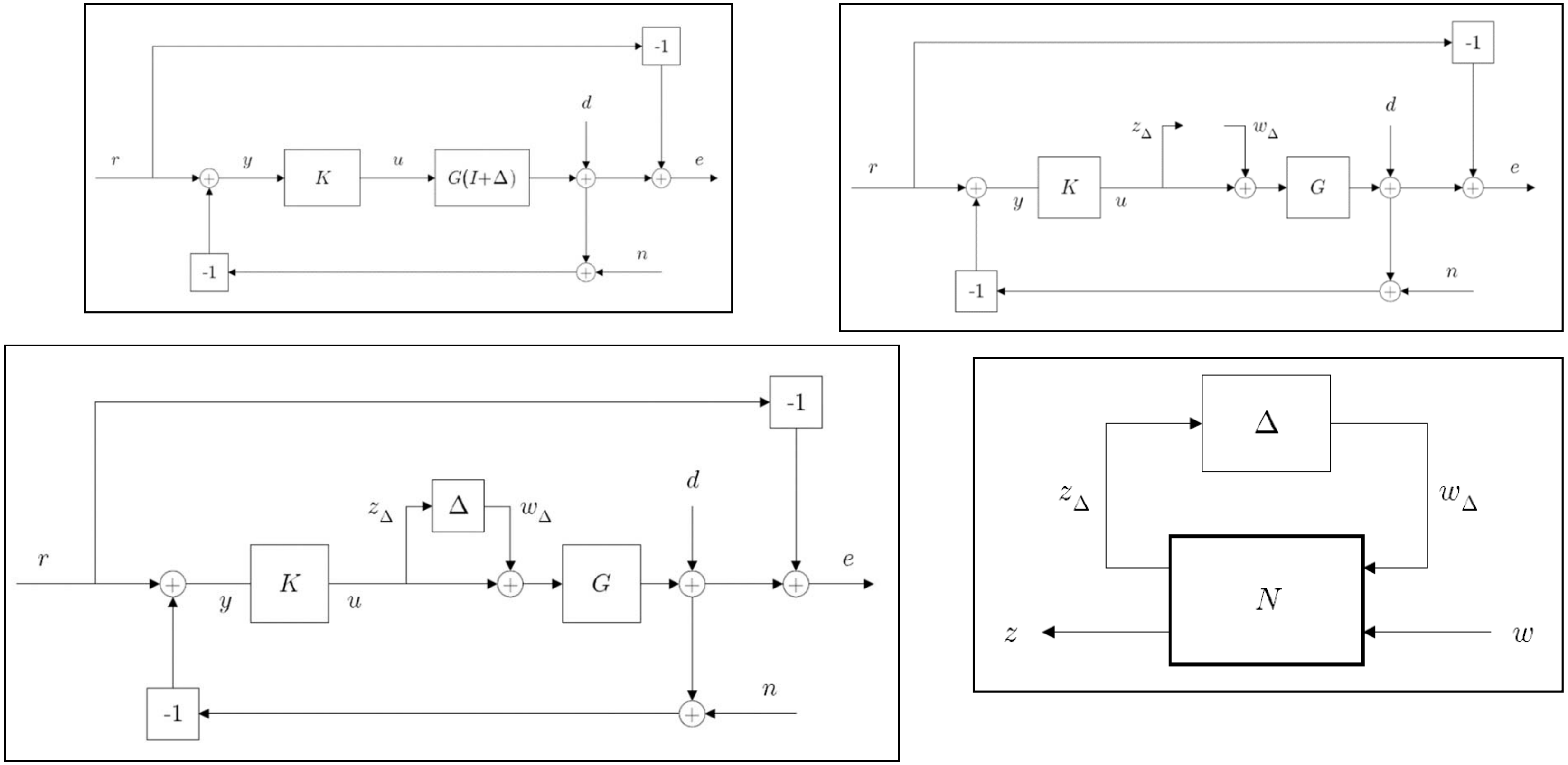
- The real system becomes (for some real parameters δ_1 and δ_2):

$$G(I+\Delta) \quad \text{with} \quad \Delta = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \qquad |\delta_1| < r, \ |\delta_2| < r$$



Uncertainty in a $\mathcal{H}_\infty$ Context

- Sequence to yield a 2 – block structure



$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}; N_{11} = -(I + KG)^{-1} KG$$

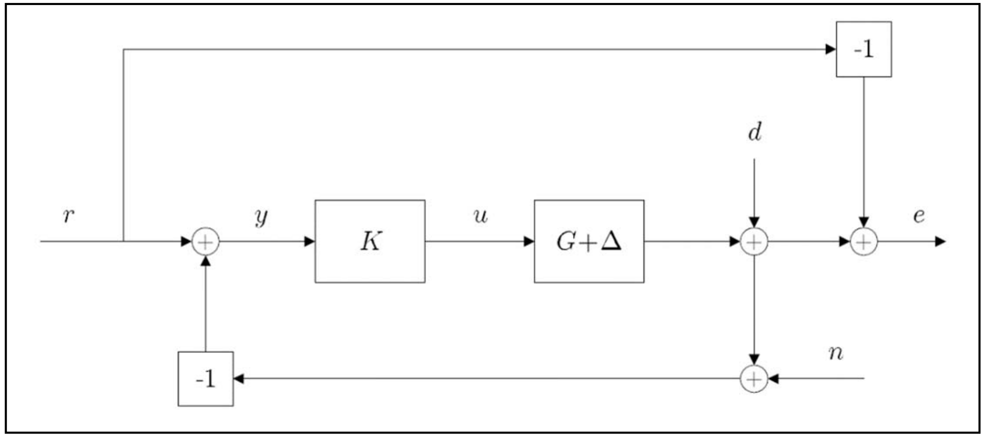
- 3 – Block Closed Loop System $z = [\square] w$

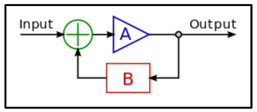
$$\begin{cases} z_\Delta = N_{11} w_\Delta + N_{12} w \\ z = N_{21} w_\Delta + N_{22} w \end{cases} \qquad z_\Delta = \Delta w_\Delta \qquad \boxed{z = \left[N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1} N_{12}\right] w} \qquad w = \begin{bmatrix} d \\ n \\ r \end{bmatrix}$$

$$N_{11} = -(I + KG)^{-1} KG = M$$

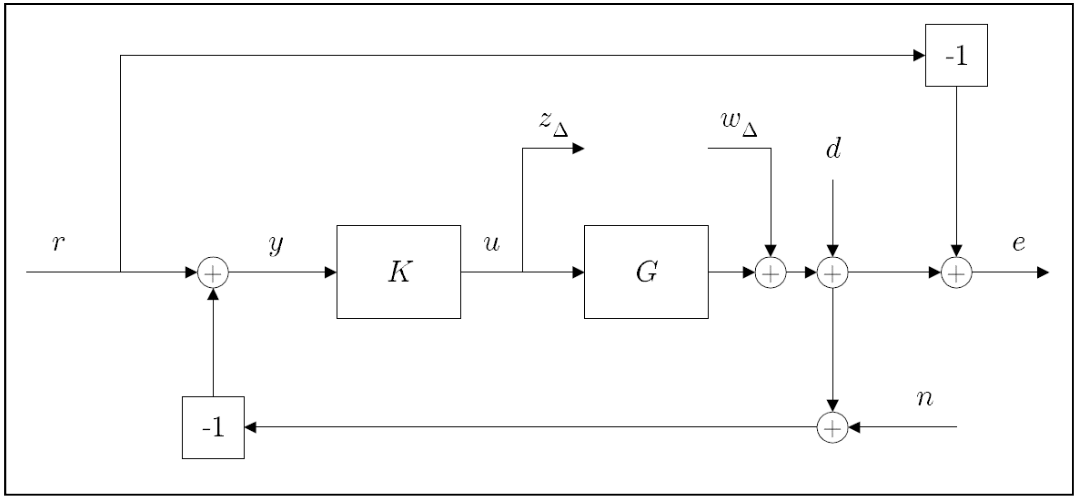
Example 2: (Scherer p. 33)

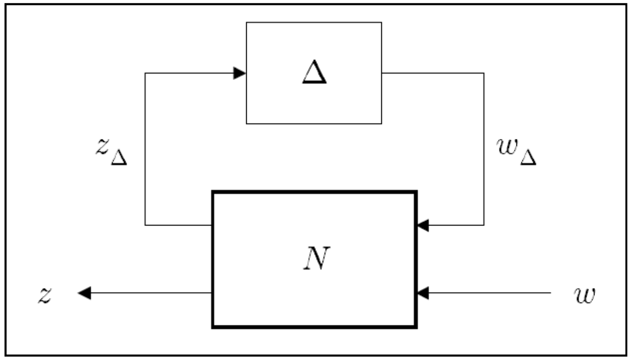
- Nominal plant and controller: $G(s) = \dfrac{200}{10s+1} \cdot \dfrac{1}{(0.5s+1)^2}$ $\qquad K(s) = \dfrac{0.1s+1}{(0.65s+1)(0.03s+1)}.$
  - Additive uncertainty: $\tilde{G}(s) = G(s) + \Delta(s); \forall \Delta(s) \in \left\{ \left|\Delta(j\omega)\right| < 1, \omega \in \Re; \Delta(j\omega) \in \mathbb{R}\mathcal{H}_\infty \right\}$





Uncertainty in a $\mathcal{H}_\infty$ Context





- 3 – Block Closed Loop System $z = \left[\square\right] w$

$$\begin{cases} z_\Delta = N_{11} w_\Delta + N_{12} w \\ z = N_{21} w_\Delta + N_{22} w \end{cases} \qquad z_\Delta = \Delta w_\Delta \qquad \boxed{z = \left[N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1} N_{12}\right] w} \qquad w = \begin{bmatrix} d \\ n \\ r \end{bmatrix}$$

$$N_{11} = -(I + KG)^{-1} K = M$$

Uncertainty in a $\mathcal{H}_\infty$ Context

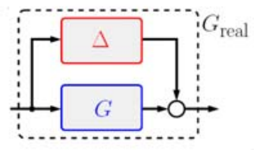
- From examples, it shows that we can always rewrite uncertainties in a 2 – Block format consisting of the uncertainty block and the controlled model block

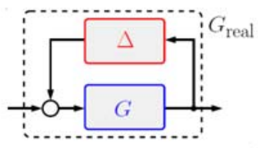
  - **Unstructured Uncertainty**: unknown nature, modeled by a generic frequency dependent matrix. Uncertainty $\Delta(s)$ weighted with known frequency dependent matrices $W_1(s)$, and $W_2(s)$ such that:

$$\tilde{G}(s) = G_{real}(s) = \left[G(s) \cup \Delta(s)\right], \Delta(s) = W_1(s)\Delta(s)W_2(s)$$
$$\left\{\Delta(s) \in \mathbb{R}H_\infty \mid \|\Delta\|_\infty < 1\right\}$$

1. The first equation indicates that the uncertainty could be additive and/or multiplicative, located at the input and or output of the model $G(s)$
2. The constraint on Δ is to be a stable, bounded, and in general full complex matrix (second equation)
3. The weights should be incorporated into the model itself

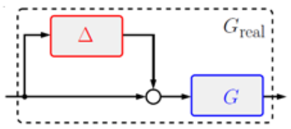
$$\tilde{G} = G + \Delta$$

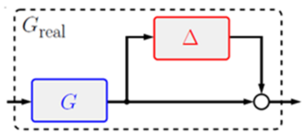




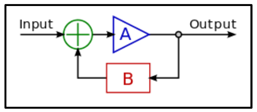
$$\tilde{G} = G(I + \Delta G)^{-1}$$

$$\tilde{G} = G(I + \Delta)$$





$$\tilde{G} = (I + \Delta)G$$



Uncertainty in a $\mathcal{H}_\infty$ Context

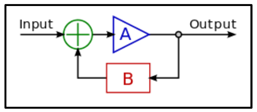
- **Structured Uncertainty**: specified nature, specific frequency dependent model. Modeling follows the format seen before with the difference being the structure of the uncertainty block, which now contains real parameters and transfer functions in a specific (structured) uncertainty matrix.

$$\tilde{G}(s) = \left[G(s) \cup \Delta(s)\right], \Delta(s) = W_1(s)\Delta(s)W_2(s)$$
$$\Delta(s) = diag\left[\delta_i^r I \quad \dots \quad \delta_j^c(s) I \quad \dots\right]$$
$$\left\{\delta_i^r \in \mathbb{R} \mid \left|\delta_i^r\right| < 1; \delta_j^c(s) \in \mathbb{R}H_\infty \mid \left\|\delta_j^c(s)\right\|_\infty < 1; \Delta_i(s) \in \mathbb{R}H_\infty \mid \left\|\Delta_i(s)\right\| < 1\right\}$$

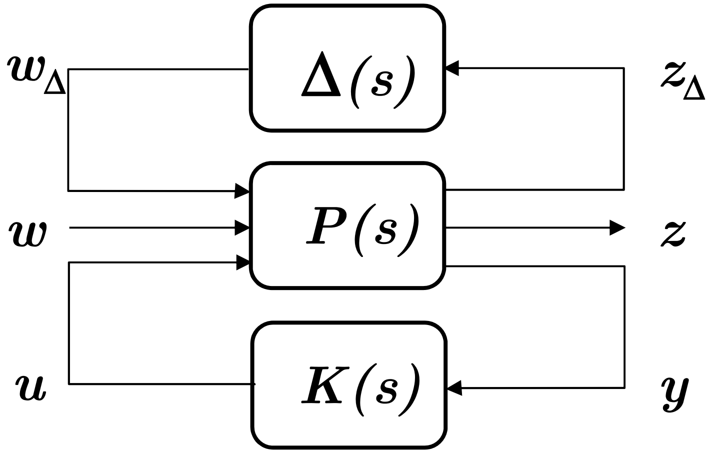
1. Real uncertain parameters are bounded
2. Complex uncertain functions are stable and norm bounded

- The combined uncertainty is structured (since it is expressed by a diagonal norm – limited matrix)

(*)
$$\Delta(s) \in \mathbf{\Delta}; \mathbf{\Delta} := \left\{ \Delta = \begin{bmatrix} \delta_1^r & & & & & & & & \\ & \dots & & & & & & & \\ & & \delta_r^r & & & & & & \\ & & & \delta_1^c & & & & & \\ & & & & \dots & & & & \\ & & & & & \delta_q^c & & & \\ & & & & & & \Delta_1 & & \\ & & & & & & & full & \\ & & & & & & & & \Delta_m \end{bmatrix} \left| \left|\delta_i^r\right| < 1, \ \left\|\delta_i^c\right\|_\infty < 1, \ \left\|\Delta_i\right\|_\infty < 1 \ \text{stable} \right. \right\}$$



Uncertainty in a $\mathcal{H}_\infty$ Context



- $\Delta(s)$ stable, and norm – bounded $\|\square\|_\infty < 1$
- $P(s)$ includes performance and uncertainty weights
- $K(s)$ stabilizing controller to be designed for nominal **AND** uncertain Closed Loop stability and performance

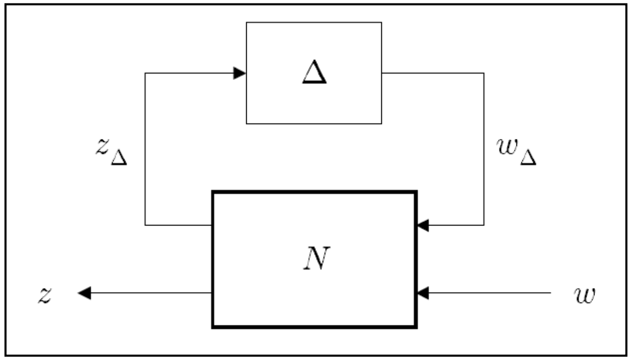
$$\begin{bmatrix} z_\Delta \\ z \\ y \end{bmatrix} = P(s)\begin{bmatrix} w_\Delta \\ w \\ u \end{bmatrix}; P(s) = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

- In the absence of uncertainty (nominal $\Delta = 0$), a solution can be found using $\mathcal{H}_\infty$ synthesis and γ – iteration or as a mixed sensitivity problem

- **Tools:**
  1. Small Gain Theorem
  2. Structured Singular Value μ

Uncertainty in a $\mathcal{H}_\infty$ Context

Small Gain Theorem



$$\mathbb{F}_L(P,K) = N = \begin{bmatrix} N_{11} = \boldsymbol{M} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

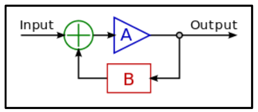
$$z = \left[N_{22} + N_{21}\Delta(I - M\Delta)^{-1}N_{12}\right]w$$

- Assume the uncertainty block (structured and unstructured) is stable and bounded as: $\Delta \in \boldsymbol{\Delta}\mathbb{C}^{p\times q}; \left\|\Delta\right\|_\infty < \gamma$
- From previous results:

Theorem 12 : (Scherer, p. 63). If $K(s)$ stabilizes $P(s)$, and $I\text{-}M\Delta$ has a proper and stable inverse for all Δ stable and norm – bounded, then $K(s)$ robustly stabilizes:

$$\mathbb{F}_U(\Delta, P); \forall\Delta \in \boldsymbol{\Delta} \qquad N \Rightarrow P(s)$$

Theorem 13: (Scherer, p. 65). Suppose $M(s)$ is a proper and stable transfer matrix. If $\det[I - M\Delta] \neq 0$ then $I\text{-}M\Delta$ has a proper and stable inverse for all Δ stable and norm – bounded uncertainties $\boldsymbol{\Delta}$

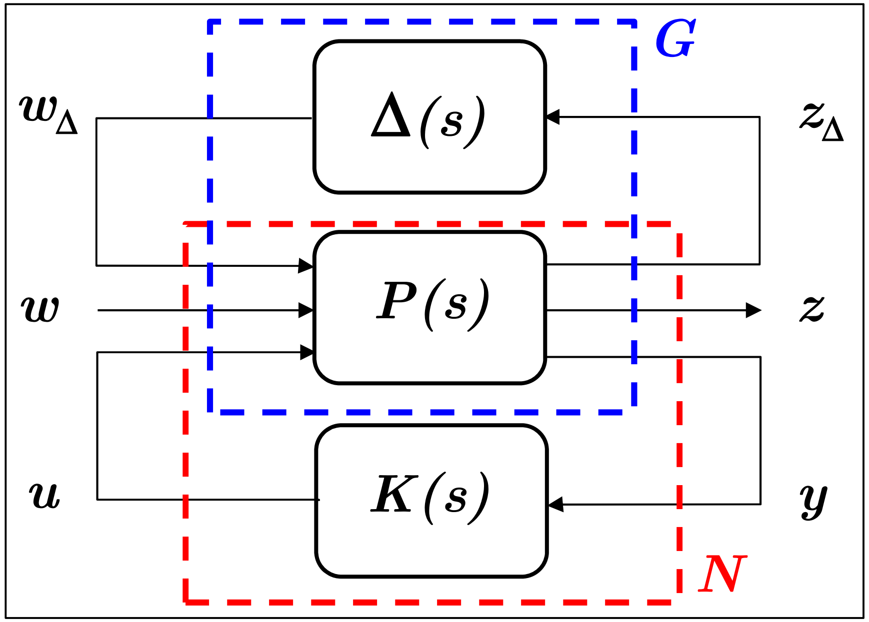


Note: See other statement of the theorem (Zhou, Mackenroth, Levretski)

2. μ Analysis and Synthesis

µ

- **Question:** Is it possible to devise a general procedure that provides the synthesis of a robust controller (with respect to unstructured and structured uncertainties), and at the same time can verifiy and guarantee stability and performance?



- **Robustness Analysis is a 2 – Block representation, which includes the closed loop controller in $N(s)$:**

$$\begin{bmatrix} z_\Delta \\ z \end{bmatrix} = N(s) \begin{bmatrix} w_\Delta \\ w \end{bmatrix}$$

$$N(s) = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix}$$

- **Robustness Synthesis is a 2 – Block representation, which includes the uncertainty structure in $G(s)$:**

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(s) \begin{bmatrix} w \\ u \end{bmatrix}$$

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$$

μ

- Linear Fractional Transformation (upper, lower) techniques are used to establish the appropriate transfer matrices $N(s)$ and $G(s)$, including all weights.

- Robustness Analysis:

$$\begin{bmatrix} z_\Delta \\ z \end{bmatrix} = N(s) \begin{bmatrix} w_\Delta \\ w \end{bmatrix}$$

$$N(s) = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix}$$

- Robustness Synthesis:

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(s) \begin{bmatrix} w \\ u \end{bmatrix}$$

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$$

$$N_{zw}(s) = \left\{ N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1} N_{12} \right\}$$

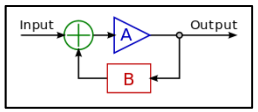
$$G_{zw}(s) = \left\{ G_{11} + G_{12}K(I - G_{22}K)^{-1} G_{21} \right\}$$

$$N_{zw}(s) = \mathbb{F}_U \left\{ N(P,K), \Delta \right\}$$

$$N_{11}(s) = M(s)$$

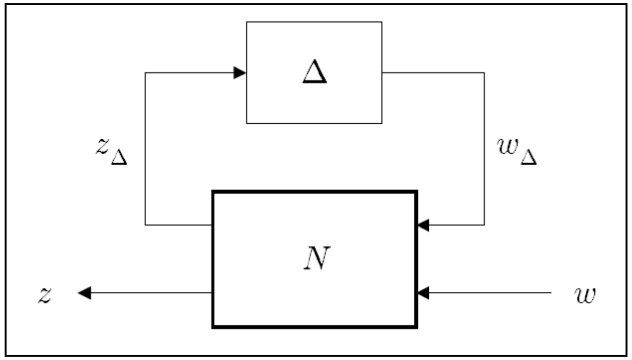
$$G_{zw}(s) = \mathbb{F}_L \left\{ G(P,\Delta), K \right\}$$

$$G_{22}(s) = P(s)$$



μ

- **Definition of Robust Stability (RS)**: The controller $K(s)$ within the feedback $N(s)$ stabilizes the nominal system ($\Delta = 0$) as well the closed loop uncertain 2 – block structure, for ALL uncertainties assumed in block Δ belonging to the class (*).



$$z = \left[N_{22} + N_{21}\Delta(I - M\Delta)^{-1}N_{12}\right]w$$

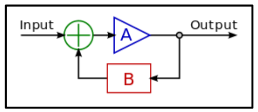
- Critical relationship between $\boldsymbol{RS}$ and $I\text{-}M\Delta$ to have a proper and stable inverse or, equivalently:

$$\det[I - M\Delta] \neq 0$$

- **Logical Procedure for introducing the structured singular value (SSV or μ)**

  - Consider the set of structured uncertainties given by **(*)**
  - Scale the set Δ with a parameter r, which in practice shrinks or expands the set $\boldsymbol{\Delta}$ **->** $\boldsymbol{r\Delta}$
  - Determine the largest r^* that keeps $\det[I - M\Delta] \neq 0 \; \forall \Delta \in r\Delta$

$$N_{11} = -(I + KG)^{-1}K = M$$



μ

- **Analytically:**

$$r^* = \sup\left\{ r \mid \det(I - M\Delta) \neq 0; \forall \Delta \in r\boldsymbol{\Delta} \right\}$$

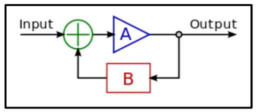
  - **Note:** For small r, $\det(I - M\Delta) \neq 0$ for any $\Delta \in r\boldsymbol{\Delta}$. As r increases, we might find some Δ for which the matrix $I - M\Delta$ is singular. If $r^* = \infty$ this does not happen. Otherwise r^* is the finite critical value up to which we can assure non singularity.

- **Definition 1:** The structured singular value of the complex matrix $M(\mathrm{s})$ with respect to the set $\boldsymbol{\Delta}$ is defined as:

$$\mu_{\boldsymbol{\Delta}} = \frac{1}{r^*} = \frac{1}{\sup\left\{ r \mid \det(I - M\Delta) \neq 0; \forall \Delta \in r\boldsymbol{\Delta} \right\}}$$

$$r \uparrow \Rightarrow \mu \downarrow$$

- The theory of SSV includes several fundamental statements and algebraic properties that can be used in the context of robust control.



Alternate Definition of Structured Singular Value:

$$det\,[I - M(j\omega)\Delta(j\omega)] \neq 0,\ \ \forall \omega \in \mathcal{R},\ \ \forall \Delta$$

- Recall the general Expression of Uncertainty:

$$\mathbf{\Delta} = \{diag[\delta_1 I_{r_1}, \cdots, \delta_s I_{r_s}, \Delta_1, \cdots, \Delta_f] : \delta_i \in \mathcal{C}, \Delta_j \in \mathcal{C}^{m_j \times m_j}\}$$

where $\sum_{i=1}^{s} r_i + \sum_{j=1}^{f} m_j = n$ with n is the dimension of the block Δ. We also assume the set of $\mathbf{\Delta}$ is bounded. And, we may thus define a normalized set of structured uncertainty by

$$\mathbf{B\Delta} := \{\Delta :\ \overline{\sigma}(\Delta) \leq 1,\ \ \Delta \in \mathbf{\Delta}\}$$

Definition 3.3. *For $M \in \mathcal{C}^{n\times n}$, the structured singular value $\mu_\Delta(M)$ of M with respect to $\mathbf{\Delta}$ is the number defined such that $\mu_\Delta^{-1}(M)$ is equal to the smallest $\overline{\sigma}(\Delta)$ needed to make $(I - M\Delta)$ singular (rank deficiency). That is*

$$\mu_\Delta^{-1}(M) := \min_{\Delta \in \mathbf{\Delta}} \{\overline{\sigma}(\Delta) :\ \ det(I - M\Delta) = 0\} \qquad (3.12)$$

If there is no $\Delta \in \mathbf{\Delta}$ such that $det(I - M\Delta) = 0$, then $\mu_\Delta(M) := 0$. ■

μ

- **Theorem (19, Scherer):** Let $M(s)$ be a complex matrix and $\mathbf{\Delta}$ an arbitrary (open) set of complex matrices. Then:
  - $\mu_\Delta(M) \leq 1$ implies that $I - M\Delta$ is non-singular for all $\Delta \in \mathbf{\Delta}$.
  - $\mu_\Delta(M) > 1$ implies that there exists a $\Delta \in \mathbf{\Delta}$ for which $I - M\Delta$ is singular.

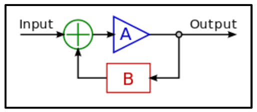
- **Proof:** by contradiction (see Scherer notes)

  - Unlike the Small Gain Theorem statement, SSV depends on M as well as Δ,
  - An analytical closed form solution of SSV is not possible,
  - Efficient computation of upper and lower limits is possible numerically.

- **Theorem (20 Scherer):** Let $M(s)$ be a complex matrix and $\mathbf{\Delta}$ an arbitrary (open) set of complex matrices. Then:
  - $\mu_\Delta(M) \leq \gamma_1$ implies that $I - M\Delta$ is non-singular for all $\Delta \in \gamma_1^{-1}\mathbf{\Delta}$.
  - $\mu_\Delta(M) > \gamma_2$ implies that there exists a $\Delta \in \gamma_2^{-1}\mathbf{\Delta}$ for which $I - M\Delta$ is singular.

  - **Proof:** it follows from μ property for which scaling of M or Δ scales μ of the same amount

$$\mathbf{\Delta}_\gamma = \left\{\Delta \in \mathbb{RH}_\infty^{p\times q} \mid \|\Delta\|_\infty < \gamma\right\}$$



Theorem 19 Proof:

Proof. Let us first assume that $\mu_{\boldsymbol{\Delta_c}}(M_c) \leq 1$. This implies that $r_* \geq 1$. Suppose that there exists a $\Delta_0 \in \boldsymbol{\Delta_c}$ that renders $I - M_c\Delta_0$ singular. Since $\boldsymbol{\Delta}_c$ is open, Δ_0 also belongs to $r\boldsymbol{\Delta}_c$ for some $r < 1$ that is close to 1. By the definition of r_*, this implies that r_* must be smaller than r. Therefore, we conclude that $r_* < 1$ what is a contradiction.

Suppose now that $\mu_{\boldsymbol{\Delta_c}}(M_c) > 1$. This implies $r_* < 1$. Suppose $I - M_c\Delta_c$ is non-singular for all $\Delta_c \in r\boldsymbol{\Delta}_c$ for $r = 1$. This would imply (since r_* was the largest among all r for which this property holds) that $r_* \geq r = 1$, a contradiction.

■

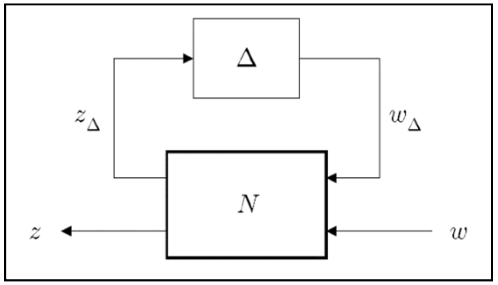
- **Theorem (21, Scherer):** $I - M\Delta$ has a proper and stable inverse for all $\Delta \in \boldsymbol{\Delta}$ if and only if:
  - $\mu_{\Delta}(M(s)) \leq 1$ for all $\omega \in [0, \infty)$

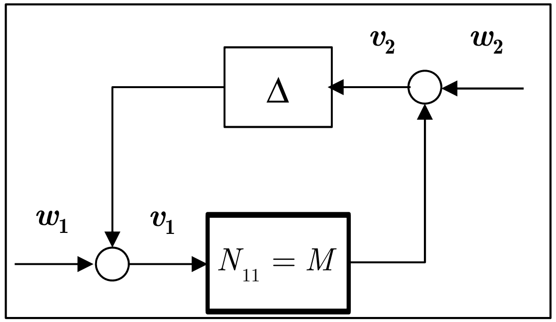
Fundemental result of robust stability:

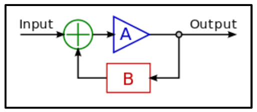
- **Corollary (22 Scherer):** If $K(s)$ stabilizes $P(s)$, and if
  - $\mu_{\Delta}(M(s)) \leq 1$ for all $\omega \in [0, \infty)$

  Then $K(s)$ robustly stabilizes $\mathbb{F}_L\{G(P, K)\}$ with respect to all $\boldsymbol{\Delta}$

- 2 – Block equivalence in terms of robust internal stability







Robustness in MIMO Systems

- **NOTE:** 2- Block Diagram Equivalence. Signals $(\boldsymbol{u},\ \boldsymbol{y})$ on the left diagram are the same as signals $(\boldsymbol{v}_1,\ \boldsymbol{v}_2)$ on right diagram

Figure 1

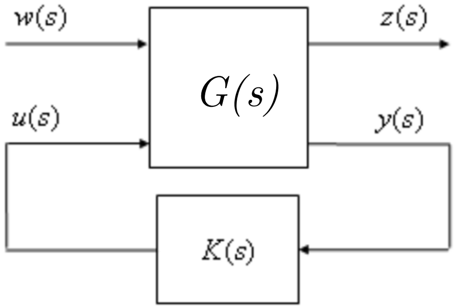
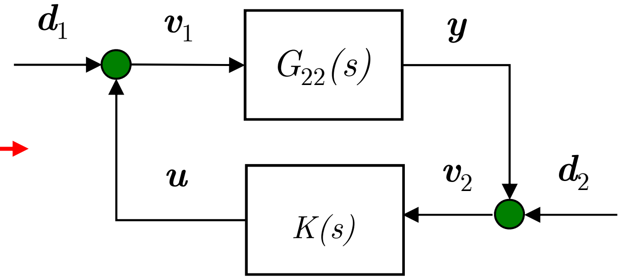


Figure 2



- From Figure 2 (irrespective of d_i), follows:

$$\begin{bmatrix} I & -D_k \\ -D_{22} & I \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \end{bmatrix} + \begin{bmatrix} 0 & C_k \\ C_2 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{q} \end{bmatrix}$$

- Compute the closed loop transfer matrix from disturbance to output:

$$\begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} = T(s)\begin{bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}\begin{bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \end{bmatrix} \quad \textbf{(a)}$$

(b)

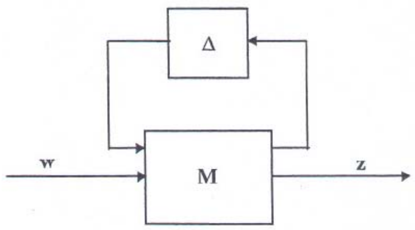
$$A_c = \left[\begin{array}{c|c} A & B_2C_k \\ \hline 0 & A_k \end{array}\right] + \left[\begin{array}{c} B_2D_k \\ \hline B_k \end{array}\right] R^{-1}\left[\begin{array}{c|c} C_2 & D_{22}C_k \end{array}\right] \qquad C_c = D_c\begin{bmatrix} 0 & C_k \\ C_2 & 0 \end{bmatrix}$$

$$B_c = \begin{bmatrix} B_1 & 0 \\ 0 & B_k \end{bmatrix} D_c \qquad D_c = D_{11} + D_{12}D_kR^{-1}D_{21}$$

- **Lemma: Since the two closed loop transfer matrices are the same, the system in Figure 1 is well-posed and internally stable, if and only if the system in Figure 2 has the same properties.**

- When *M* is an interconnected transfer matrix, the structured singular value, with respect to Δ, is defined by

$$\mu_\Delta(M(s)) := \sup_{\omega \in \mathcal{R}} \mu_\Delta(M(j\omega))$$



- Correspondingly, the uncertainty set may be defined as

$$\mathcal{M}(\mathbf{\Delta}) := \{\Delta(\cdot) \in \mathcal{RH}_\infty : \ \Delta(j\omega) \in \mathbf{\Delta} \ \mathit{for\ all}\ \omega \in \mathcal{R}\}$$

- The reciprocal of the structured singular value denotes a frequency dependent stability margin. The larger is µ, the smaller the perturbation that makes $\det(I\text{-}M\Delta) = 0$. The robust stability result with regard to structured uncertainty is now given in the following theorem:

Theorem *Let the nominal feedback system* $(M(s))$ *be stable and Let* $\beta > 0$ *be an uncertainty bound, i.e.,* $\|\Delta\|_\infty \leq \beta, \forall\ \Delta(\cdot) \in \mathcal{M}(\mathbf{\Delta})$. *The perturbed system of Figure is robustly stable, with respect to* $\mathbf{\Delta}$, *if and only if* $\mu_\Delta(M(s)) < \frac{1}{\beta}$. ■

It is obvious that if the uncertainty lies in the unit ball $\mathbf{B\Delta}$, the robust stability condition is then $\mu_\Delta(M(s)) < 1$.

μ

- $\mu[M(j\omega)]$ is frequency dependent and is calculated at “each” frequency over a reasonable range. Therefore, the computation of the structured singular value is not an easy task. Practical computation is based on its properties, and on the evaluation of its lower and upper bounds (see Mackenroth text section 13.3.2)

  - **Property 1:** The structured singular value is monotone in the set Δ

$$\Delta_1 \subset \Delta_2; \left\{\Delta_1, \Delta_2 \in \boldsymbol{\Delta}; M \in \mathbb{C}^{p\times q}\right\} \Rightarrow \mu_{\Delta_1}(M) \le \mu_{\Delta_2}(M)$$

  - **Property 2:** The structured singular value is invariant with respect to scalar scaling

$$\mu(\alpha M) = \left|\alpha\right| \mu(M); \forall \alpha \in \mathbb{C}$$

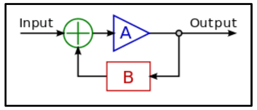
  - **Property 3:** From the definition of structured singular value

$$\Delta_1 = \left\{\delta I_n : \delta \in \mathbb{C}\right\} \Rightarrow \mu(M) = \rho(M); \rho(M) = \max_i \left|\lambda_i(M)\right|$$

ρ Is the spectral radius

$$\Delta_2 \in \mathbb{C}^{n\times n} \Rightarrow \mu(M) = \sigma_{\max}(M)$$

  - **Property 4:** If the uncertainty Δ contains only real blocks $\Delta = \left\{\delta I_n : \delta \in \mathbb{R}\right\} \Rightarrow \mu(M) = \rho_{\mathbb{R}}(M)$ and μ is discontinuous; if the uncertainty Δ contains only complex blocks, then the mapping M -> μ is continuous



- **Lemma (Mackenroth, p. 398):** Given a complex matrix $M(\mathrm{s})$, and an uncertainty set Δ given by repeated scalar blocks and repeated full blocks as:

$$\Delta \in \mathbb{C}^{n\times n} = \left\{ diag\left[\delta_1 I_{r_1}, \ldots, \delta_s I_{r_S}, \Delta_1, \ldots, \Delta_F\right] : \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j \times m_j} \right\} \qquad (*)$$

$$\sum_{i=1}^{S} r_i + \sum_{j=1}^{F} m_j = n$$

  - Then, the following holds: $\boxed{\rho(M) \le \mu_\Delta(M) \le \sigma_{\max}(M)}$ **(**)**

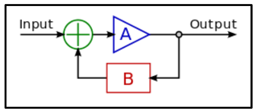
- **Bounds on $\mu_\Delta M(j\omega)$:** Numerical computation of bounds (**) requires tighter gap between extrema, in order to guarantee that robust stability is maintained (Doyle & Young, 1990, IEEE CDC)

  - **Lower Bound:** Procedure recalls Theorem 20, in particular: $\gamma_0 \le \mu_\Delta(M)$ implies that there exists a $\Delta \in \gamma_0^{-1}\,\mathbf{\Delta}$ for which $I - M\Delta$ is singular.

Suppose we find a pair $(\gamma_0,\ \omega_0)$ such that the inequality holds. Then there exists some $\Delta_0 \in \gamma_0^{-1}\,\mathbf{\Delta}$ that makes $I - M(j\omega_0)\Delta_0 = 0$. This is a non convex optimization to yield the highest possible γ_0.

  - **Upper Bound:** Procedure recalls Theorem 20, in particular: $\mu_\Delta(M) \le \gamma_1$ implies that $I - M\Delta$ is non singular for all $\Delta \in \gamma_1^{-1}\,\mathbf{\Delta}$

Note that for full complex uncertainty a sufficient condition is already known from the small gain theorem.



μ

- **Scaling:** The main idea is to get sharper bounds than those given by **(**)** by appropriate scaling of $M(s)$
  - Consider the square case $m = n$ and the uncertainty set $\boldsymbol{\Delta}$ given by **(*)**.
  - Define the following sets $\mathcal{U}$ and $\mathcal{D}$ (frequency dependent matrices)

$$\mathcal{U} = \left\{ \mathcal{U} \in \boldsymbol{\Delta} : UU^{*} = I_{n} \right\}$$

$$\mathcal{D} = \left\{ D = diag\left[D_{1}, \ldots, D_{S}, d_{1}\, \mathrm{I}_{\mathrm{m}_{1}}, d_{F}\, \mathrm{I}_{\mathrm{m}_{F}} \right] : D_{i} \in \mathbb{C}^{r_i \times r_i}, D_{i} = D_{i}^{*} > 0, d_{j} > 0 \right\}$$

  - The following equations hold for every : $\Delta \in \boldsymbol{\Delta}, U \in \mathcal{U}, D \in \mathcal{D}$

$$U\mathcal{U} = \mathcal{U} = \mathcal{U}U$$

$$\sigma_{\max}(U\Delta) = \sigma_{\max}(\Delta U) = \sigma_{\max}(\Delta)$$

$$D\Delta = \Delta D$$

- **Main Result :** (Mackenroth Corollary 13.3.1., Zhou Section 10.2.2.)

$$\max_{U \in \mathcal{U}} \rho(MU) \le \mu_{\Delta}(M) \le \inf_{D \in \mathcal{D}} \left[\sigma_{\max}(DMD^{-1}) \right]$$

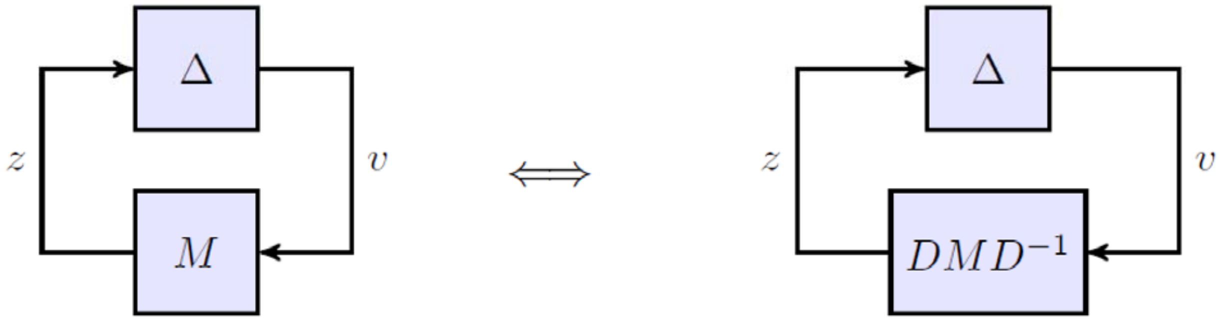
Lower Bound yields local minima, since is a non convex optimization. (Doyle & Young, 1990, IEEE CDC)

Upper bound search is convex. Equality is achieved only if 2S+F ≤ 3 (Doyle & Young, 1990, IEEE CDC) .

- **Scaling:** The main idea is to get sharper bounds than those given by **(**)** by appropriate scaling of $M(s)$

If an invertible matrix D commutes with all $\Delta \in \boldsymbol{\Delta}$, then,

$$\mu_{\boldsymbol{\Delta}}(M) = \mu_{\boldsymbol{\Delta}}(DMD^{-1})$$

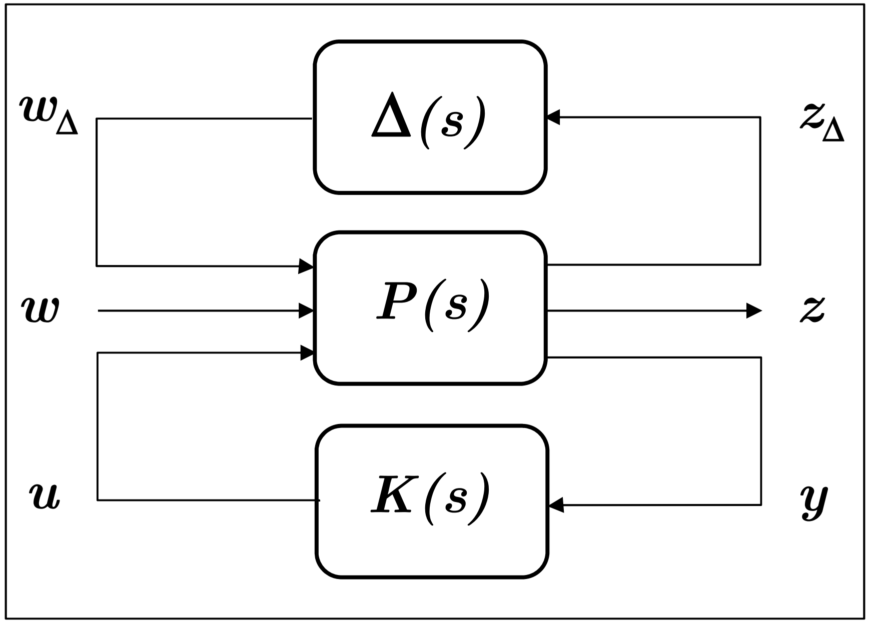


A μ upper bound (for robust performance):

$$\mu_{\tilde{\boldsymbol{\Delta}}}(M) \leq \inf_{D \in \mathcal{D}} \overline{\sigma}\left(DMD^{-1}\right)$$

Notation: $\mathcal{D} = \{\, D \mid D\Delta D^{-1} \in \boldsymbol{\Delta} \text{ for all } \Delta \in \boldsymbol{\Delta} \,\}$

- **General Problem Setup:** (Performance and Uncertainty frequency shaping weights added to $P(\mathrm{s})$ if appropriate)



We can now address the global Robust Control Problem
by defining the solution steps for:

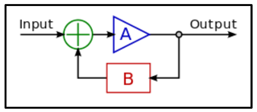
1. Nominal Stability
2. Nominal Performance
3. Robust Stability
4. Robust Performance

- **Nominal Stability:**
  - For $\Delta = 0$, conditions are known for determining a controller $K(s)$, such that the feedback loop is well posed, and internally stable

$$K(s) = \left[\begin{array}{c|c} A + B_2K_1 + K_2C_2 + K_2D_{22}K_1 & -K_2 \\ \hline K_1 & 0 \end{array}\right]$$

$$T_{zw}(s) \in \Re H_\infty$$

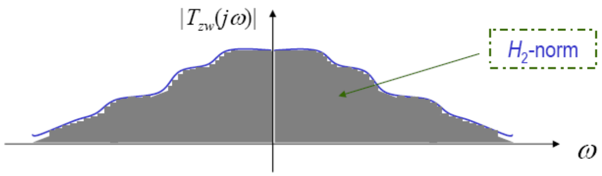
$$\{P_{zw} + P_{zu}K(I\text{-}P_{yu}K)^{-1}P_{yw}\} = T_{zw}(s)$$



❑ Nominal Stability and Performance (Optimal Control):

- LQR, LQG are obtained when we minimize the system's $\mathcal{H}_2$ norm

Graphically,

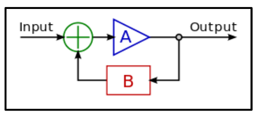


Note: The H_2-norm is the total energy corresponding to the impulse response of $T_{zw}(s)$. Thus, minimization of the H_2-norm of $T_{zw}(s)$ is equivalent to the minimization of the total energy from the disturbance w to the controlled output z.

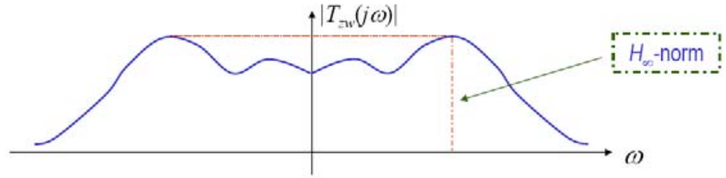
$$\begin{cases} A^T X_2 + X_2 A - X_2 B_2 R^{-1} B_2^T X_2 + Q = 0 \\ Y_2 A^T + A Y_2 - Y_2 C_2^{\ T} V^{-1} C_2 + B_1 W B_1^T = 0 \end{cases}$$

$$\begin{cases} F_2 = -R^{-1} B_2^T X_2 \\ L_2 = -Y_2 C_2^T V^{-1} \end{cases}$$

$$\begin{cases} u = F_2 q \\ \dot{q} = \left(A + B_2 F_2 + L_2 C_2 + L_2 D_{22} F_2\right) q - L_2 y \end{cases}$$



- $\mathcal{H}_\infty$ - Control is obtained when we minimize the system's $\mathcal{H}_\infty$ norm



Note: The H_∞-norm is the worst case gain in $T_{zw}(s)$. Thus, minimization of the H_∞-norm of $T_{zw}(s)$ is equivalent to the minimization of the worst case (gain) situation on the effect from the disturbance w to the controlled output z.

$$K_{sub}(s) = \left(\begin{array}{c|c} A_\infty & L_Z \\ \hline K_\infty & 0 \end{array}\right)$$

$$A_\infty = A + \gamma^{-2} X B_1 B_1^T X - B_2 (D_{12}^T D_{12})^{-1} B_2^T X - L_Z C_2$$

$$K_\infty = -(D_{12}^T D_{12})^{-1} B_2^T X$$

$$L_Z = Z \tilde{C}_2^T (\tilde{D}_{21} \tilde{D}_{21}^T)^{-1}$$

$$Z = Y (I - \gamma^{-2} X Y)^{-1}$$

- The numerical computationn is known as γ - iteration

μ Analysis and Synthesis

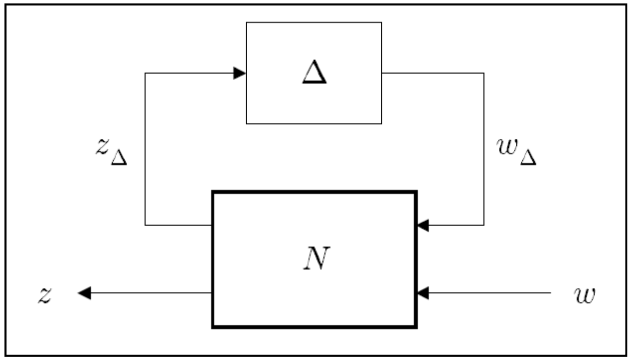
Robust Stability:

- For unstructured uncertainties, the Small Gain Theorem gives necessary and sufficient conditions in terms of the infinity Norm of the Complementary Sensitivity Matrix with respect to stable and norm bounded Uncertainties.

$$\left\| N(s) = F_l(P,K) \right\|_\infty = \left\| \left\{ N_{22} + N_{21} K (I - N_{11} K)^{-1} N_{12} \right\} \right\|_\infty \leq \gamma \qquad \Delta \in \mathbb{C}^{p \times q}; \left\| \Delta \right\|_\infty < \gamma^{-1}$$

$$N_{11} = M$$

- For structured (as well as unstructured) uncertainties, the same result is achieved using the structured singular Value μ.



$$\mu_\Delta[M(s)] := \sup_{\omega \in \mathbb{R}} \mu_\Delta[M(s)] \leq \inf_{D \in \mathcal{D}} \left[\sigma_{\max}(DMD^{-1}) \right] \leq \gamma \quad \left\| \Delta \right\|_\infty < \gamma^{-1}$$

$$\Delta \in \mathbb{C}^{n \times n} = \left\{ diag\left[\delta_1 I_{r_1}, \ldots, \delta_s I_{r_S}, \Delta_1, \ldots, \Delta_F \right] : \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j \times m_j} \right\}$$

$$\sum_{i=1}^{S} r_i + \sum_{j=1}^{F} m_j = n$$

- **For convenience, we can use a norm – bounded subset of Δ, such that the upper bound is 1 rather than γ, as follows:**

$$\boldsymbol{B\Delta} = \left\{ \Delta \in \boldsymbol{\Delta} : \sigma_{\max}(\Delta) \leq 1 \right\}$$

μ Analysis and Synthesis

Robust Performance:

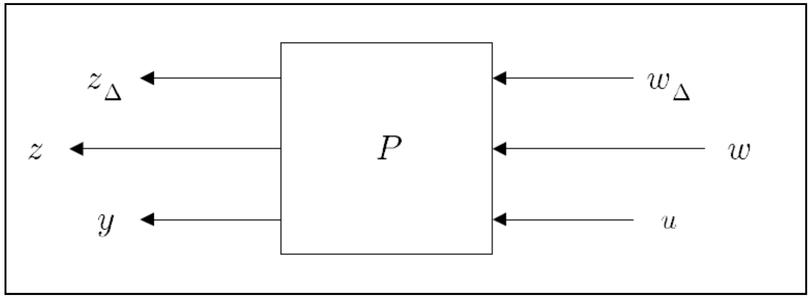
- Robust Performance is measured by the Infinity Norm of the feedback system, as seen by the uncertainty. The feedback system includes therefore the uncertainty set. That is:

$$\left\|F_u(N,\Delta)\right\|_\infty \leq \gamma \qquad \left\|\Delta\right\|_\infty < \gamma^{-1}$$

- Robust Performance can be seen as a particular case of Robust Stability: Consider the uncertain 2 Block, and create a feedback with a fictitious unstructured Uncertainty Δ_p

$$\tilde{\Delta} \in \tilde{\mathbf{\Delta}} := \{diag\{\Delta,\ \Delta_p\} : \Delta \in \mathbf{B\Delta},\ \|\Delta_p\|_\infty \leq 1\}$$

1. The first step is to guarantee the conditions of nominal performance of the general open interconnection below:



$$\begin{bmatrix} z_\Delta \\ z \\ y \end{bmatrix} = P \begin{bmatrix} w_\Delta \\ w \\ u \end{bmatrix} \qquad P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \qquad \textbf{(1rp)}$$

$$\begin{cases} u = Ky \\ w_\Delta = \Delta z_\Delta \\ \Delta \in \boldsymbol{B\Delta} \end{cases} \qquad \textbf{(2rp)}$$

μ Analysis and Synthesis

- Nominal uncontrolled interconnection (no uncertainty):

$$P_0 = S(0,P) = \begin{bmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{bmatrix}$$

- Nominal controlled interconnection (no uncertainty, stable closed loop system):

$$\left\{ z = S(P_0,K)w; S(P_0,K) = P_{22} + P_{23}K(I - P_{33}K)^{-1}P_{32} \right\}$$

  - One possible solution is an $\mathcal{H}_\infty$ controller $K(\mathrm{s})$, where solution is achieved by requiring:

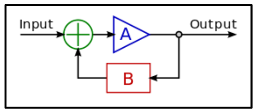
$$\left\| S(P_0,K) \right\|_\infty \leq 1$$

2. The second step is to guarantee robust performance:

- Uncertain uncontrolled interconnection (loop closed over uncertainty):

$$P_\Delta = S(\Delta,P) = \begin{bmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{bmatrix} + \begin{bmatrix} P_{21} \\ P_{31} \end{bmatrix} \Delta(I - P_{11}\Delta)^{-1} \begin{bmatrix} P_{12} & P_{13} \end{bmatrix}$$

  - The controller $K(\mathrm{s})$ should stabilize $P_\Delta = S(\Delta,\ P)$ and $\left\| S(P_\Delta,K) \right\|_\infty \leq 1; \forall \Delta \in \mathbf{\Delta}$ **(3rp)**



µ Analysis and Synthesis

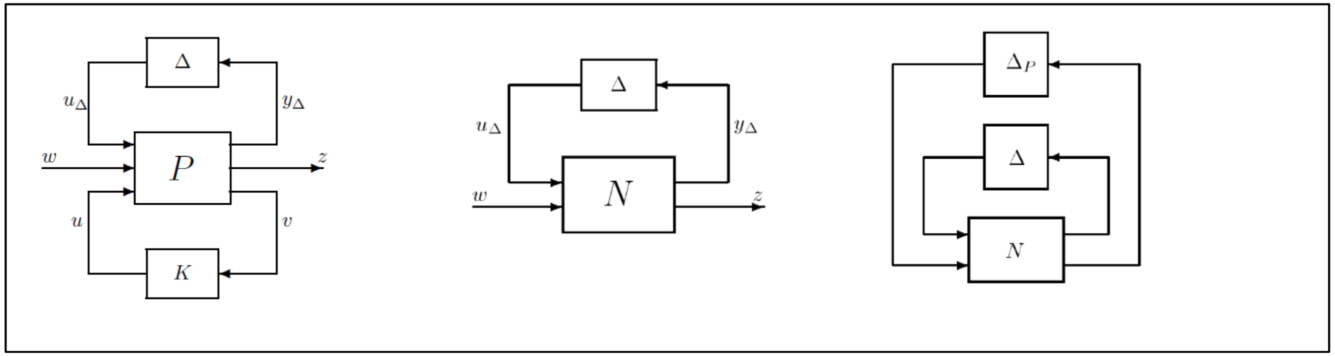
- **RP Analysis**: Given $K(\mathrm{s})$, test whether (3rp) is satisfied
- **RP Synthesis**: Find $K(\mathrm{s})$ such that (3rp) is satisfied

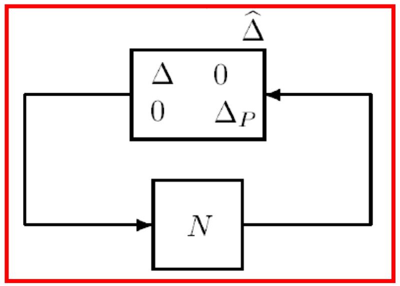
$$\left\| S(P_{\Delta}, K) \right\|_{\infty} \leq 1; \forall \Delta \in \mathbf{\Delta} \qquad \text{(3rp)}$$

3. The third step is to perform the robust performance analysis:

➡ **The 'trick' is the addition of a fictitious unstructured uncertainty Δ_{P} of dimensions corresponding to the sizes of z and w, and to solve the equivalent robust stability problem of the resulting interconnection.**

- The fictitious uncertainty is fed back to the standard interconnection





- ❑ $w = \Delta_{\mathrm{P}} z$

- With the previous structure N, we can show that robust performance of the feedback loop M with uncertainty Δ, is equivalent to robust stability of the feedback loop N with uncertainty $\hat{\Delta}$

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} M & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

❑ Main Theorem (Scherer, p. 114; Skogestad, p. 317)

- Consider the upper LFT representing a controlled uncertain system: $F_u(N,\Delta) = S(P_\Delta, K); \Delta \in \boldsymbol{\Delta}$
- Assume nominal stability ($\boldsymbol{NS}$), that is the block $N(\mathrm{s})$ is well-posed and internally stable with some admissible stabilizing controller $K(\mathrm{s})$,
- Then the controller $K(\mathrm{s})$ provides robust performance ($\boldsymbol{RP}$), that is:

$$\left\|F_u(N,\Delta)\right\|_\infty = \left\|S(P_\Delta, K)\right\|_\infty = \left\|S(S(\Delta, P), K)\right\|_\infty \leq 1; \forall \left\|\Delta\right\|_\infty, \Delta \in \boldsymbol{\Delta}$$

- Provided the following bound is satisfied:

$$\boxed{\mu_{\hat{\Delta}}[N(j\omega)] \leq 1; \left\{ \forall \omega \in \mathbb{R} \mid \hat{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix} \right\}}$$

- **Comments:**

1. The condition $\mu_{\hat{\Delta}}[N(j\omega)] \le 1;$ allows testing $\left\| F_u(N,\Delta) \right\|_\infty \le 1$ for all Δ,

   it is therefore a worst case analysis

2. The condition $\mu_{\hat{\Delta}}[N(j\omega)] \le 1;$ requires Δ_P be unstructured, for which the following SSV property holds:

$$\sigma_{\max}(N_{22}) = \mu_{\Delta_p}[N_{22}]$$

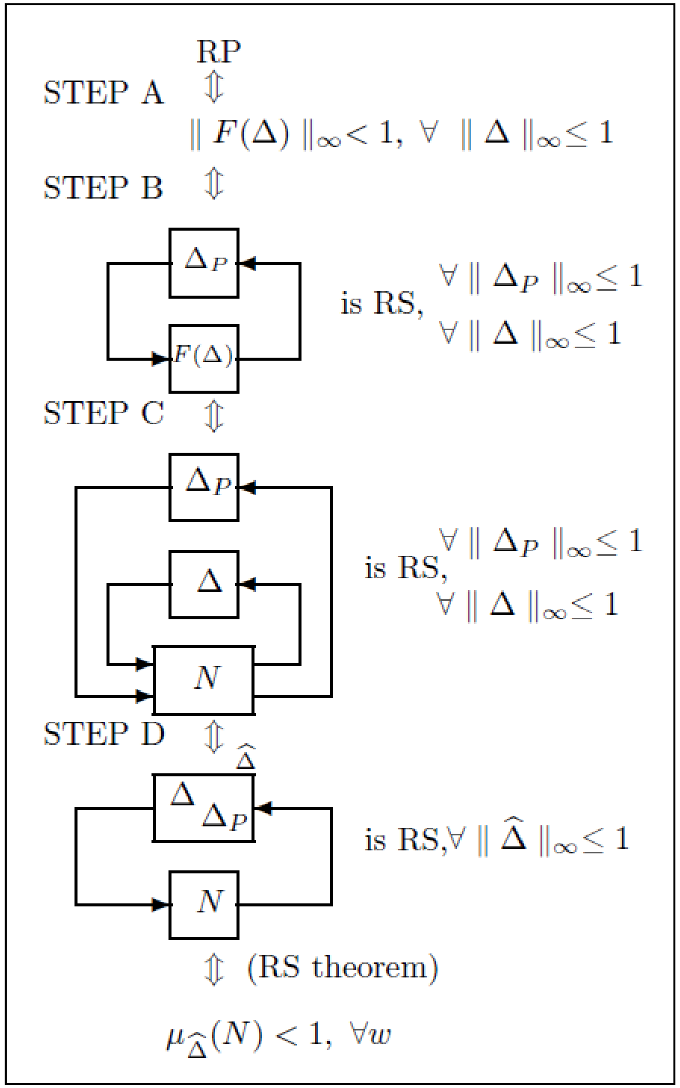
3. From properties of SSV, we have:

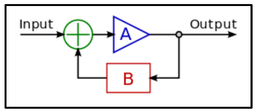
$$\underbrace{\mu_{\hat{\Delta}}[N]}_{\boldsymbol{RP}} \ge \max\left\{ \underbrace{\mu\Delta[N_{11}]}_{\boldsymbol{RS}}, \underbrace{\mu_{\Delta_P}[N_{22}]}_{\boldsymbol{NP}} \right\}$$

   This implies that robust stability ($\boldsymbol{RS}$) and nominal performance($\boldsymbol{NP}$) are satisfied by the robust performance ($\boldsymbol{RP}$) requirement. However, nominal stability ($\boldsymbol{NS}$) needs to be checked separately.

4. A more general form of the theorem can be found in: (Zhou, Th. 11.7, p. 284)

µ Analysis and Synthesis





- **Sketch of Proof 1:**

  - **STEP A:** Statement of robust performance Theorem

  - **STEP B:** From Small Gain Theorem, for unstructured uncertainty, for robust stability we require $\|M\|_\infty = \|N_{11}\|_\infty \leq 1$

    Therefore the equivalence:

    $$\left[\left\|F_u(N,\Delta)\right\|_\infty \leq 1\right]^{RP} \Leftrightarrow \left[\left\|F_u, \Delta_P\right\|_\infty \leq 1\right]^{RS}$$

  - **STEP C:** Introduce the new interconnection $\hat{\Delta}$

  - **STEP D:** compute the robust stability with the new interconnection, which yields:

    $$\mu_{\hat{\Delta}}[N(j\omega)] \leq 1;$$

Sketch of Proof 2:

- From definition of SSV:

$$\mu_{\hat{\Delta}}[N(j\omega)] < 1 \Leftrightarrow \det\left[(I - N(j\omega)\hat{\Delta}(j\omega)\right] \neq 0 \left\{\forall \hat{\Delta}, \sigma_{\max}(\hat{\Delta}(j\omega)) \leq 1\right\}$$

- From Schur's formula of a determinant of a partitioned matrix:

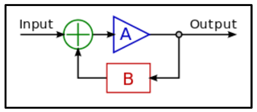
$$\det(I - N\hat{\Delta}) = \det\left\{\left[\begin{array}{c|c} I - N_{11}\Delta & -N_{12}\Delta_P \\ \hline -N_{21}\Delta & I - N_{22}\Delta_P \end{array}\right]\right\} = \ldots = \det(I - N_{11}\Delta) \cdot \det\left[(I - F_u(N,\Delta)\Delta_P\right]$$

- Since the above expression must be $\neq 0$ for each frequency:

$$\det(I - N_{11}\Delta) \neq 0; \forall \Delta \Leftrightarrow \mu_{\Delta}(N_{11}) < 1; \forall \omega$$ Which is a robust stability ($\boldsymbol{RS}$) requirement for all Δ

- And for all $\hat{\Delta}$:

$$\det\left[(I - F_u(N,\Delta)\Delta_P\right] \neq 0; \forall \Delta_P \Leftrightarrow \mu_{\Delta_p}(F_u) < 1 \Leftrightarrow \sigma_{\max}(F_u) < 1$$ robust performance ($\boldsymbol{RP}$) definition

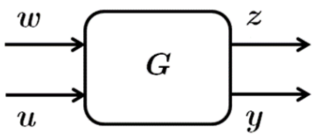


μ Analysis and Synthesis

❑ Summary:

$$\begin{bmatrix} z \\ y \end{bmatrix} = G \begin{bmatrix} w \\ u \end{bmatrix}$$

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

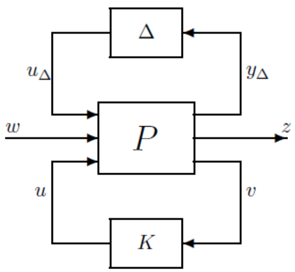


Class of Uncertainty

$$\Delta(s) \in \boldsymbol{\Delta}; \boldsymbol{\Delta} := \left\{ \Delta = \begin{bmatrix} \delta_1^r & & & & & & & \\ & \ddots & & & & & & \\ & & \delta_r^r & & & & & \\ & & & \delta_1^c & & & & \\ & & & & \ddots & & & \\ & & & & & \delta_q^c & & \\ & & & & & & \Delta_1 & \\ & & & & & & & \ddots \\ & & & & & & & & \Delta_m \end{bmatrix} \middle| \; \left|\delta_i^r\right| < 1, \; \left\|\delta_i^c\right\|_\infty < 1, \; \left\|\Delta_i\right\|_\infty < 1 \;\; \text{stable} \right\}$$

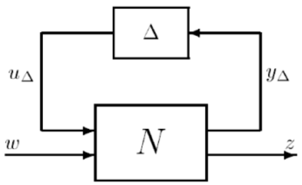
$$\begin{bmatrix} z_\Delta \\ z \\ y \end{bmatrix} = P \begin{bmatrix} w_\Delta \\ w \\ u \end{bmatrix} \qquad P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

Includes shaping and uncertainty weights



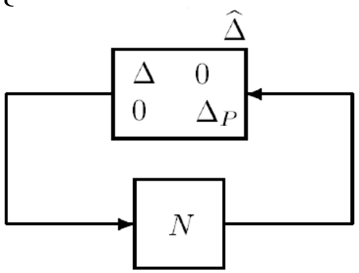
$$\begin{cases} z = P_{11} w + P_{12} u \\ v = P_{21} w + P_{22} u \end{cases}$$

$$z = \left[P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21} \right] w$$



$$\begin{cases} z_\Delta = {\color{red} N_{11}} w_\Delta + N_{12} w \\ z = N_{21} w_\Delta + N_{22} w \end{cases}$$

$$\begin{cases} N_{11} = M \\ N_{22} = \left[P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21} \right] \end{cases}$$



$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} {\color{red} M} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

μ Analysis and Synthesis

Appropriate Linear Fractional Transformations:

$$F_u(N,\Delta)=\left\{N_{22}+N_{21}\Delta(I\text{-}N_{11}\Delta)^{-1}N_{12}\right\};N_{22}=F_l(P,K)=\left[P_{11}+P_{12}K(I-P_{22}K)^{-1}P_{21}\right]$$

- **Nominal Stability – NS** : the system $N(\mathrm{s})$ is internally stable, for instance with proper choice of $K(\mathrm{s})$:

$$\left\|N_{22}\right\|_\infty < 1$$

- **Nominal Performance – NP** :

$$\sigma_{\max}\left[N_{22}\right]=\mu_{\Delta_P}<1;\forall\omega$$ + nominal stability

- **Robust Stability – RS** :

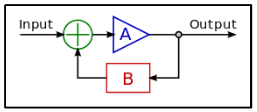
$$\mu_{\Delta}\left[N_{11}\right]<1;\forall\omega$$ + nominal stability

- **Robust Performance – RP** :

$$\mu_{\hat{\Delta}}\left[N\right]<1;\left\{\forall\omega,\hat{\Delta}=\begin{bmatrix}\Delta & 0\\ 0 & \Delta_P\end{bmatrix}\right\}$$ + nominal stability

Implies NP and RS

$$\underbrace{\mu_{\hat{\Delta}}[N]}_{RP}\geq\max\left\{\underbrace{\mu\Delta[N_{11}]}_{RS},\underbrace{\mu_{\Delta_P}[N_{22}]}_{NP}\right\}$$



D-K Iteration

- Design a controller $K(s)$ in the closed loop structure:

$$M(j\omega) = F_L(G,K) = \left\{G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}\right\}$$

where:

- Nominal Performance only, $\Delta = 0$:

$$G = \begin{bmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{bmatrix}$$

- Robust Stability only:

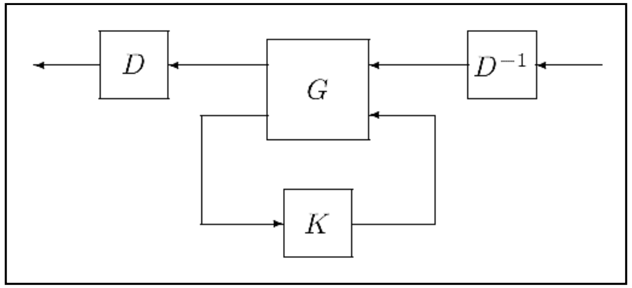
$$G = \begin{bmatrix} P_{11} & P_{13} \\ P_{31} & P_{33} \end{bmatrix}$$

- Robust Performance:

$$G = P = \left[\begin{array}{cc:c} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ \hdashline P_{31} & P_{32} & P_{33} \end{array}\right]$$

D-K Iteration

- ***D − K* Iteration:** A numerical procedure for determining an upper bound of the scaled SSV by iteratively selecting scaling matrix D and the controller $K(\mathrm{s})$



J.C. Doyle. Structured uncertainty in control system design. In *Proceedings of the 24 IEEE Conference on Decision and Control*, pages 260–265, December 1985.

- The procedure is based on the main equivalence:

$$\left\| F_U \left\{ [F_L[(P(j\omega), K(j\omega)], \Delta(j\omega) \right\} \right\|_\infty \le 1$$
$$\Updownarrow$$
$$\mu[F_L(G,K)] \le 1$$
$$\Updownarrow$$
$$\min_{K(s)\,stab} \sup_{\omega\in\mathbb{R}} \inf_{D,D^{-1}\in\mathcal{D}} \left[D\cdot F_L(G,K)\cdot D^{-1} \right]$$

- **Notes:**
  - The numerical iteration sequentially selects D, and $K(\mathrm{s})$ until μ can't be decreased any longer,
  - The minimization is convex in D and $K(\mathrm{s})$ separately, but not simultaneously,
  - From properties of SSV, we have also:

$$\hat{G} = \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix} G \begin{bmatrix} D^{-1} & 0 \\ 0 & I \end{bmatrix}$$

D-K Iteration

Step by Step Procedure:

1. Start with an initial guess for D, usually $D = I$.

$$D = diag\left[D_1, \dots, D_S, d_1 \mathrm{I}_{m_1}, d_F \mathrm{I}_{m_F}\right] : D_i \in \mathbb{C}^{r_i \times r_i}, D_i = D_i^* > 0, d_j > 0$$

2. Fix D, and solve and $\mathcal{H}_\infty$ optimization for $K(s)$ (or other controller) on $\hat{G}(s)$

$$K(s) = \arg \inf_{K(s)} \left\| F_L(\hat{G}, K) \right\|_\infty$$

3. Fix $K(s)$, and calculate D scales for μ upper bound

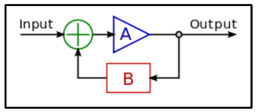
$$\inf_{D(\omega) \in \mathcal{D}} \sigma_{\max} [D(\omega) F_L(G, K) D(\omega)^{-1}]$$

4. Approximate frequency data $D(\omega)$ with a stable, minimum phase function:

$$D(\omega) \approx \hat{D}(\omega) \in \mathbb{R}H_\infty$$

5. Go to Step 2, and repeat until

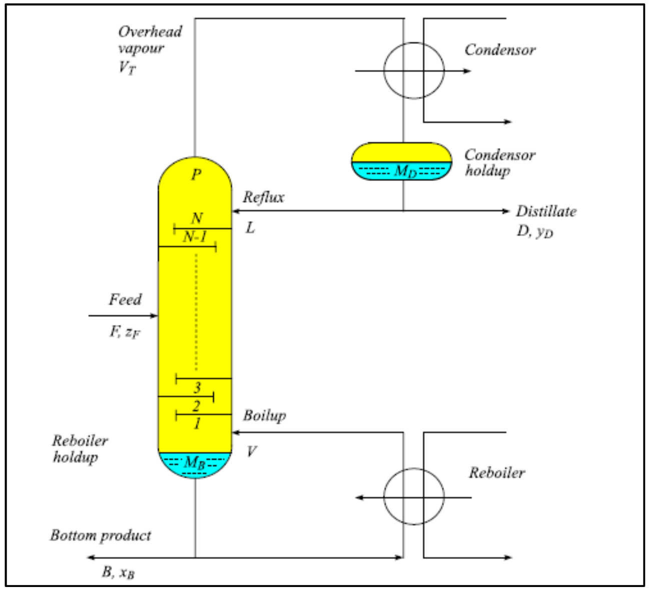
   - Prespecified tolerance is achieved, or
   - maximum iteration number is reached, or
   - $$\min_{K(s)\, stab} \sup_{\omega \in \mathbb{R}} \inf_{D, D^{-1} \in \mathcal{D}} \left[D \cdot F_L(G, K) \cdot D^{-1} \right] < 1$$

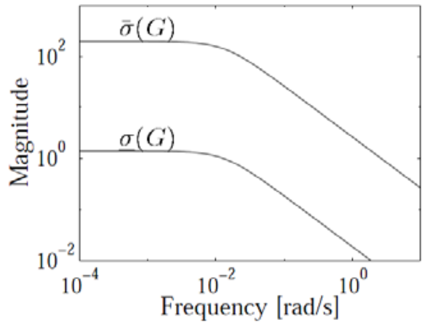


D-K Iteration

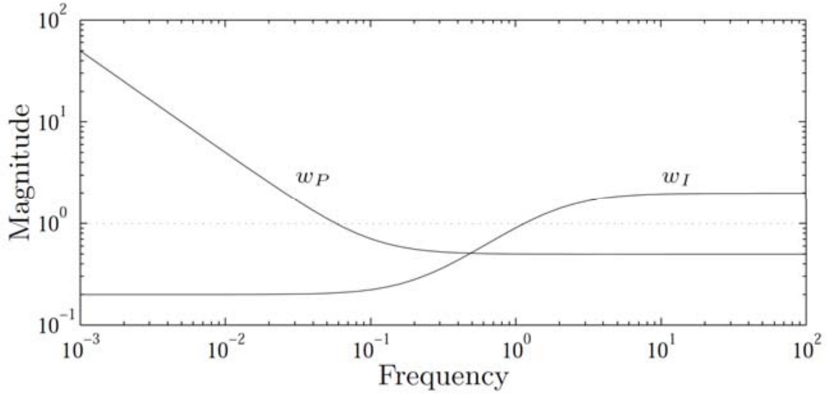
❑ Distillation Column Example:

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}$$



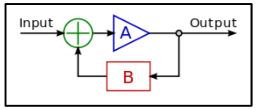


$$P = \begin{bmatrix} 0 & 0 & W_I \\ W_P G & W_P & W_P G \\ -G & -I & -G \end{bmatrix}$$



$$T_I = KG(I+KG)^{-1},\ S = (I+GK)^{-1}$$

$$N = \begin{bmatrix} w_I T_I & w_I K S \\ w_P S G & w_P S \end{bmatrix}$$

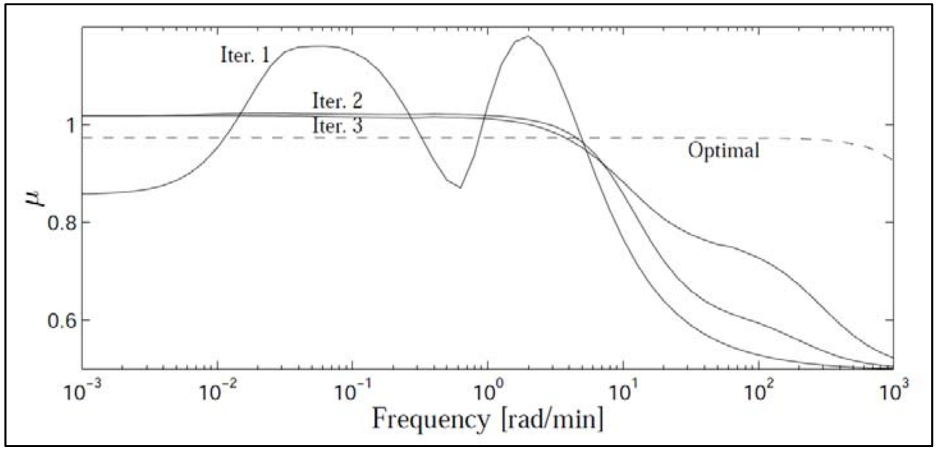


D-K Iteration

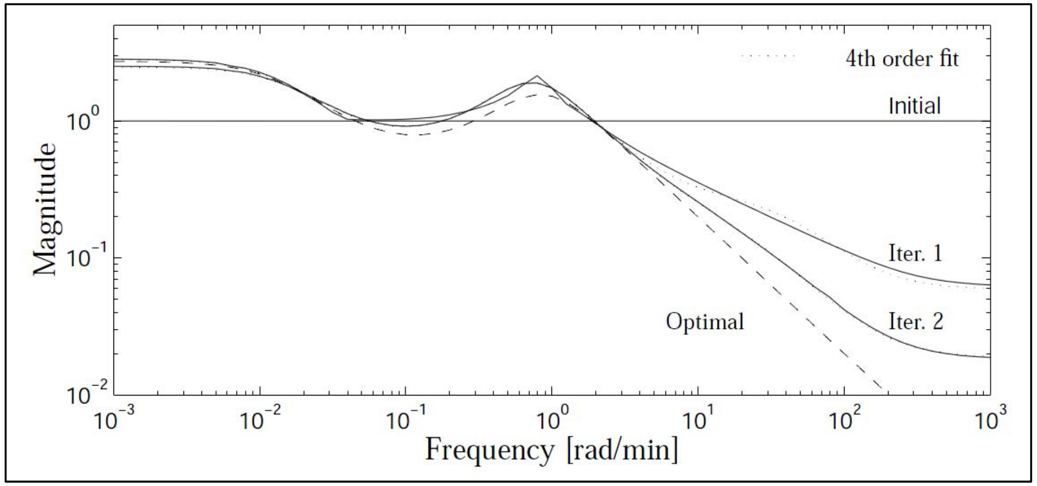
Then the block-structure is defined; it consists of two 1×1 blocks to represent Δ_I and a 2×2 block to represent Δ_P. The scaling matrix D for DND^{-1} then has the structure $D = \text{diag}\{d_1, d_2, d_3 I_2\}$ where I_2 is a 2×2 identity matrix, and we may set $d_3 = 1$. As initial scalings we select $d_1^0 = d_2^0 = 1$. P is then scaled with the matrix $\text{diag}\{D, I_2\}$ where I_2 is associated with the inputs and outputs from the controller (we do not want to scale the controller).

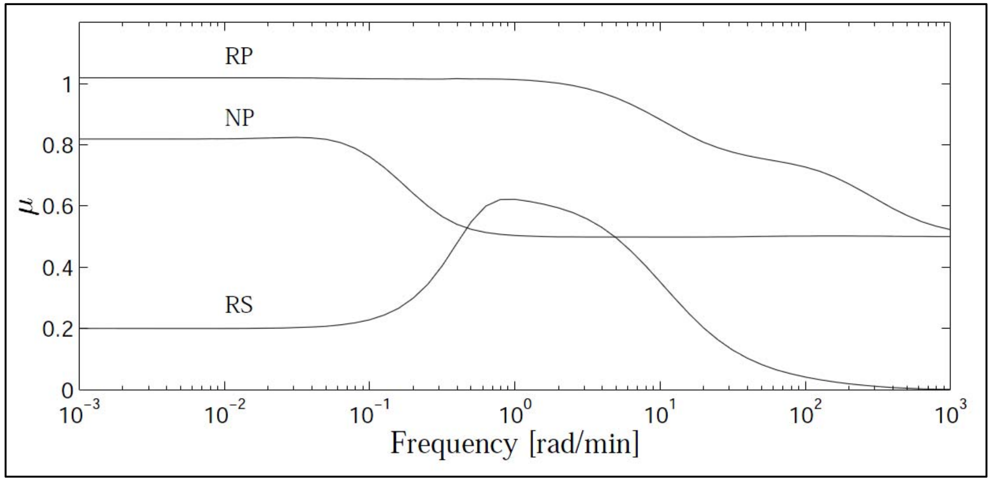
Step 1: With the initial scalings, $D^0 = I$, the H_∞ software produced a 6 state controller with $\gamma = 1.18$ and also $\mu = 1.18$

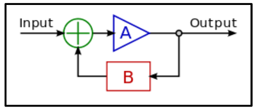
After 3 iterations, the resulting controller with 22 states gives a peak μ-value of 1.019, using 2 scaling matrices $D^1(s)$ and $D^2(s)$



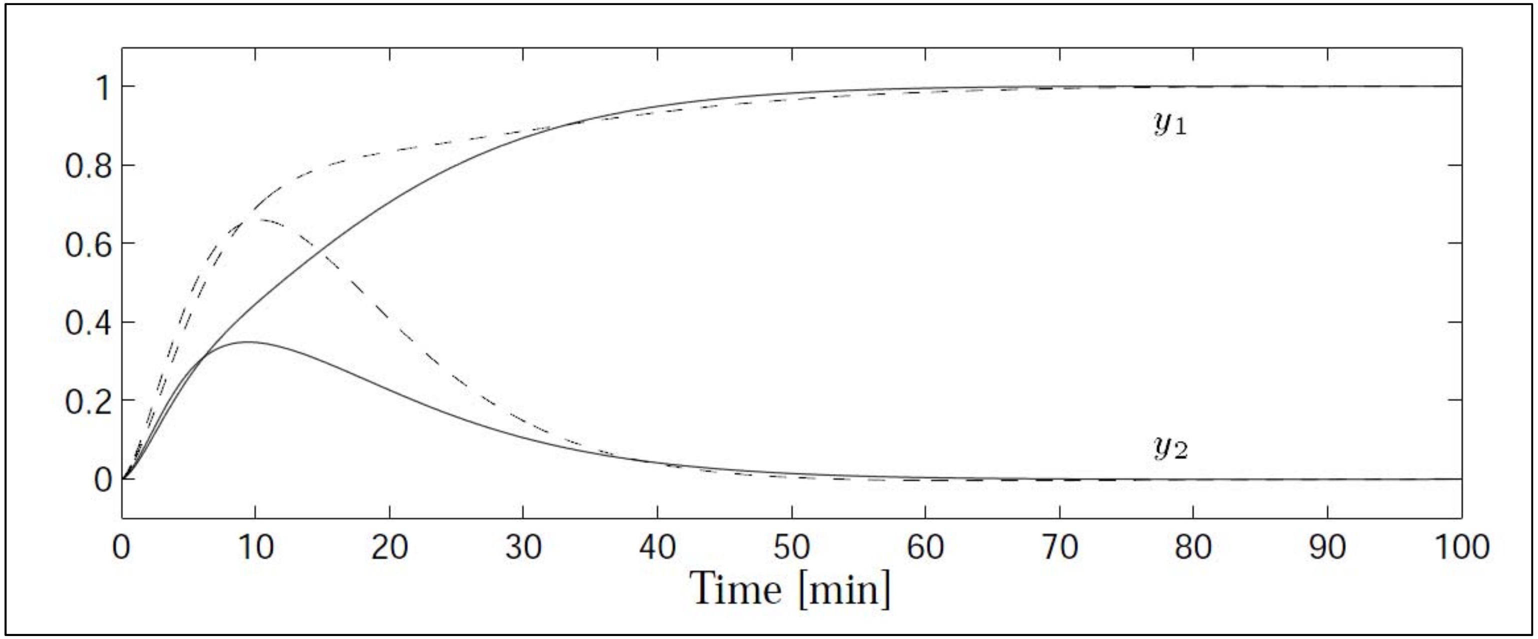
D-K Iteration

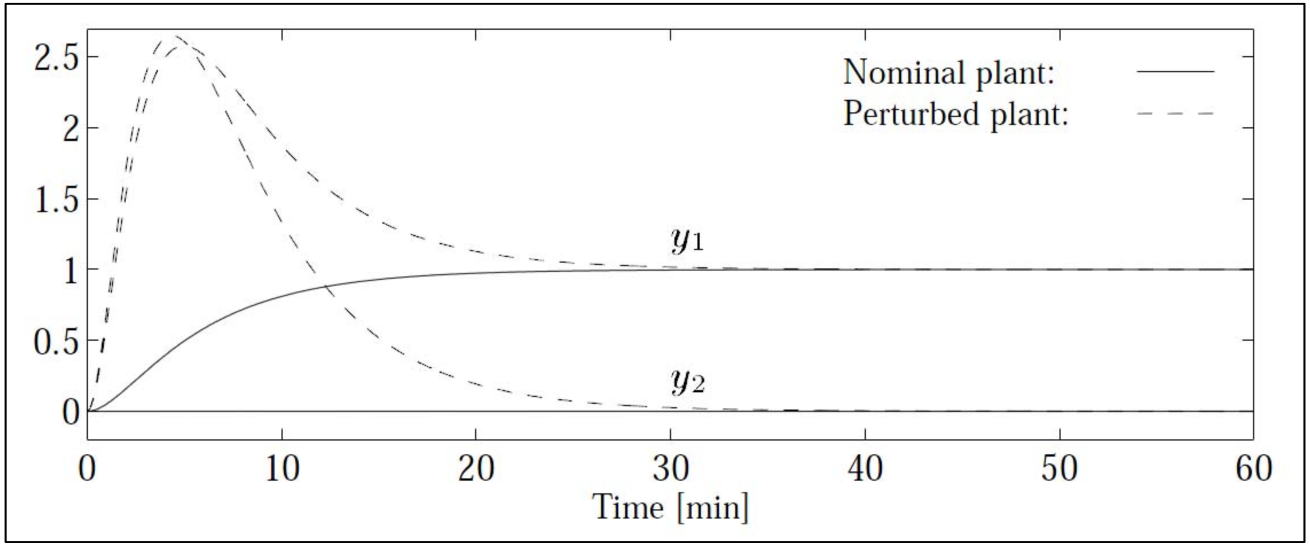


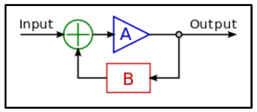




D-K Iteration







Matlab commands

[K,CLPERF,INFO] = musyn(P,NMEAS,NCONT) also returns a struct array INFO where INFO(j) contains the results of the j-th D-K iteration. The fields of INFO are:

K,gamma   Controller and its scaled H-inf performance
KInfo   Synthesis data (see HINFSYN)
PeakMu   Robust performance of LFT(P,K)
DG   D,G scalings from robust performance analysis
dr,dc,PSI   Rational fit of D,G data (see reference pages)
FitOrder   D,G fit orders
PeakMuFit   Scaled H-inf performance with fitted D,G.

[STABMARG,WCU] = robstab(USYS) calculates the robust stability margin
for the uncertain system USYS (USS or UFRD)

[PERFMARG,WCU] = robgain(USYS,GAMMA) calculates the robust performance
margin for the uncertain system USYS and the performance level GAMMA.

```
D-K ITERATION SUMMARY:
-----------------------------------------------------------------
                       Robust performance               Fit order
-----------------------------------------------------------------
  Iter          K Step       Peak MU       D Fit             D
    1           2.954        2.452         2.483            16
    2           1.145        1.143         1.153            18
    3           1.086        1.086          1.09            18
    4           1.082        1.081         1.083            18
    5           1.085        1.085         1.086            18

Best achieved robust performance: 1.08
```

Controller Reduction

Controller Complexity

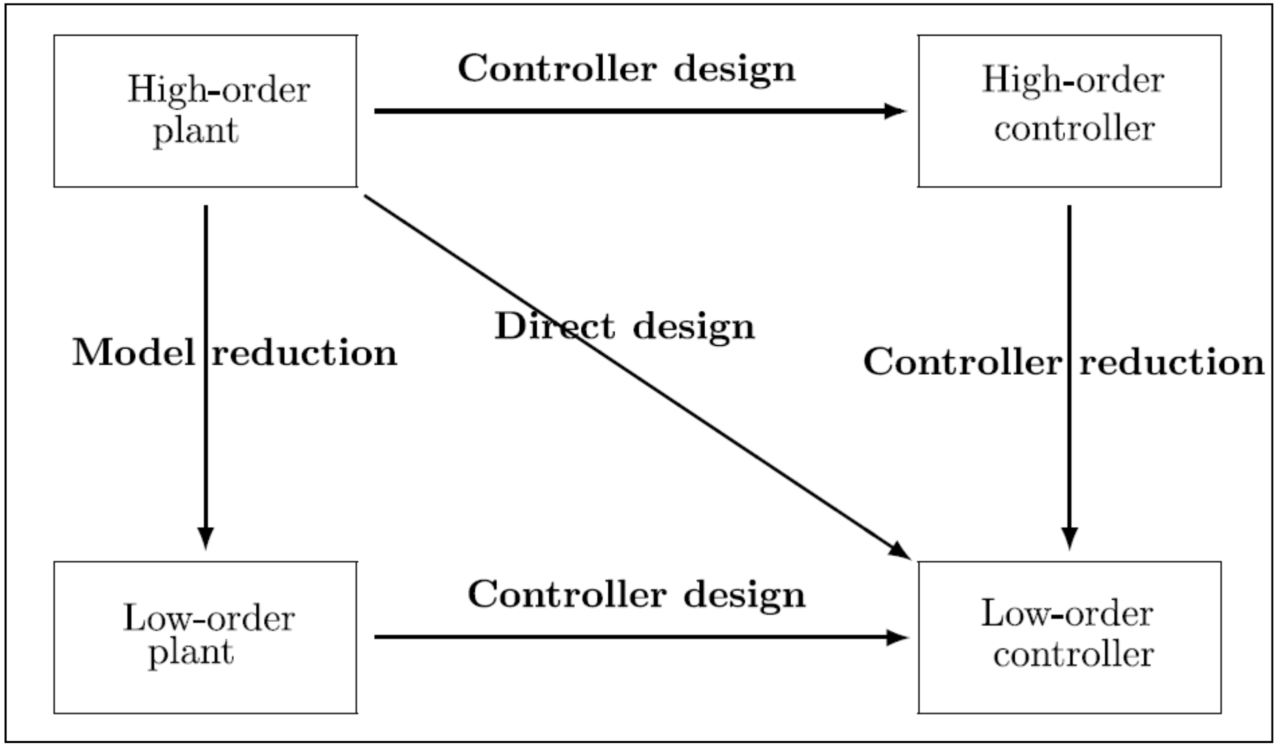
- Due to increasing demands on quality and productivity of industrial systems and with deeper understanding of these systems, mathematical models derived to represent the system dynamics are more complete, usually of multi-input-multi-output form, and are of high order. Consequently, the controllers designed are complex.
- The order of such controllers designed using, for instance, the $\mathcal{H}_\infty$ **optimization** approach or the **μ -synthesis**, is higher than, or at least similar to, that of the plant.
- On the other hand, in the implementation of controllers, high-order controllers will lead to high cost, difficult commissioning, poor reliability and potential problems in maintenance.
- Lower order controllers are always welcomed by practicing control engineers. Hence, how to obtain a low-order controller for a high-order plant is an important and interesting task.

Traditional Approaches

1. plant model reduction followed by controller design;
2. controller design followed by controller-order reduction; and,
3. direct design of low-order controllers.

Approaches (1) and (2) are widely used and can be used together. When a controller designed using a robust design method, Approach (1) would usually produce a stable closed loop, though the reduction of the plant order is likely to be limited. In Approach (2), there is freedom in choosing the final order of the controller, but the stability of the closed-loop system should always be verified. The third approach usually would heavily depend on some properties of the plant, and require numerous computations.

- **Complexity Reduction Schemes:**



- The central problem we address is: given a high-order linear time-invariant stable model G, find a low-order approximation G_a such that the infinity ($\mathcal{H}_\infty$ or $\mathcal{L}_\infty$) norm of the difference, $\|G - G_a\|_\infty$, is small.

Controller Reduction

Truncation and Residualization:

- Let (A, B, C, D) be a minimal realization of a stable system $G(s)$, and partition the state vector x, of dimension n, into $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where x_2 is the vector of $n-k$ states which we wish to remove. With appropriate partitioning of A, B and C, the state-space equations become

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \qquad \begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1 u \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2 u \\ y &= C_1 x_1 + C_2 x_2 + Du \end{aligned}$$

- Several Approaches are available in the literature, see for instance:
  - Yousuff, A., ‘Controller Reduction by Component Cost Analysis ‘, IEEE TR-AC, Vol. 29, No. 6, June 1984.
  - Grigoriadis, K., ‘Model Reduction of Large Scale Systems Using Approximate Component Cost Analysis’, American Control Conference, San Diego, CA, June 1999.

- **Main Point**: provide truncation of fast modes using Lyapunov’s techniques and/or Singular Perturbations

Truncation and Residualization:

- A k-th order truncation of the realization $G \stackrel{s}{=} (A, B, C, D)$ is given by $G_a \stackrel{s}{=} (A_{11}, B_1, C_1, D)$. The truncated model G_a is equal to G at infinite frequency, $G(\infty) = G_a(\infty) = D$, but apart from this there is little that can be said in the general case about the relationship between G and G_a.

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad B = \begin{bmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_n^T \end{bmatrix} \quad C = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

Then, if the λ_i are ordered so that $|\lambda_1| < |\lambda_2| < \cdots$, the fastest modes are removed from the model after truncation. The difference between G and G_a following a k-th order model truncation is given by

$$G - G_a = \sum_{i=k+1}^{n} \frac{c_i b_i^T}{s - \lambda_i} \qquad \|G - G_a\|_\infty \le \sum_{i=k+1}^{n} \frac{\bar{\sigma}(c_i b_i^T)}{|Re(\lambda_i)|}$$

In truncation, we discard all the states and dynamics associated with x_2. Suppose that instead of this we simply set $\dot{x}_2 = 0$, i.e. we *residualize* x_2, in the state-space equations. One can then solve for x_2 in terms of x_1 and u, and back substitution of x_2, then gives

$$\begin{aligned} \dot{x}_1 &= (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u \\ y &= (C_1 - C_2A_{22}^{-1}A_{21})x_1 + (D - C_2A_{22}^{-1}B_2)u \end{aligned}$$

Balanced Realizations:

- **Definition:** In words only: A balanced realization is an asymptotically stable minimal realization in which the controllability and observability Gramians are equal and diagonal.

- Let $(A;B;C;D)$ be a minimal realization of a stable, rational transfer function matrix $G(s)$, then $(A;B;C;D)$ is called balanced if the solutions to the following Lyapunov equations:

$$\begin{aligned} AP + PA^T + BB^T &= 0 \\ A^TQ + QA + C^TC &= 0 \end{aligned}$$

are $P = Q = diag(\sigma_1, \sigma_2, \ldots, \sigma_n) \triangleq \Sigma$, where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$. P and Q are the controllability and observability Gramians, also defined by

$$\begin{aligned} P &\triangleq \int_0^\infty e^{At}BB^Te^{A^Tt}dt \\ Q &\triangleq \int_0^\infty e^{A^Tt}C^TCe^{At}dt \end{aligned}$$

Σ is therefore simply referred to as the Gramian of $G(s)$. The σ_i are the ordered Hankel singular values of $G(s)$, more generally defined as $\sigma_i \triangleq \lambda_i^{\frac{1}{2}}(PQ), i = 1, \ldots, n$. Notice that $\sigma_1 = ||G||_H$, the Hankel norm of $G(s)$.

Controller Reduction

- So what is so special about a balanced realization? In a balanced realization the value of each σ_i is associated with a state x_i of the balanced system. And the size of σ_i is a relative measure of the energy contribution that x_i makes to the input-output behaviour of the system.

- Therefore if $\sigma_1 >> \sigma_2$, then the state x_1 affects the input-output behaviour much more than x_2, or indeed any other state because of the ordering of the singular values. After balancing a system, each state is just as controllable as it is observable, and a measure of a state's joint observability and controllability is given by its associated Hankel singular value.

Let the balanced realization (A, B, C, D) of $G(s)$ and the corresponding Σ be partitioned compatibly as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \ C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

where $\Sigma_1 = diag(\sigma_1, \sigma_2, \ldots, \sigma_k)$, $\Sigma_2 = diag(\sigma_{k+1}, \sigma_{k+2}, \ldots, \sigma_n)$ and $\sigma_k > \sigma_{k+1}$.

Note: The McMillan degree of a transfer-function matrix is the total number of poles in the diagonal elements of the matrix in its McMillan form. This number determines the order of any minimal state-space realization of the transfer-function matrix or the minimal order of coprime matrix-fraction models.

Controller Reduction

- **Balanced residualization**. In balanced truncation above, we discarded the least controllable and observable states corresponding to Σ_2. In balanced residualization, we simply set to zero the derivatives of all these states. The method was introduced by Fernando and Nicholson (1982) who called it a singular perturbational approximation of a balanced system. The resulting balanced residualization of $G(s)$ is (A_r, B_r, C_r, D_r)

$$
\begin{aligned}
A_r &\triangleq A_{11} - A_{12}A_{22}^{-1}A_{21} \\
B_r &\triangleq B_1 - A_{12}A_{22}^{-1}B_2 \\
C_r &\triangleq C_1 - C_2A_{22}^{-1}A_{21} \\
D_r &\triangleq D - C_2A_{22}^{-1}B_2
\end{aligned}
$$

- **Balanced truncation**. The reduced order model given by (A_{11}, B_1, C_1, D) is called a *balanced truncation* of the full order system $G(s)$. This idea of balancing the system and then discarding the states corresponding to small Hankel singular values was first introduced by Moore (1981).

Theorem 11.1 *Let $G(s)$ be a stable rational transfer function with Hankel singular values $\sigma_1 > \sigma_2 > \ldots > \sigma_N$ where each σ_i has multiplicity r_i and let $G_a^k(s)$ be obtained by truncating or residualizing the balanced realization of $G(s)$ to the first $(r_1 + r_2 + \ldots + r_k)$ states. Then*

$$\|G(s) - G_a^k(s)\|_\infty \leq 2(\sigma_{k+1} + \sigma_{k+2} + \ldots + \sigma_N).$$

Optimal Hankel Norm Approximation (see Skogestad text, Theorem 11.2, p. 464):

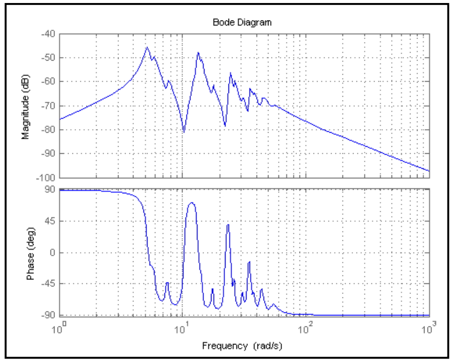
- In this approach to model reduction, the problem that is directly addressed is the following: given a stable model $G(s)$ of order (McMillan degree) n, find a reduced order model $G_h^k(s)$ of degree k such that the Hankel norm of the approximation error, $\|G(s) - G_h^k(s)\|_H$, is minimized.

  The Hankel norm of any stable transfer function $E(s)$ is defined as

$$\|E(s)\|_H \triangleq \rho^{\frac{1}{2}}(PQ)$$

where P and Q are the controllability and observability Gramians of $E(s)$. It is also the maximum Hankel singular value of $E(s)$. So in the optimization we seek an error which is in some sense closest to being completely unobservable and completely uncontrollable, which seems sensible.

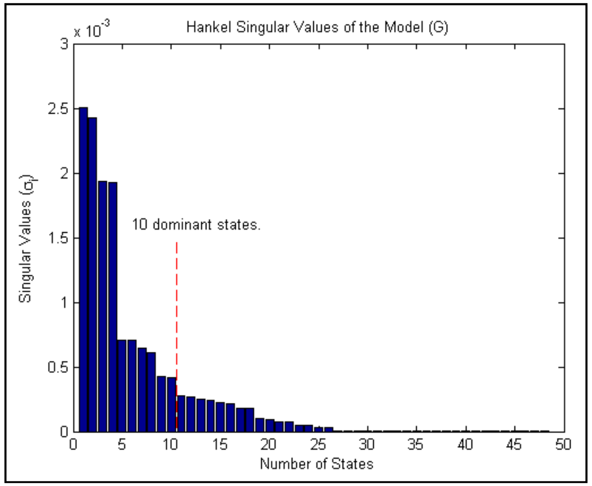
Model Reduction Example (Matlab, Robust Control Toolbox)

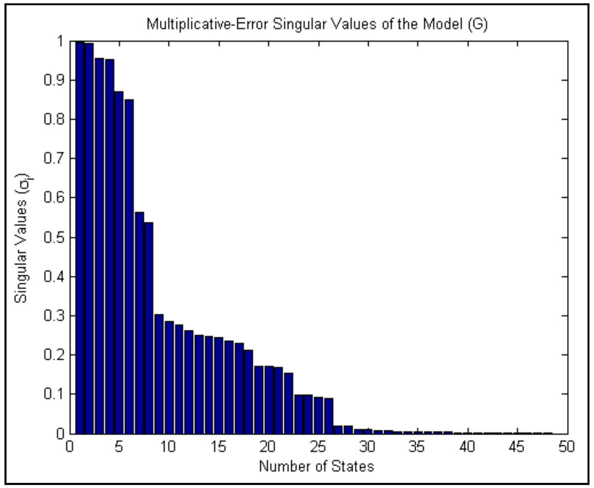


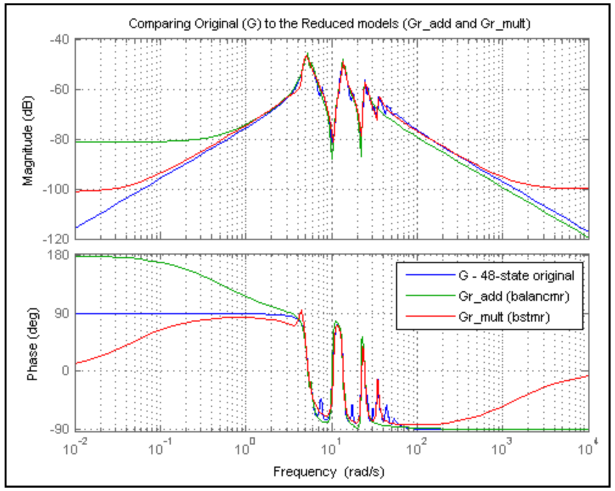
SISO System:
1 input
1 output
48 States

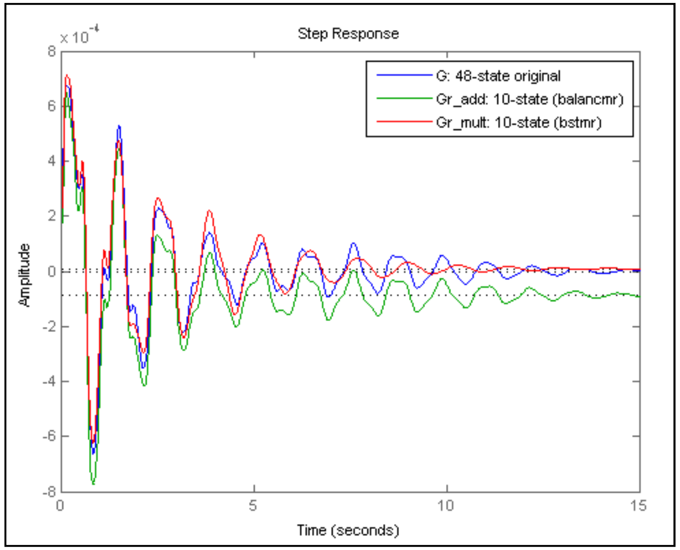
Controller Reduction

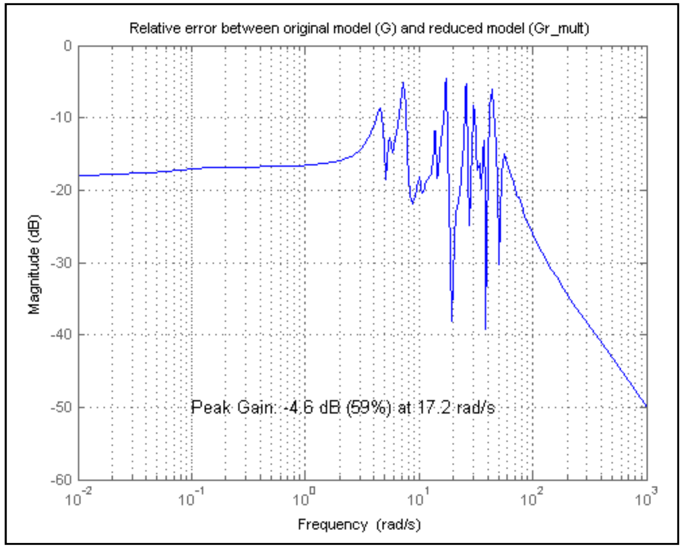
- **Model Reduction Example (Matlab, Robust Control Toolbox)**











Controller Reduction

- Specified Maximum Error Level (5% in this example) yields a 34th order model