Chapter 3: MIMO Tools

Review of Chapter 2

Basics of Systems Theory
- State variable representation
- Structural properties
- MIMO properties and comparison with SISO systems

MIMO Control Approaches
- Pole placement
- Linear Quadratic Optimal Control (LQR)
- State Reconstruction and Estimation

References:
1. Handouts
2. ME851 notes, 16.30 MIT notes
3. Skogestad – “MULTIVARIABLE FEEDBACK CONTROL Analysis and design”, chapter 5 (relevant parts)
Chapter 3: MIMO Tools

- **Geometry**
  - Norms of Vectors, Matrices, SVD
  - Measures of Signals, Systems, and Performance

- **MIMO Block Diagram Algebra**
  - LFT Representation of MIMO Systems
  - Structural Properties

- **Frequency Shaping Design for MIMO systems**
  - Singular Values and Transfer Matrices
  - MIMO Nyquist
  - Small Gain Theorem
  - Bounder Real Lemma

• Zhou: Robust and Optimal Control, Chapters 4, 5
• Mackenroth: Robust Control Systems, Chapters 5, 6, App. A
• Skogestad: Multivariable Feedback Control, Chapters 3, 4, 6, App. A
“Of course I am an experienced space scientist- I know a lot about Banach spaces, Hilbert spaces, Lebesgue spaces, Sobolev spaces.....”
Multivariable systems analysis and control rely heavily on geometric measures of performance, which can be represented in a very general way by norms of elements of appropriate Vector Spaces.

**Definition:** A vector space $V$, is a closed set of elements (vectors), for which the following axioms hold:

- $v_1, v_2 \in V \Rightarrow v_1 + v_2 \in V$
- $v_1, v_2 \in V \Rightarrow v_1 + v_2 = v_2 + v_1$
- $v_1 \in V, \alpha \in \mathbb{R} \Rightarrow \alpha v_1 \in V$
- $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$
- $\exists 0 : v_1 + 0 = v_1 \in V$

**Definition:** A vector subspace $W$, is a subset of a vector space $V$, for which the same axioms hold:

- Example: $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3, w = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3 \quad W \subset V$
Linear dependence: given a set of \( n \) vectors \( v_i \) and a set of \( n \) scalars \( \alpha_i \), the set linearly independent if and only if:

\[
\alpha_1 v_1 + \ldots + \alpha_n v_n = 0 \Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n \neq 0
\]

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{bmatrix} = 0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 \neq 0
\]

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
2 \\
8 \\
0
\end{bmatrix}
\]

\[
\alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_2 = -2\alpha_1
\]

Dimension of a vector (sub)space: The dimension of a (sub)space is given by the maximum number of linear independent vectors belonging to the (sub)space.

- Such set is called BASIS of the (sub)space.
- All vectors of a (sub)space can be found as a linear combination of the basis vectors.

Span: given \( p \) vectors belonging to a vector space \( V \) \( v_1, \ldots, v_p \)

The subspace generated by these vectors (all possible linear combinations) is called span and its basis contains at the most \( p \) elements: \( \text{span} \left( v_1, \ldots, v_p \right) \)

Definition – Two vectors are orthogonal if their inner product is equal to 0

\[
\langle v_1, v_2 \rangle \triangleq v_1^T \cdot v_2 = 0 = \left\| v_1 \right\| \left\| v_2 \right\| \cos \alpha
\]
A matrix $A$ can relate an input signal $x$ with an output signal $y$ as follows:

$$x^{n \times 1} \rightarrow A(m, n) \rightarrow y^{m \times 1} = Ax^{n \times 1}$$

Definition (Geometry): Let associate a constant matrix $A^{m \times n}$ to two vectors $x$ and $y$, such that:

$$y = Ax, x \in V^n, y \in V^m$$

Then $A$ a linear operator of the vector $x$ in $y$

Definition (Algebra): $y = Ax$ represents a linear algebraic system whose solution is a vector $x$, given the vector $y$. If $y = 0$ the system is said to be homogeneous.

$y = Ax$ defines the **Fundamental Theorem of Linear Algebra**, that is the existence and relationship of the 4 fundamental subspaces introduced by $A$:

<table>
<thead>
<tr>
<th>name of subspace</th>
<th>definition</th>
<th>containing space</th>
<th>dimension</th>
<th>basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>column space, range or image</td>
<td>$\text{im}(A)$ or $\text{range}(A)$</td>
<td>$\mathbb{R}^m$</td>
<td>$r$ (rank)</td>
<td>The first $r$ columns of $U$</td>
</tr>
<tr>
<td>nullspace or kernel</td>
<td>$\text{ker}(A)$ or $\text{null}(A)$</td>
<td>$\mathbb{R}^n$</td>
<td>$n - r$ (nullity)</td>
<td>The last $(n - r)$ columns of $V$</td>
</tr>
<tr>
<td>row space or coimage</td>
<td>$\text{im}(A^T)$ or $\text{range}(A^T)$</td>
<td>$\mathbb{R}^n$</td>
<td>$r$ (rank)</td>
<td>The first $r$ columns of $V$</td>
</tr>
<tr>
<td>left nullspace or cokernel</td>
<td>$\text{ker}(A^T)$ or $\text{null}(A^T)$</td>
<td>$\mathbb{R}^m$</td>
<td>$m - r$ (corank)</td>
<td>The last $(m - r)$ columns of $U$</td>
</tr>
</tbody>
</table>
Some physical interpretations of: \( y = Ax, x \in V^n, y \in V^m \)

\[
A_{m \times n} x = y
\]

\( x \in \mathbb{R}^4 \)

\( y \in \mathbb{R}^6 \)

- \( x_j \) is external force/torque applied at some point/direction/axis
- \( y \in \mathbb{R}^6 \) is resulting total force & torque on body

\[
A_{m \times n} x = y
\]

\( x \in \mathbb{R}^6 \)

\( y \in \mathbb{R}^4 \)

- \( x_i \) is external force applied at some node, in some fixed direction
- \( y_i \) is (small) deflection of some node, in some fixed direction
• Recall the linear transformation: $Ax = y, A(m \times n)$ 
  $a_1x_1 + a_2x_2 + \ldots + a_nx_n = y$

• The vector $y$ is a linear combination of the columns of $A$, therefore

  $y \in V := \text{span}\{a_1, a_2, \ldots, a_n\}$

**Definition:** The subspace formed by the columns of $A$ is called Image Space or Range Space $= \mathbb{R}(A)$. Since $A$ has $n$ columns and $m$ rows, the Range Space is a subspace in $\mathbb{R}^m$

  the range of $A \in \mathbb{R}^{m \times n}$ is defined as

  $\mathcal{R}(A) = \{Ax \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$

**Theorem:** The non homogeneous system has a solution (non trivial) if and only if $y$ is a linear combination of a basis of the Range Space of $A$:

  $\text{Rank}(A) = \text{Rank}(A \mid y)$
Chapter 3: MIMO Tools: Geometry, Linear Algebra

- The Range Space is a subspace in $\mathbb{R}^2$ of $\mathbb{R}^4$
- The system has a solution for any $y$ linear combination of a basis of the Range Space

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \Rightarrow A\mathbf{x} = \mathbf{y}$$

$$y = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} x_1 = 1 - x_3 = 0 \\ x_2 = -x_1 - 2x_3 = -2 \\ x_3 = 1 \end{bmatrix}$$

- The following system has no solution (i.e. Only the trivial solution)

$$\forall \mathbf{y} = \alpha \mathbf{a}_1 + \beta \mathbf{a}_2 \Rightarrow \mathbf{x} \neq 0$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \gamma \neq 0$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \delta \neq 0$$

$$\mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} x_1 = -x_3 = 0 \\ x_2 = -x_1 - 2x_3 = 0 \\ x_3 = 1 - x_1 = 0 \end{bmatrix}$$
• Consider now the homogeneous system

\[ Ax = 0 \]

- **Definition**: The solution \( x_h \) of the homogeneous system belongs to a subSpace called **Null Space** of \( A \) (or Kernel) = \( N(A) \). The Null Space of \( A \) (mxn) has dimensions n and it always contains at least the zero vector.

the nullspace of \( A \in \mathbb{R}^{mxn} \) is defined as

\[ N(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \} \]

- **Comment**: Similar to the previous non homogeneous case the solution to an homogeneous system depends now on the relationship between the Rank of the matrix \( A \) and the dimension of its Null Space \( N(A) \).
Chapter 3: MIMO Tools: Geometry, Linear Algebra

\[
A \mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \cdot \mathbf{x} = 0 \quad \text{Rank}(A): r = 2 \quad \mathbf{x}_h = \alpha \begin{bmatrix} 0 \\ 0 \end{bmatrix} 
\]

\[
A \mathbf{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \cdot \mathbf{x} = 0 \quad \text{Rank}(A): r = 2 \quad \mathbf{x}_h = 0 \quad \text{Rank}(A): r = 1 \quad \mathbf{x}_h = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

- **Rank Theorem:** It relates the Rank \( \gamma \), the dimension of the Range Space \( \dim \mathbf{R}(A) \) and the dimension of the Null Space \( \dim \mathbf{N}(A) \).

\[
\gamma + \dim \mathbf{N}(A) = \dim \mathbf{R}(A)
\]

- Given the system \( \mathbf{y} = A \mathbf{x} \), with \( m \) equations and \( n \) unknowns. Assume the Rank of \( A \) be equal to \( \gamma \). Then the system has \( \infty^{(n-\gamma)} \) solutions obtained adding to one particular solution all the solutions of the homogeneous system.
**Basis, Rank, Range Space, Vector Spaces, Null Space, Matrices, and stuff.**

- Maximum Rank \( r = n \leq m \)
- Rank \( r \leq n \). The system has solution for any \( y \) linear combination of \( r \) linearly independent columns of \( A \)
- The vector \( y \) is obtained from a basis that defines a subspace of dimensions equal to the rank \( A \)

- Maximum Rank \( r = m \leq n \)
- If the rank is maximum \( (r = m) \) the system has always solution, since \( y \) is always a linear combination of the columns of \( A \)
- If the rank is \( r < m \), the system has solution if and only if \( y \) is a linear combination of a basis of the Range Space of \( A \)

- Maximum Rank \( r = n \)
- Rank \( r = n \). The system has 1 solution if and only if \( y \) is a linear combination of the columns of \( A \)
- The Rank is \( r < n \). The system has a solution if \( y \) is a linear combination of a basis of the \( \text{span}(A) \) of dimension \( r \)
Chapter 3: MIMO Tools: Geometry, Linear Algebra

**Conclusion:** The solution exists if and only if $y \in \mathbb{R}(A)$ and can be written as: $x = x_p + x_h = x_p + N(A)$. The solution is unique if and only if $N(A)$ is empty (except for the zero vector), that is $\gamma = n$.

- $m > n$: more equations than unknowns, the system is overconstrained. Happens in, e.g., estimation problems, where one tries to estimate a small number of parameters from a lot of experimental measurements. In such cases the problem is typically inconsistent, i.e., $y \notin \mathcal{R}(A)$. So one is interested in finding the solution that minimizes some error criterion.

- $m < n$: more unknown than equations, the system is underconstrained. Happens in, e.g., control problems, where there may be more than one way to complete a desired task. If there is a solution $x_p$ (i.e., $Ax_p = y$), then typically there are many other solutions of the form $x = x_p + x_h$, where $x_h \in N(A)$ (i.e., $Ax_h = 0$). In this case it is desired to find the solution that minimizes some cost criterion.
“What really pisses me off is them always saying: ‘Let epsilon go to zero...’, ‘Suppose epsilon vanishes...’, ‘As epsilon becomes infinitesimally small...’, and NEVER ‘How is poor epsilon? Hope he’s doing well. He’s sooo cute! May epsilon grow big and strong!’”
Chapter 3: MIMO Tools: Norms

- **Point of understanding (SISO system)**
  \[ G(s) = \frac{Y(s)}{U(s)} \]
  \[ L(s) = G(s)K(s) \]
  \[ T(s) = \frac{L(s)}{1 + L(s)} \]
  \[ S(s) = \frac{1}{1 + L(s)} \]
  \[ S_u(s) = K(s)S(s) \]
  - The dynamic representation is given by SCALAR complex functions identified by a magnitude and a phase

- **Point of understanding (MIMO system)**
  - The dynamic representation is given by MATRICES of complex functions which need to be characterized by similar variables
  \[ L(s) = G(s)K(s) = \text{?} = K(s)G(s) \]
  \[ T(s) = L(s)[1 + L(s)]^{-1} = \text{?} = [1 + L(s)]^{-1} L(s) \]
  \[ S(s) = [1 + L(s)]^{-1} \]
  \[ |L(s)| = \text{?} \]
  \[ |S(s)|_{\text{max}} = \text{?} \]
Chapter 3: MIMO Tools: Norms

- **AGAIN:** Multivariable systems analysis and control rely heavily on geometric measures of performance, which can be represented in a very general way by norms of elements of appropriate Vector Spaces.

- **Why do we want Norms?:** to have a single number that gives a measure / quantification of the “size” of a vector, matrix, signal, or system.

- a norm is a function that assigns a strictly positive length or size to each vector in a vector space, except for the zero vector, which is assigned a length of zero.

- **Definition:** A Norm in a vector space $V$ is a scalar function mapping $V \rightarrow [0, \infty)$, which satisfies:
  - Definiteness (1, 2): $\|v\| \geq 0; \|v\| = 0$ iff $v = 0$
  - Homogeneity (3): $\|\alpha v\| = |\alpha| \|v\| \forall (u, v) \in V, \alpha \in (\mathbb{C})$
  - Triangle Inequality (4): $\|u + v\| \leq \|u\| + \|v\|$
Chapter 3: MIMO Tools: Norms

- A vector \( p \) norm is defined in a general vector space \( \mathbb{R}^n \)

\[
\| \mathbf{v} \|_p = \left[ \sum_i |v_i|^p \right]^{1/p} \quad \mathbf{v} = [v_1, v_2, \ldots, v_n]
\]

- Euclidean norm: \( \| \mathbf{v} \|_2 = \left( |v_1|^2 + \ldots |v_n|^2 \right)^{1/2} = \sqrt{\mathbf{v}^T \mathbf{v}} = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \)

- The Classical Euclidean Norm is also called an Inner Product Induced Norm, from its definition, it generalizes the concept of vector angle in \( n \) – dimensions (Hilbert Space)

\[
\| \mathbf{v} \|_2 := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \quad \langle \mathbf{v}_1, \mathbf{v}_2 \rangle := \mathbf{v}_1^T \mathbf{v}_2 = \sum_{i=1}^n v_i^j v_j^i \quad \cos \angle(\mathbf{v}_1, \mathbf{v}_2) = \frac{\langle \mathbf{v}_1, \mathbf{v}_2 \rangle}{\| \mathbf{v}_1 \|_2 \| \mathbf{v}_2 \|_2}
\]

- 1-norm \( \| \mathbf{v} \|_1 = |v_1| + \ldots |v_n| \)

- \( \infty \)-norm: \( \| \mathbf{v} \|_\infty = \max \{ |v_k|; k = 1, \ldots, n \} \)
Chapter 3: MIMO Tools: Norms

Matrix Norms

- **Definition**: Given a general complex matrix $A$, a norm on $A$ is a matrix norm if, in addition to the four norm properties defined previously for vector norms, it also satisfies the multiplicative property:

$$\|AB\| \leq \|A\|\|B\|$$

- **Example**: A matrix norm of common use is the Frobenius Norm (also called Euclidean Matrix Norm):

$$\|A\|_F = \left[\text{Trace}(A^*A)\right]^{1/2} \quad \langle A, B \rangle := \text{Trace} A^*B = \sum_{i=1}^{n} \sum_{j=1}^{m} \bar{a}_{ij}b_{ij} \quad \forall A, B \in \mathbb{C}^{n \times m}$$

- **Concept**: An induced matrix norm or induced matrix $p$ – norm represents a matrix norm associated to a specific vector $p$-norm.

  - A complex matrix $A^{m \times n}$ maps a complex vector space $\mathbb{C}^n$ equipped with a $p$ – norm, into a complex vector space $\mathbb{C}^m$ also equipped with a $p$ – norm.
  - It is introduced to indicate the maximum gain (or the maximum amplification) of a vector signal through a matrix.
Chapter 3: MIMO Tools: Norms

\[ z = A x \]

- Equivalent Representations:
  \[ \frac{\|z\|_p}{\|x\|_p} = \frac{\|Ax\|_p}{\|x\|_p} ; z = Ax \]
  \[ \|A\|_{ip} = \max_{\|x\|_p \leq 1} \|Ax\|_p = \max_{\|x\|_p = 1} \|Ax\|_p \]

- An important induced matrix \( p \)-norm is the 2 – norm also called **Spectral Norm:**
  \[ \|A\|_2 = \sqrt{\lambda_{\text{max}}(A^* A)} \quad A^* = A^T \quad \text{if } A \text{ is a real matrix} \]

- **Definition_Spectral Radius:** given a square real or complex matrix \( A \) with eigenvalues \( \lambda_i \), its spectral radius \( \rho(A) \) is defined as:
  \[ \rho(A) := \max_i |\lambda_i(A)| \]

  - **Theorem:** for any matrix or induced matrix norm, the following holds:
    \[ \rho(A) \leq \|A\|_{ip} \]
Chapter 3: MIMO Tools: SVD

Singular Values and Singular Value Decomposition

- Singular values provide a fundamental tool in defining a “measure” of a complex matrix and with their relationship to eigenvalues and induced matrix norms

- **Definition:** A square complex matrix $D$ is Unitary, if $D^*D=DD^*=I$

- **Definition:** A square complex matrix $D$ is Hermitian, if $D=D^*$, where $D^*$ is the conjugate transpose

**Theorem:** Given a complex rectangular matrix $A \in \mathbb{C}^{m \times n}$ there exist unitary matrices $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$, such that:

$$A = U \Sigma V^*; \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_p \end{bmatrix}$$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0, \quad p = \min\{m, n\}.$$

- **Note:** $\text{Rank}(A) = p$

- **Definition:** a complex square matrix $U$ is unitary if its conjugate transpose $U^*$ is also its inverse—that is, if $U^*U=UU^*=I$
Chapter 3: MIMO Tools: SVD

- **Standard Nomenclature:**
  
  \( u_i \), Left singular vectors
  
  \( v_i \), Right singular vectors
  
  \( \sigma_i \), Singular Values

  \( \sigma_1 = \bar{\sigma} = \sigma_{\text{max}} \) Maximum Singular Value

  \( \sigma_p = \bar{\sigma} = \sigma_{\text{min}} \) Minimum Singular Value

- Singular values are good measures of the size of a matrix
- Singular vectors are good indications of strong/weak input or output directions.

\[
\gamma(A) := \frac{\bar{\sigma}(A)}{\sigma(A)} \quad \text{Condition Number, for a square matrix}
\]

\[
\begin{align*}
\bar{\sigma}(A) & = \|A\|_2 \quad \text{norm}(A, 2) \text{ or } \max(\text{svd}(A)) \\
\|A\|_1 & = \text{norm}(A, 1) \\
\|A\|_\infty & = \text{norm}(A, 'inf') \\
\|A\|_F & = \text{norm}(A, 'fro') \\
\|A\|_{\text{sum}} & = \text{sum} \left( \text{sum}(\text{abs}(A)) \right) \\
\|A\|_{\text{max}} & = \max(\max(\text{abs}(A))) \quad (\text{which is not a matrix norm}) \\
\rho(A) & = \max(\text{abs}(\text{eig}(A))) \\
\rho(|A|) & = \max(\text{eig}(\text{abs}(A))) \\
\gamma(A) & = \bar{\sigma}(A)/\sigma(A) \quad \text{cond}(A)
\end{align*}
\]
Chapter 3: MIMO Tools: SVD

Depending on the rank of $A$, we have different matrix sizes for $U$, $\Sigma$, and $V$.

Thin SVD

$$A = \hat{U} \hat{\Sigma} V^*$$

Full SVD

$$A = U \Sigma V^*$$

Matrices without full rank

$$A = \hat{U} \hat{\Sigma} V^*$$

The statement

$$[U,S,V] = \text{svd}(X)$$

produces

$$U = \begin{bmatrix} -0.1525 & -0.8226 & -0.3945 & -0.3800 \\ -0.3499 & -0.4214 & 0.2428 & 0.8007 \\ -0.5474 & -0.0201 & 0.6979 & -0.4614 \\ -0.7448 & 0.3812 & -0.5462 & 0.0407 \end{bmatrix}$$

$$S = \begin{bmatrix} 14.2691 & 0 \\ 0 & 0.6268 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.6414 & 0.7672 \\ -0.7672 & -0.6414 \end{bmatrix}$$
Chapter 3: MIMO Tools: SVD


- Geometric Interpretation of singular values and singular vectors

  ![Geometric Interpretation Diagram](image)

  Every matrix $A \in \mathbb{R}^{m \times n}$ maps the unit hyper sphere in $\mathbb{R}^n$ to a hyper ellipsoid in $\mathbb{R}^m$. The ellipsoid is a sphere stretched by factors $\sigma_1, \ldots, \sigma_n$ in orthogonal directions called principal semi axes $u_1, \ldots, u_n$. The mapping relationship is given by:

  $$S = \{x \in \mathbb{R}^n \mid \|x\| = 1\}, AS = \{Ax \mid x \in S\}$$

- Mapping sequence:
  - Rotation by $V$ (unitary)
  - Diagonal scaling by $\sigma_i$ (ordered)
  - Rotation by $U$ (unitary)

  ![Mapping Sequence Diagram](image)
Chapter 3: MIMO Tools: SVD

- **Singular Values and Spectral Norm:**

  From the definition of SVD decomposition, for a square complex matrix \( A \in \mathbb{C}^{m \times m} \):

  \[
  A = U \Sigma V^* = \sum_{i=1}^{r} \sigma_i u_i v_i^* \quad \Rightarrow \quad AA^* = U \Sigma V^* V \Sigma^* U^* = U \Sigma \Sigma^* V^* = A^* A \quad \Rightarrow \quad \sigma_i^2 = \lambda(A^* A) = \lambda(AA^*)
  \]

  \[
  \sigma_1 = \sigma_{\text{max}} = \bar{\sigma} = \sqrt{\lambda_{\text{MAX}}(A^* A)} = \|A\|_2 = \max_{\|x\|_{l^2} \neq 0} \frac{\|Ax\|}{\|x\|}
  \]

  Similarly:

  \[
  \|A\|_F = \sqrt{\sum_i \sigma_i^2(A)}
  \]
Chapter 3: MIMO Tools: SVD

- **Singular Values Algebra**
  - $\sigma(M) > 0 \iff M$ non-singular
  - $\sigma(AB) \leq \sigma(A)\sigma(B)$
  - $\sigma(A)\sigma(B) \leq \sigma(AB)$ if $A$ is non-singular
  - $\sigma(A)\sigma(B) \leq \sigma(AB)$ if $B$ is non-singular

  - $\|M\|_{\text{max}} \leq \sigma(M) \leq \sqrt{mn}\|M\|_{\text{max}}$
  - $\bar{\sigma}(M) \leq \sqrt{\|M\|_{1-\text{ind}}\|M\|_{\infty-\text{ind}}}$
  - $\frac{1}{\sqrt{n}}\|M\|_{\infty-\text{ind}} \leq \bar{\sigma}(M) \leq \sqrt{m}\|M\|_{\infty-\text{ind}}$
  - $\frac{1}{\sqrt{m}}\|M\|_{1-\text{ind}} \leq \bar{\sigma}(M) \leq \sqrt{n}\|M\|_{1-\text{ind}}$
  - $\max\{\bar{\sigma}(M), \|M\|_{F}, \|M\|_{1-\text{ind}}, \|M\|_{\infty-\text{ind}}\} \leq \|M\|_{\text{sum}}$
  - $\|M\|_{F} = \sqrt{\sum_i \sigma_i(M)^2}$

  - $\sigma(M) \leq |\lambda_i(M)| \leq \sigma(M) \quad \forall i$
  - $\bar{\sigma}(MT) = \bar{\sigma}(M)$
  - $\bar{\sigma}(M^{-1}) = (\sigma(M))^{-1}, \sigma(M^{-1}) = (\bar{\sigma}(M))^{-1}$ if $M$ non-singular
  - $\rho(M) \leq \|M\|$ for any matrix norm
  - $\bar{\sigma}(M) \leq \|M\|_{F} \leq \sqrt{\min\{m, n\}\bar{\sigma}(M)}$

  - $|\sigma(A) - \sigma(B)| \leq \sigma(A + B) \leq \sigma(A) + \sigma(B)$
  - $\sigma(A) - \sigma(B) \leq \sigma(A + B) \leq \sigma(A) + \sigma(B)$
  - $\sigma(A) - 1 \leq \sigma(I + A) \leq \sigma(A) + 1$
  - $\sigma(A) - 1 \leq \left(\sigma((I + A)^{-1})\right)^{-1} \leq \sigma(A) + 1$
Chapter 3: MIMO Tools: SVD

• Example

\[ \sigma^2_{\text{max}}(A)I - A^*A = \sigma^2_{\text{max}}(A)I - V\Sigma U^* U\Sigma V^* = \sigma^2_{\text{max}}(A)I - V\Sigma^2 V^* \]

\[ \sigma^2_{\text{max}}(A)I - V\Sigma^2 V^* = \sigma^2_{\text{max}}(A)VIV^* - V\Sigma^2 V^* = V[\sigma^2_{\text{max}}(A)I - \Sigma^2]V^* \geq 0 \]

• Note that, given some positive scalar \( \gamma > 0 \), since \( \gamma I - A^*A \geq 0 \) it must be \( \sigma_{\text{max}}(A) < \gamma \)

• Therefore the largest singular value must be the smallest among all positive numbers \( \gamma \) that satisfy

\[ \gamma I - A^*A \geq 0 \]

This inequality may be used as an equivalent definition of the largest singular value. This relationship will play a role in solving \( \mathcal{H}_\infty \) control problem and other issues later.
Due to the linear mapping property, and the definition of spectral norm, the maximum and minimum singular values indicate the maximum amplification and maximum attenuation of $A$, scaled to the unit sphere.

If referred to a complex matrix function of frequency, i.e. a transfer matrix $G(j\omega)$, they provide bounds on its global amplification and/or attenuation, once the signals are scaled to the unit sphere.

$$\hat{G}(s) = \begin{bmatrix} \frac{10(s+1)}{s^2+0.2s+100} & \frac{1}{s+1} \\ \frac{s+2}{s^2+0.1s+10} & \frac{5(s+1)}{(s+2)(s+3)} \end{bmatrix}$$
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Example: Linearized lateral dynamics F - 14

Nonlinear Dynamic Equations
\[ \dot{y} = \begin{bmatrix} \dot{\beta} \\ \dot{p} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} (\cos \theta \sin \phi) \nu + (\cos \phi \cos \theta) \sin \phi \sin \theta - (\sin \theta \sin \phi) \sin \phi \cos \theta \\ (\sin \theta \cos \phi) \nu + (\cos \phi \cos \theta) \sin \phi \cos \theta - (\sin \theta \sin \phi) \sin \theta \cos \phi \\ (\sin \phi) \nu + (\cos \phi) \sin \theta \cos \phi - (\cos \phi \sin \theta) \sin \phi \cos \phi \end{bmatrix} \]

State Vector, 6 components

\[ x_{st} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \]

Side Velocity Cross-axle Time
Body – Axis Roll Rate Body – Yaw Rate Roll Angle about Body x Axis Yaw Angle about Inertial x Axis

Dutch-roll mode is primarily described by stability-axis yaw rate and sideslip angle

Roll and spiral mode are primarily described by stability-axis roll rate and roll angle

Linearized equations allow the three modes to be studied

\[ y(s) = \begin{bmatrix} r(s) \\ \beta(s) \\ p(s) \\ \phi(s) \end{bmatrix} = G^{4x2}(s) \begin{bmatrix} \delta_A(s) \\ \delta_R(s) \end{bmatrix} = G^{4x2}(s)u(s) \]
NOTE: Potential limitations in the use of singular values of a frequency dependent complex matrix?
The most important objective of a control system is to achieve certain performance specifications in addition to providing the internal stability.

- **Traditional Measures**
  - Stability, pole/eigenvalue location
  - Steady state error to command
  - Transient response tracking/Sensitivity
  - Disturbance attenuation
  - Bandwidth and response velocity
  - Overshoot
  - Control Limitations
  - ...

One way to describe the performance specifications of a control system is in terms of the size of certain signals of interest. For example, the performance of a tracking system could be measured by the size of the tracking error signal.

The use of norms of signals and systems is a very convenient way to formalize performance and stability requirements in the context linear quadratic measures of optimality.
• Consider a scalar function over the real numbers $x(t) := \mathbb{R}^1 \to \mathbb{R}^1$, we define two standard metrics used to ‘measure’ the function:

1. **Instantaneous Power at time** $t_i$, $x^2(t_i)$
2. **Energy as average value over time** $E = \int_0^\infty x^2(t)dt$

Definition: A **Lebesgue Space** $\mathcal{L}_p$ is a function Space of all the square measurable integrable functions $x(t)$ (in the $\mathcal{L}_p \mathbb{R}^+$ space) with an associated $p$ – norm (as extension of vector norms).

$$\|x(t)\|_{\mathcal{L}_p} = \begin{cases} \left[ \int_{-\infty}^{\infty} |x(t)|^p \, dt \right]^{1/p} & 1 \leq p < \infty, \\ \sup_{t \in \mathbb{R}} |x(t)| & p = \infty \end{cases}$$

• **Note that the Lebesgue Space is a Hilbert Space**
Chapter 3: MIMO Tools: Performance Measures

- The previous norms can be easily extended to vector functions \( x(t) \) (vector signals, in \( \mathbb{R}^n \) or \( \mathbb{C}^n \) whose components may or may not be weighted). For instance:

\[
\|x(t)\|_{L^p} = \left( \int_{-\infty}^{\infty} \|x(t)\|^p \, dt \right)^{\frac{1}{p}}
\]

- We can now represent Energy and Instantaneous Power (peak value) measures of a time varying signal using the \( L^2 \) and \( L^\infty \) norms respectively. From the previous definition:

\[
\|x(t)\|_{L^2} = \left( \int_{-\infty}^{\infty} \|x(t)\|^2 \, dt \right)^{\frac{1}{2}}
\]

\[
\|x(t)\|_{L^\infty} = \sup_{t \in \mathbb{R}} \|x(t)\|
\]

Note the equivalence with the corresponding vector norm

\[
\|x(t)\|_2 := \sqrt{\langle x^*(t), x(t) \rangle} = \left[ \int_{-\infty}^{\infty} x^*(t)x(t) \, dt \right]^{\frac{1}{2}}
\]

\[
\|x(t)\|_\infty = \sup_{t \in \mathbb{R}} \max_i \{x_i(t)\}
\]
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• Example:

\[ z(t) = \begin{bmatrix} e^{-t} \\ 2e^{-3t} \end{bmatrix}, \quad t \geq 0 \]

\[ \|z(t)\|_2 = \left( \int_0^\infty (z^T(t) \cdot z(t))^{\frac{1}{2}} \right) = \left( \int_0^\infty \left[ e^{-t} - 2e^{-3t} \right] \cdot \left[ e^{-t} - 2e^{-3t} \right] dt \right)^{\frac{1}{2}} = \left( \int_0^\infty [e^{-2t} + 4e^{-6t}] dt \right)^{\frac{1}{2}} = \sqrt{\frac{7}{6}} \]

\[ \|z(t)\|_{\infty} = \sup_{t} \|z(t)\| = \sup_{t} \| \frac{1}{2} \| = 2 \]

• In the analysis and design of linear systems, the relationship between the time domain and frequency domain characteristics plays an especially important role. In system analysis, conditions for the robust stability and robust performance are described, in most cases, as specifications on the frequency response. However, in system design the state space in the time domain is more convenient. Therefore, it is indispensable to transform frequency domain conditions into equivalent time domain conditions (and vice versa).
Measure of signals in the frequency domain are obtained from Parseval’s theorem.
Measure of signals in the frequency domain are obtained from **Parseval’s theorem**

- Parseval’s theorem gives the relationship between the squared integral of a time function and that of its Fourier transform, namely, the energy in the time domain is equal to the energy in the frequency domain.

\[
\hat{f}(j\omega) = \mathcal{F}[f(t)] = \int_0^\infty f(t)e^{-j\omega t} dt, \quad f(t) = \mathcal{F}^{-1}[\hat{f}(j\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(j\omega)e^{j\omega t} d\omega.
\]

\[
f_1(t) * f_2(t) = \int_0^\infty f_1(t - \tau)f(\tau)d\tau, \quad \mathcal{F}[f_1(t) * f_2(t)] = \hat{f}_1(j\omega)\hat{f}_2(j\omega).
\]

- What is the difference between Fourier Transform and Laplace Transform from a Control theory point of view?

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Theorem 8.1 (Parseval’s theorem) Assume that signal vectors \( f(t), f_1(t), f_2(t) \in \mathbb{R}^n \) have Fourier transforms \( \hat{f}(j\omega), \hat{f}_1(j\omega), \hat{f}_2(j\omega) \), respectively. Then, there hold the following integral relationships:

1. The inner product in the time domain is equal to that in the frequency domain, that is,

\[
\int_0^{\infty} f_1^T(t) f_2(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}_1^*(j\omega) \hat{f}_2(j\omega) \, d\omega.
\]  
(8.8)

2. The two-norm in the time domain is equal to that in the frequency domain:

\[
\int_0^{\infty} \|f(t)\|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\hat{f}(j\omega)\|^2 \, d\omega.
\]  
(8.9)

\[\int_0^{\infty} \|f(t)\|^2 \, dt\] on the left-hand side of (8.9) represents the energy of signal \( f(t) \). In this sense, \( \|\hat{f}(j\omega)\|^2 \) on the right-hand side can be regarded as the energy density at the frequency \( \omega \). Therefore, \( \|\hat{f}(j\omega)\|^2 \) is also called the energy spectrum.

- Recall the measure of Instantaneous Power (peak value) and Energy measures of a time varying signal using the \( L_2 \) and \( L_\infty \) norms.
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\[ \|x(t)\|_{L^2} = \left[ \int_{0}^{\infty} x^*(t)x(t)dt \right]^{\frac{1}{2}} \]

\[ \|x(j\omega)\|_{L^2} = \left[ \int_{-\infty}^{\infty} x^*(j\omega)x(j\omega)\omega d\omega \right]^{\frac{1}{2}} \]

- From Parseval's theorem, the following identity holds:

\[ \int_{-\infty}^{\infty} x(t)x^*(t)dt = \int_{-\infty}^{\infty} x(j\omega)x(-j\omega)\omega d\omega \]

- Extension definition to vector valued functions:

\[ \langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f^*(t)g(t)dt = \int_{-\infty}^{\infty} \text{trace}[f^*(t)g(t)]dt \]

\[ \|x(t)\|_{L^2} = \|x(j\omega)\|_{L^2} \]

- Similar to vector norms \( \Leftrightarrow \) norms of signals, we have a relationship between Induced norms \( \Leftrightarrow \) Norms of systems.

- Consider a linear MIMO system with transfer matrix \( G(s) \) and corresponding impulse response vector \( g(t) \), from Parseval’s theorem the following holds in certain conditions:

\[ \|g(t)\|_{L^2} := \left[ \int_{-\infty}^{\infty} \text{trace}[g^*(t)g(t)]dt \right]^{\frac{1}{2}} = \left[ \int_{-\infty}^{\infty} |g(t)|^2 dt \right]^{\frac{1}{2}} = \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}[G^*(j\omega)G(j\omega)]d\omega \right]^{\frac{1}{2}} = \|G(j\omega)\|_{L^2} \]

What is this??
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- **$\mathcal{L}_2$-induced norm**: Suppose $G(s)$ is a rational matrix representing a linear dynamic system. The norm of the system induced by the $\mathcal{L}_2$ - norm of $g(t)$ exists if and only if $G(s)$ is proper and stable (has no poles in the closed right-half plane). In that case the $\mathcal{L}_2$ - norm equals the $\mathcal{H}_2$-norm of the transfer matrix:

$$\|g(t)\|_{\mathcal{L}_2} = \left[ \int_{-\infty}^{\infty} \text{trace} \left( g^*(t)g(t) \right) dt \right]^{\frac{1}{2}}$$

$$\|G(j\omega)\|_{\mathcal{H}_2} = \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace} \left( G^*(j\omega)G(j\omega) \right) d\omega \right]^{\frac{1}{2}}$$

- The function Space of all proper and stable (analytic on the closed RHP) transfer matrices $G(s)$ is called Hardy Space

- **$\mathcal{L}_\infty$-induced norm**: The norm of a system $G(s)$ induced by the $\mathcal{L}_\infty$-norm, if it exists, is called $\mathcal{H}_\infty$-norm and it is given equivalently by:

$$\|g(t)\|_{\mathcal{L}_\infty} = \sup_{t \in \mathbb{R}} \|g(t)\| = \max_{i=1,...,m} \int_{-\infty}^{\infty} \left| \sum_{j=1}^{k} g_{ij}(t) \right| dt = \|G(j\omega)\|_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} \sigma_{\max} \left[G(j\omega)\right]$$
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\[ \|G(j\omega)\|_{H_\infty} = \sup_{\omega \in \mathbb{R}} \sigma_{\max} [G(j\omega)] \]

Hence, the \( H_\infty \)-norm of a system describes the maximum energy gain of the system and is decided by the peak value of the largest singular value of the frequency response matrix over the whole frequency axis. This norm is called the \( H_\infty \)-norm, since we denote by \( H_\infty \) the linear space of all stable linear systems.

**Summary**

- the \( H_2 \) norm is the energy of the output of a linear system \( G(s) \) when the input is the impulse.
- Given a linear system \( G(s) \), assuming it exists, reduction of the \( H_2 \) norm implies reduction of all singular values:
  \[ \|G(s)\|_2 \downarrow \Leftrightarrow [\sigma_i(G)], i = 1, \ldots, n \downarrow \]
- the \( H_\infty \) norm of a linear system \( G(s) \) is the maximum amplification (peak power) of the output over a normed sinusoidal input limited to the unit sphere.
- Given a linear system \( G(s) \), assuming it exists, reduction of the \( H_\infty \) norm implies reduction of the maximum singular value:
  \[ \|G(s)\|_\infty \downarrow \Leftrightarrow \sup[\sigma_{\max}(G(j\omega))] \downarrow \]
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- Example:

\[
G(s) = \frac{1}{s + a}
\]

\[
\|G(s)\|_2 = \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2 \, d\omega \right\}^{1/2} = \left\{ \frac{1}{2\pi a} \tan^{-1} \left( \frac{\omega}{a} \right) \right\}\bigg|_{-\infty}^{+\infty}^{1/2} = \sqrt{\frac{1}{2a}}
\]

\[
\|G(s)\|_\infty = \max_\omega |G(j\omega)| = \max_\omega \frac{1}{(\omega^2 + a^2)^{1/2}} = \frac{1}{a}
\]

- Static Gain of the System (max amplitude of Bode plot in this case)

\[
\text{ GS = nd2sys}([10,10],[1,0.2,100]);
\]
\[
\text{ G1=G1nd2sys(1,1,1));}
\]
\[
\text{ G2=nd2sys([1,2],[1,0.1,10]));}
\]
\[
\text{ G2=nd2sys([1,5],[1,1,5,6]));}
\]
\[
\text{ G=abs(cayr(G1,G2,abv(G11,G21,abv(012,622)));}
\]
\[
\text{ w=logspace(0.2,200);}
\]
\[
\text{ Gf=frep(G,w));}
\]
\[
\text{ [u,s,v]=svd(G2);}
\]
\[
\text{ subplot(liv,lm'',s);grid}
\]
\[
\text{ plvnorm(s)}
\]

- Example: \( G(s) = \begin{bmatrix} \frac{10(s + 1)}{s^2 + 0.2s + 100} & \frac{1}{s + 1} \\ \frac{s + 1}{s + 2} & \frac{5(s + 1)}{s^2 + 0.1s + 10} \end{bmatrix} \)

\[
\|G(s)\|_\infty \approx 32.861004645043288
\]

\[
\|G(s)\|_\infty \approx 48.285975309513439
\]
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Computation of the System’s $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Norms.

Computation of the System’s $\mathcal{H}_2$-Norm

\[
\|G(j\omega)\|_2 = \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}[G^*(j\omega)G(j\omega)]d\omega \right]^{\frac{1}{2}}
\]

- **Theorem:** The $\mathcal{H}_2$-norm is finite if and only if the matrix $G(s)$ is strictly proper and stable, which means:

\[
\lim_{s \to \infty} G(s) = G(\infty) = 0 \iff D = 0
\]

- **Lemma:** Consider a stable strictly proper system ($A$ Hurwitz) with TFM $G(s)$. Then:

\[
\|G\|_2 = \sqrt{\text{tr}(B^*L_0B)} = \sqrt{\text{tr}(CL_CC^*)}
\]

with $L_0$ and $L_c$ Observability and Controllability Gramians, solutions of:

\[
\begin{cases}
A^*L_0 + L_0A + C^*C = 0 \\
AL_c + L_c A^* + BB^* = 0
\end{cases}
\]
• **Proof:** since the system is stable, and strictly proper, we have $A < 0$, and the impulse response is:

$$g(t) = \begin{cases} 
-Ce^{At}B, & t \geq 0 \\
0, & t < 0 
\end{cases}, \quad g^*(t) = g^T(t)$$

• From the definition of $H_2$-Norm:

$$\|G(j\omega)\|_2^2 = \|g(t)\|_2^2 = \int_0^\infty tr[g^*(t)g(t)]dt = \int_0^\infty tr[g(t)g^*(t)]dt$$

• Substituting for the impulse response $g(t)$ from the above, we obtain:

$$\|G\|_2^2 = \int_0^\infty tr[B^*e^{A^t}C^*Ce^{At}B]dt = \int_0^\infty tr[Ce^{At}BB^*e^{A^t}C^*]dt$$

$$\|G\|_2^2 = tr\left\{B^*\left[\int_0^\infty e^{A^t}C^*Ce^{At}dt\right]B\right\} = tr\left\{C\left[\int_0^\infty e^{At}BB^*e^{A^t}dt\right]C^*\right\}$$

$$\|G\|_2^2 = tr\left[B^*L_0B\right] = tr\left[CL_CC^*\right] \quad \text{q.e.d.}$$

"Quod Erat Demonstrandum"
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• Example:

\[
G(s) = \frac{10(s + 1)}{s^2 + 0.2s + 100} \frac{1}{s + 2} \frac{1}{5(s + 1)} \frac{1}{s^2 + 0.1s + 10} \frac{1}{(s + 2)(s + 3)}
\]

```matlab
>> G11=nd2sys([10,1],[1,0.2,100]);
>> G12=nd2sys([1,1]);
>> G21=nd2sys([1,2],[1,0.1,10]);
>> G22=nd2sys([5,5],[1,5,6]);
>> G=bsd(sbs(G11,G21),sbs(G12,G22));

>> eig(A)
ans =
  -0.100000000000000 + 9.9999999749375i
  -0.100000000000000 - 9.9999999749375i
  -0.050000000000000 + 3.16188235752475i
  -0.050000000000000 - 3.16188235752475i
  -2.000000000000000 + 0.000000000000000
  -2.000000000000000 - 0.000000000000000
  -2.000000000000000 + 0.000000000000000
  -2.000000000000000 - 0.000000000000000

>> [A,B,C,D]=unpak(G)
>> h2norm(G)
1.621e+001
```
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**Computation of the System’s $\mathcal{H}_\infty$-Norm**

\[ \|G(j\omega)\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\max} \left( G(j\omega) \right) \]

- The $\mathcal{H}_\infty$-norm of a system provides the least upper bound on the system gain, where the gain is defined in terms of the signal’s $L_2$-norm. Its computation is therefore only numerical.

- There are several formal results needed to arrive at the computational procedure for finding the least upper bound (sup of the largest singular value) of a $\mathcal{H}_\infty$-norm.

- Given a MIMO LTI system with input $u(t)$ and impulse response $g(t)$. The following holds:

  \[ \|g(t) \otimes u(t)\|_2 = \int_0^t [g(t - \tau)u(\tau)d\tau] \leq \|G\|_\infty \cdot \|u(t)\|_2 \]

  from which

  \[ \|G\|_\infty = \sup_{\omega \neq 0} \frac{\|g(t) \otimes u(t)\|_2}{\|u(t)\|_2} \]

- Given two linear systems connected in series, $G_1(s)$, and $G_2(s)$, their $\mathcal{H}_\infty$-norm is bounded:

  \[ \|G_1G_2\|_\infty \leq \|G_1\|_\infty \cdot \|G_2\|_\infty \]

*Kronecker Product*
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- **Theorem 1:** The $L_2$-norm of the output $y(t)$ is bounded by the product of the system’s $H_\infty$-norm and the $L_2$-norm of the input $u(t)$

  \[ \|y(t)\|_2 \leq \|G\|_\infty \cdot \|u(t)\|_2 \]

  **Proof:** We wish to prove: \[ \|y(t)\|_2 = \|Gu(t)\|_2 \leq \|G\|_\infty \cdot \|u(t)\|_2 \]

  - From the definition of $L_2$-norm: \[ \|y(t)\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} \left[ y(j\omega) y(j\omega) \right] d\omega} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \|y(j\omega)\|_2^2} d\omega \]

  - but we know that: \[ \|y(j\omega)\|_2 \leq \max_{\omega} \|G(j\omega)\|_2 \|u(j\omega)\|_2 \]

  - therefore: \[ \|y(t)\|_2 \leq \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \max_{\omega} \|G(j\omega)\|_2^2 \|u(j\omega)\|_2^2} d\omega \]

  - Since the maximum singular value at a given frequency is bounded above by the supremum over all frequencies,

    \[ \|y(t)\|_2 \leq \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \max_{\omega} \|G(j\omega)\|_2^2 \|u(j\omega)\|_2^2} d\omega \leq \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \sup_{\omega} \max_{\omega} \|G(j\omega)\|_2^2 \|u(j\omega)\|_2^2} d\omega \]

    - thus: \[ \|y\|_2 \leq \|G\|_\infty \|u\|_2 \]
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- **Theorem 2**: Given some non-zero input vector \( u(t) \), the following holds:

\[
\|G\|_{\infty} = \sup_{u \neq 0} \frac{\|Gu\|_2}{\|u\|_2}
\]

**Proof**: No Proof

- **Theorem 3**: The following holds:

\[
\|G_1 G_2\|_{\infty} \leq \|G_1\|_{\infty} \cdot \|G_2\|_{\infty}
\]

**Proof**: We know that for some non zero input \( u(t) \neq 0 \):

\[
\|G_1 G_2 u\|_2 \leq \|G_1\|_{\infty} \cdot \|G_2 u\|_2 \leq \|G_1\|_{\infty} \cdot \|G_2\|_{\infty} \cdot \|u\|_2
\]

- Rearranging and taking the sup of both sides:

\[
\|G_1 G_2\|_{\infty} = \sup_{u \neq 0} \frac{\|G_1 G_2 u\|_2}{\|u\|_2} \leq \|G_1\|_{\infty} \cdot \|G_2\|_{\infty}
\]

- **Theorem 4**: Given a continuous time varying linear system \((A, B, C, D)\), it can be shown that the finite time \(\mathcal{H}_2\)-norm is finite:

\[
\|G\|_{\mathcal{H}_2[t_0, t_f]} < \infty
\]

**Proof**: No Proof
Chapter 3: MIMO Tools: Performance Measures

- Actual Computation of the $\mathcal{H}_\infty$ - norm is more complex, it requires a numerical search over all frequencies, Since:

$$\|G(j\omega)\|_\infty := \sup_{\omega \in \mathbb{R}} \max \{ G(j\omega) \}$$

- We can however compute an upper bound of the $\sup$, using the properties of Hamiltonian matrices. Recall the following results:

**Definition 1:** Given $A$, $Q$, and $R$ real square $nxn$ matrices with $Q$, and $R$ symmetric. The associated algebraic Riccati equation is given by:

$$A^T X + XA + XRX + Q = 0$$

**Definition 2:** Given an algebraic Riccati equation, the associated $2nx2n$ Hamiltonian matrix is:

$$\mathcal{H} := \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix}$$

- Property of a Hamiltonian matrix: The eigenvalue spectrum of $\mathcal{H}$ is symmetric about the imaginary axis.

- Example: LQR Problem

$$A^T S + S A + Q - S B R^{-1} B^T S = 0$$
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■ Bounded Real Lemma

■ Theorem (Zhou, “Essentials of Robust Control”): The $\mathcal{H}_\infty$-norm of a stable proper system is bounded by a positive scalar $\gamma > 0$, that is:

$$\|G(s)\|_\infty < \gamma, G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

if and only if:

$$\sigma_{\max}(D) < \gamma$$

and the 2nx2n Hamiltonian Matrix $\mathcal{H}$ has no eigenvalues on the imaginary axis.

$$\mathcal{H} = \begin{bmatrix} A + BR^{-1}D^T C & BR^{-1}B^T \\ -C^T(I + DR^{-1}D^T)C & -A^T - C^T DR^{-1}B^T \end{bmatrix}$$

$R = (\gamma^2 I - D^T D)$

■ Proof: We wish to prove that, for a stable, proper system:

$$G(j\omega) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$\|G(j\omega)\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\max}[G(j\omega)] = \sup_{\omega \in \mathbb{R}} \sigma_{\max}[C(j\omega I - A)^{-1}B + D] < \gamma$$

■ Taken from Zhou, Doyle: “Essentials of Robust Control”, Prentice Hall, 1997
Recall from the definition of singular values:

\[
\lim_{\omega \to \infty} [G(j\omega)] = D \Rightarrow \lim_{\omega \to \infty} \sigma_{\text{max}} [G(j\omega)] = \sigma_{\text{max}} [D]
\]

Therefore:

\[
\left\|G(j\omega)\right\|_{\infty} = \sup_{\omega} \sigma_{\text{max}} [G(j\omega)] \geq \sigma_{\text{max}} [D]
\]

\[
\sigma_i = \sqrt{\lambda_i(G^*G)} = \sqrt{\lambda_i(GG^*)}
\]

The \( \mathcal{H}_\infty \) - norm may be less than \( \gamma \), only if: \( \sigma_{\text{max}} [D] < \gamma \)

But from the definition of singular values: \( \sigma_{\text{max}} [D] < \gamma \) \text{ If and only if: }

\[
\lambda(G^*G) < \gamma^2 \quad \text{All eigenvalues are strictly real}
\]

From Cayley – Hamilton Theorem: \( \lambda(G^*G) < \gamma^2 \) implies: \( \lambda(\gamma^2 I - G^*G) > 0 \)
• Therefore: \( \lambda(\gamma^2 I - G^*G) > 0 \) holds with the previous assumption:

\[
\begin{cases}
\omega \to \infty \\
\sigma_{\text{max}}[D] < \gamma
\end{cases}
\]

• The eigenvalues are continuous functions of \( \omega \), therefore the above inequality is satisfied if and only if the eigenvalues do not cross the imaginary axis, i.e.:

\[
\det[\gamma^2 I - G^*(j\omega)G(j\omega)] \neq 0; \forall \omega \in [0, \infty)
\]

This means that \( \det[.] \) has no zeros (roots) on the imaginary axis:

• Compute the above determinant

\[
\begin{align*}
G(j\omega) &\iff [A, B, C, D] \\
G^*(j\omega) &= G^T(-j\omega) \iff [-A^T, -C^T, B^T, D^T]
\end{align*}
\]
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- The state space representation associated with the Hamiltonian matrix $\mathcal{H}$ is:

$$
\begin{align*}
\dot{x} &= \begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix} x + \begin{bmatrix} -B \\ C^T D \end{bmatrix} u \\
y &= \begin{bmatrix} D^T C & B^T \end{bmatrix} x + (\gamma^2 I - D^T D) u
\end{align*}
$$

- The zeros of the system are computed by setting the determinant equal to zero:

$$
\det \begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix} \begin{bmatrix} -B \\ C^T D \end{bmatrix} = 0 = \det[\square]
$$

- From the property of determinants:

$$
\det[\square] = \det(\gamma^2 I - D^T D) \cdot \det \left\{ sI - \begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix} - \begin{bmatrix} -B \\ C^T D \end{bmatrix} (\gamma^2 I - D^T D)^{-1} \begin{bmatrix} D^T C & B^T \end{bmatrix} \right\} = 0
$$
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- But: \( \det(\gamma^2 I - D^T D) \neq 0 \) And this implies:

\[
\det \left[ sI - \begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix} - \begin{bmatrix} -B \\ C^T D \end{bmatrix}(\gamma^2 I - D^T D)^{-1} \begin{bmatrix} D^T C & B^T \end{bmatrix} \right] = 0
\]

- The above is an eigen-equation and the system \( \gamma^2 I - G^T G \) has no zeros on the imaginary axis if and only if the Hamiltonian matrix \( \mathcal{H} \) has no eigenvalues on the imaginary axis:

\[
\mathcal{H} := \begin{bmatrix}
A + BR^{-1}D^TC & BR^{-1}B^T \\
-C^T(I + DR^{-1}D^T)C & -A^T - C^TDR^{-1}B^T \\
R &= (\gamma^2 I - D^T D)
\end{bmatrix}
\]

- The Bounded Real Lemma gives an algorithm that can be used for the actual computation of the bound \( \gamma \) (see Matlab toolboxes on robust control).

Algorithm:
1. Select \( \gamma > 0 \)
2. Find \( \text{Re}\{\lambda_i(\mathcal{H})}\) 
3. If \( \text{Re}\{\lambda_i(\mathcal{H})\} \neq 0, \gamma_{i+1} < \gamma \) goto 2 
4. else \( \gamma_{i+1} > \gamma \) goto 2 
5. Stop \( \gamma = \gamma_{\text{min}} \)
6. \( \|G\|_\infty \leq \gamma_{\text{min}} \)

Remark: From \( \|G\|_\infty < \gamma \) we have \( \|\gamma^{-1}G\|_\infty < 1 \), therefore if we let \( \gamma = 1 \), we can use \( \|G\|_\infty < 1 \).
Chapter 3: MIMO Tools: Performance Measures

**H∞ Computation with Matlab**

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} = A \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} + B \begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}, \quad D = 0,
\]

\[
A = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-k_1 & -k_1 & \frac{b_1}{m_1} & \frac{b_1}{m_1} \\
\frac{k_1}{m_2} & \frac{k_1}{m_2} & \frac{b_1}{m_2} & \frac{b_1}{m_2} \\
\frac{m_1}{m_2} & \frac{m_1}{m_2} & \frac{m_1}{m_2} & \frac{m_1}{m_2}
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & \frac{1}{m_2}
\end{bmatrix}.
\]

and suppose \(k_1 = 1, k_2 = 4, b_1 = 0.2, b_2 = 0.1, m_1 = 1,\) and \(m_2 = 2\) with appropriate units.

**H2 Computation with Matlab**
Chapter 3: MIMO Tools: Performance Measures

\[ G = \text{pck}(A,B,C,D); \]
\[ \text{hinfnorm}(G,0.0001) \text{ or } \text{linfnorm}(G,0.0001) \% \text{ relative error} \leq 0.0001 \]
\[ w = \text{logspace}(-1,1,200); \% 200 \text{ points between } 1 = 10^{-1} \text{ and } 10^{1} \]
\[ Gf = \text{frsp}(G,w); \% \text{ computing frequency response} \]
\[ [u,s,v] = \text{svd}(Gf); \% \text{ SVD at each frequency} \]
\[ \text{vplot}(\text{’liv, Im’, s}, \text{ grid} \% \text{ plot both singular values and grid} \]

\[ \|G(s)\|_\infty = 11.47 = \text{ the peak of the largest singular value Bode plot in} \]

Since the peak is achieved at \( \omega_{\text{max}} = 0.8483 \), exciting the system using the following sinusoidal input

\[
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix} = \begin{bmatrix}
0.9614 \sin(0.8483t) \\
0.2753 \sin(0.8483t - 0.12)
\end{bmatrix} \rightarrow
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
11.47 \times 0.9614 \sin(0.8483t - 1.5483) \\
11.47 \times 0.2753 \sin(0.8483t - 1.4283)
\end{bmatrix}
\]
Note: Alternate Statement of BRL (for clarity):

By Parseval’s theorem, the $H_\infty$ norm can be characterized in the time domain as

$$\|G\|_\infty = \sup \left\{ \frac{\|z\|_2}{\|v\|_2} : v \neq 0 \right\}$$

It follows that for any $\gamma > 0$, $\|G\|_\infty < \gamma$ if and only if

$$J_\infty(G, \gamma) := \max_v \left[ \|z\|_2^2 - \gamma^2 \|v\|_2^2 \right]$$

$$= \max_v \int_0^\infty \left[ z^T(t) z(t) - \gamma^2 v^T(t) v(t) \right] dt$$

$$< \infty$$

**Theorem 2.1** The system $G(s)$ with state-space representation (2.62) has $H_\infty$-norm less than $\gamma$, $\|G\|_\infty < \gamma$, if and only if $\gamma^2 I - D^T D > 0$ and the Riccati equation

$$A^T S_\infty + S_\infty A + (S_\infty B + C^T D)(\gamma^2 I - D^T D)^{-1}(B^T S_\infty + D^T C) + C^T C = 0$$

has a bounded positive semidefinite solution $S_\infty$ such that the matrix $A + B(\gamma^2 I - D^T D)^{-1}(B^T S_\infty + D^T C)$ has all eigenvalues in the left half plane.
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Different perspectives of "Systems"
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Linear Fractional Transformations 2x2 Block Representation

Consider a closed loop MIMO LTI System with output feedback dynamic compensator (Optimal Controllers can be casted in the form below):

\[
\begin{align*}
\dot{x} &= Ax + Bu + Dw \\
y &= Cx, z = Ex \\
\dot{q} &= Kq + Ly \\
u &= Mq + Ny
\end{align*}
\]

- \( w \) = exogenous input (disturbances, noise, reference signals, commands, ...)
- \( u \) = control input
- \( y \) = feedback output (measurements to controller)
- \( z \) = regulated output (all variables of interest for the closed loop performance)
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Feedback the controller to get the closed loop structure:

\[
P(s) = \begin{bmatrix} P_{zw}(s) & P_{zu}(s) \\ P_{yw}(s) & P_{yu}(s) \end{bmatrix} = \begin{bmatrix} E[sI - A]^{-1} D \\ E[sI - A]^{-1} B \\ C[sI - A]^{-1} D \\ C[sI - A]^{-1} B \end{bmatrix}
\]

\[
K(s) = K_{uy}(s) = M[sI - K]^{-1} L
\]
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\[
\begin{align*}
\begin{cases}
z = P_{zw}(s)w + P_{zu}(s)u \\
y = P_{yw}(s)w + P_{yu}(s)u \\
u = K(s)y
\end{cases}
\end{align*}
\]

\[
\begin{align*}
z &= P_{zw}(s)w + P_{zu}(s)K(s)y \\
y &= P_{yw}(s)w + P_{yu}(s)K(s)y \\
[I - P_{yu}(s)K(s)]y &= P_{yw}(s)w
\end{align*}
\]

\[
z = P_{zw}(s)w + P_{zu}(s)K(s)[I - P_{yu}(s)K(s)]^{-1}P_{yw}(s)w
\]

\[
z = \left\{ P_{zw} + P_{zu}(I-P_{yu})^{-1}P_{yw} \right\} w = T_{zw}(s)w
\]
• **Example:** One Degree of Freedom Feedback Configuration

The first step is to identify the loop signals of interest \((w, u, z, y)\)

\[
w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \end{bmatrix} = \begin{bmatrix} d \\ r \\ n \\ \end{bmatrix}; \quad z = e = y - r; \quad v = r - y_m = r - y - n
\]

\[
z = y - r = Gu + d - r = Iw_1 - Iw_2 + 0w_3 + Gu
\]
\[
v = r - y_m = r - Gu - d - n = -Iw_1 + Iw_2 - Iw_3 - Gu
\]
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\[
\begin{bmatrix}
  z \\
  v
\end{bmatrix} = P(s) \begin{bmatrix}
  w \\
  u
\end{bmatrix}; P = \begin{bmatrix}
  I & -I & 0 & G(s) \\
  -I & I & -I & -G(s)
\end{bmatrix}
\]

% Uses the Mu-toolbox
systemnames = 'G'; % G is the SISO plant.
inputvar = '[d(1); r(1); n(1); u(1)]'; % Consists of vectors w and u.
input_to_G = '[u]';
outputvar = '[G+d-r; r-G-d-n]'; % Consists of vectors z and v.
sysoutname = 'P';
sysic;

**sysic command in Matlab**
**Example:** LQR Representation in P-K Form

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= x
\end{align*}
\]

\[
J = \int_0^\infty [x^T Q x + u^T R u] dt = \int_0^\infty [z^T z + u^T R u] dt \\
u = -Kx = R^{-1} B^T Sx
\]

\[
\begin{align*}
\dot{x} &= Ax + Iw + Bu \\
z_1 &= \sqrt{Q} x \\
z_2 &= \sqrt{R} u \\
y &= Ix
\end{align*}
\]

\[
\sqrt{Q} :\Rightarrow (\sqrt{Q})^T \sqrt{Q} = \sqrt{Q} (\sqrt{Q})^T = Q
\]

\[
\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \Rightarrow z(t) \\
P := \begin{bmatrix} A & I & B \\ \sqrt{Q} & 0 & 0 \\ 0 & 0 & \sqrt{R} \\ I & 0 & 0 \end{bmatrix}
\]

**Diagram:**

- Process:
  - Input: \( w \)
  - Output: \( z_1 \) and \( z_2 \)
  - Block Diagram: \( P(s) \)
**Example:** LQG Representation in P-K Form

\[ K(s) = K_C [sI - A + BK_C + K_F C]^{-1} K_F \]

\[
\begin{align*}
\dot{x} &= Ax + Bu + w_d \\
y &= Cx + v_n
\end{align*}
\]

\[
E\begin{bmatrix}\omega_d(t) \\
v_n(t)\end{bmatrix}
\begin{bmatrix}x(t) \\
y(t)\end{bmatrix}
= W 0 \\
0 V
\delta(t - \tau)
\]

\[ J = E \lim_{t_j \to 0} \int_{t_0}^{t_f} \left[ x^T Q x + u^T R u \right] d\tau \]

\[
\begin{bmatrix}x(t) \\
y(t)\end{bmatrix}
= -K_c \hat{x}(t) \\
\hat{x} = A\hat{x} + Bu + K_F (y - C\hat{x})
\]

\[
\dot{q} = \begin{bmatrix} A - K_F C & 0 \\
K_F C & A - BK_c \end{bmatrix} q + \begin{bmatrix} I & -K_F \\
0 & K_F \end{bmatrix} \begin{bmatrix} \omega_d \\
v_n\end{bmatrix} q = \frac{e}{\hat{x}}
\]
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- The generalized plant $P(s)$ becomes

$$P = \begin{bmatrix} A & W^{\frac{1}{2}} & 0 & B \\ Q^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & R^2 \\ C & 0 & V^{\frac{1}{2}} & 0 \end{bmatrix}$$

\[
\begin{cases}
    w_d(t) \\
    v_n(t)
\end{cases} \Rightarrow w(t) \quad K(s) = K_C \left( sI - A + BK_C + K_F C \right)^{-1} K_F
\]

\[
\begin{cases}
    \frac{1}{Q^2} x \\
    \frac{1}{R^2} u
\end{cases} \Rightarrow z(t)
\]

\[
\begin{cases}
    x = Ax + Bu + Bw_d \\
    z_1 = \frac{1}{Q^2} x \\
    z_2 = \frac{1}{R^2} u \\
    y = Cx + v_n
\end{cases}
\]

(*) Exercise: Use Matlab to write a function, which does the above.
Incorporation of shaping matrix functions in a 2–block LFT structure

- In control systems design, regulated variables are often shaped in the frequency domain to achieve desired transient response.
  - Previous example of LQR
  - Frequency shaping design (weight on sensitivity)
- Shaping functions (matrices) can be easily incorporated in a 2–Block Format

![Diagram](image_url)

- In this case, the output vector of interest \( z(t) \) includes input, output and error signals.
- Output and reference (or disturbance) constitute the input \( v(t) \) to the controller.
• $W_u(s)$, $W_T(s)$, and $W_P(s)$ are proper and stable transfer matrices selected by the designer.

\[
\begin{align*}
z_1 &= W_u u \\
z_2 &= W_T G u \\
z_3 &= W_P w + W_P G u \\
v &= -w - G u
\end{align*}
\]

• The generalized plant $P(s)$ becomes:

\[
P = \begin{bmatrix}
0 & W_u I \\
0 & W_T G \\
W_P I & W_P G \\
-I & -G
\end{bmatrix}
\]

\[
\begin{bmatrix}
z \\
v
\end{bmatrix} = P(s) \begin{bmatrix}
w \\
u
\end{bmatrix}
\]

\[
P = \begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix}
\]

\[
P_{11} = \begin{bmatrix}
0 \\
0
\end{bmatrix}, \quad P_{12} = \begin{bmatrix}
W_u I \\
W_T G
\end{bmatrix}
\]

\[
P_{21} = -I, \quad P_{22} = -G
\]

• Linear Fractional Transformations are very general block diagram algebra structures easily applicable to 2x2 blocks.
**Definition 1:** Given a complex matrix $M(s)$ in a 2x2 block form:

\[
M = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix} \in \mathbb{C}^{(p_1+p_2)\times(q_1+q_2)}
\]

Given the complex matrices:

\[
\Delta_l \in \mathbb{C}^{q_2\times p_2} \quad \Delta_u \in \mathbb{C}^{q_1\times p_1}
\]

1. We define a **Lower Fractional Transformation** with respect to $\Delta_l$ Provided the Inverse exists

\[
\Imag(M) = M_{11} + M_{12}\Delta_l(I - M_{22}\Delta_l)^{-1}M_{21}
\]

2. We define a **Upper Fractional Transformation** with respect to $\Delta_u$ Provided the Inverse exists

\[
\Imag(M) = M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12}
\]
Chapter 3: MIMO Tools: Block Algebra

• Recall the known facts:
  • A real rational matrix $G(s)$ has a real rational inverse $G^{-1}(s)$ if and only if $\text{det}[G(s)] \neq 0, \forall s \in \mathbb{C}$
  • If $G(s)$ is also proper, its inverse exists if and only if $\text{det}[G(\infty)] \neq 0$
  • A proper and stable matrix $G(s)$ has a proper and stable inverse $H(s) \Rightarrow G(s)H(s) = I$, if:
    • $\text{det}[G(\infty)] \neq 0$
    • $\text{det}[G(\infty)] \neq 0$ has no roots such that $z \notin \mathbb{C}^0 \cup \mathbb{C}^+$

**Definition 2:** A lower fractional transformation (LFT) $F(M, \Delta_l)$ is said to be well-posed or well-defined if $(I - M_{22}\Delta_l)^{-1}$ exists. Similarly for a UFT (see Zhou notes).

• Well – posedness of feedback loops is obviously critical for the actual realization of feasible closed loop controllers. Recall the general form:

$$z = \left\{ P_{zw} + P_{zu} K (I - P_{yu} K)^{-1} P_{yw} \right\} w = T_{zw}(s)w$$
Chapter 3: MIMO Tools: Block Algebra

Description of a 2 - block structure in state space format:

\[
\begin{align*}
  z &= P_{zw}(s)w + P_{zu}(s)u \\
  y &= P_{yw}(s)w + P_{yu}(s)u \\
  u &= K(s)y
\end{align*}
\]

\[ z(s) = \mathcal{Z}_1(G, K)w(s) \]

\[
\begin{align*}
  \dot{x}(t) &= ? \\
  y(t) &= ? \\
  z(t) &= ?
\end{align*}
\]

Example:

\[
\begin{pmatrix}
  w \\
  z
\end{pmatrix}
= \begin{pmatrix}
  d \\
  n
\end{pmatrix}
\]

The loop subsystems and weights have a general realization given by:

\[
P = \begin{bmatrix}
P_p & B_p \\
C_p & 0
\end{bmatrix},
F = \begin{bmatrix}
A_f & B_f \\
C_f & D_f
\end{bmatrix}
\]

\[
W_1 = \begin{bmatrix}
A_u & B_u \\
C_u & D_u
\end{bmatrix},
W_2 = \begin{bmatrix}
A_v & B_v \\
C_v & D_v
\end{bmatrix}
\]

\[
K = \begin{bmatrix}
K_A & K_B \\
K_C & K_D
\end{bmatrix}
\]

\[
\begin{align*}
  \dot{x}_p &= A_p x_p + B_p(d + u),
  y_p &= C_p x_p, \\
  \dot{x}_f &= A_f x_f + B_f(y_p + n),
  -y &= C_f x_f + D_f(y_p + n) \\
  \dot{x}_u &= A_u x_u + B_u u,
  u_f &= C_u x_u + D_u u, \\
  \dot{x}_v &= A_v x_v + B_v y_p,
  v &= C_v x_v + D_v y_p.
\end{align*}
\]
Chapter 3: MIMO Tools: Block Algebra

- Define the following State Vector:

\[ x = \begin{bmatrix} x_p \\ x_f \\ x_u \\ x_v \end{bmatrix} \]

- Eliminate \( y_p \) to get a realization for \( G(s) \):

\[
\begin{align*}
\dot{x} &= Ax + B_1w + B_2u \\
z &= C_1x + D_{11}w + D_{12}u \\
y &= C_2x + D_{21}w + D_{22}u
\end{align*}
\]

\[
A = \begin{bmatrix}
A_p & 0 & 0 & 0 \\
B_f C_p & A_f & 0 & 0 \\
0 & 0 & A_u & 0 \\
B_v C_p & 0 & 0 & A_v
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
B_p & 0 \\
0 & B_f \\
0 & 0 \\
B_v & 0
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
B_p \\
0 \\
B_u \\
0
\end{bmatrix}
\]

\[
C_1 = \begin{bmatrix}
D_v C_p & 0 & 0 & C_v \\
0 & 0 & C_w & 0
\end{bmatrix}, \quad D_{11} = 0, \quad D_{12} = \begin{bmatrix} 0 \\
D_v \end{bmatrix}
\]

\[
C_2 = \begin{bmatrix}
-D_f C_p & -C_f & 0 & 0
\end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & -D_f \end{bmatrix}, \quad D_{22} = 0.
\]

- Computational Aspects: Use the RedHeffer Star Product (Matlab Implementation).
Chapter 3: MIMO Tools: Block Algebra

\[ P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \]

\[ P = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}, \quad K = \begin{bmatrix} A_K & B_{K1} & B_{K2} \\ C_{K1} & D_{K11} & D_{K12} \\ C_{K2} & D_{K21} & D_{K22} \end{bmatrix} \]

\[ P \ast K := \begin{bmatrix} F_1(P, K_{11}) & P_{12} (I - K_{11} P_{22})^{-1} K_{12} \\ K_{21} (I - P_{22} K_{11})^{-1} P_{21} & F_u(K, P_{22}) \end{bmatrix} \]

\[ \Rightarrow P \ast K = \text{starp}(P, K, \text{dimy, dimu}) \]

\[ \Rightarrow \mathcal{F}_\ell(P, K) = \text{starp}(P, K) \]

**Summary Remarks**

\[ z = \left\{ P_{zw} + P_{zu} K (I - P_{yw}^{-1} P_{yw}) \right\} w = T_{zw}(s)w \]

\[ \begin{align*}
\dot{x} &= A x + B_1 w + B_2 u \\
 z &= C_1 x + D_{11} w + D_{12} u \\
y &= C_2 x + D_{21} w + D_{22} u
\end{align*} \]
Chapter 3: MIMO Tools: Structural Properties

- **Well – Posedness of a Feedback Loop**

  - Recall from previous result, the matrix inversion requirement for a lower LFT:

    \[
    z = \left\{ P_{zw} + P_{zu} K (I - P_{yu} K)^{-1} P_{yw} \right\} w = T_{zw}(s) w
    \]

  - Explanation using a SISO example

  ![SISO Example Diagram]

  - **Computation of the controller signal** \( u(t) \): the plant and controller are proper systems, the feedback control law however is not causal!

    \[
    u(s) = \frac{s + 2}{3} (r - n - d_o) - \frac{s - 1}{3} d_i
    \]
• **Definition:** A feedback system is said to be well-posed if all the closed loop transfer matrices are well-defined and at least proper.

- Closed loop transfer matrices (SISO example)

\[
\begin{align*}
S(s) &= \frac{1}{1 + L(s)} \\
S(s)P(s) &= K(s)S(s) \\
T(s) &= \frac{L(s)}{1 + L(s)}
\end{align*}
\]

- Consider the following 2x2 feedback configuration

\[
\hat{K}(s) = -K(s)
\]

- **Lemma (Zhou: p. 119)** The feedback configuration is well-posed iff \(1 - K(\infty)P(\infty)\) is invertible.
Chapter 3: MIMO Tools: Structural Properties

• Well-Posedness in State Space Format

Consider the feedback structure on the right. Assume that \( H_1 \) and \( H_2 \) have some state-space descriptions with inputs \( u_1, u_2 \) and outputs \( y_1, y_2 \), so that their transfer function matrices \( H_1(s) \) and \( H_2(s) \) are proper, i.e. \( H_1(\infty), H_2(\infty) \) are finite.

\[
H_1 \sim \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \quad H_2 \sim \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}
\]

Note that \( D_1 = H_1(\infty) \) and \( D_2 = H_2(\infty) \).

• Write the feedback interconnection in State Space Form (external inputs are neglected):

\[
\begin{align*}
\dot{x}_1 &= A_1 x_1 + B_1 u_1 \\
y_1 &= C_1 x_1 + D_1 u_1 \\
\dot{x}_2 &= A_2 x_2 + B_2 u_2 \\
y_2 &= C_2 x_2 + D_2 u_2 
\end{align*}
\]
Chapter 3: MIMO Tools: Structural Properties

- The interconnection constraints on the loop signal are expressed by the following equations:

\[
\begin{align*}
    u_1 &= r_1 + y_2 = r_1 + C_2 x_2 + D_2 u_2 \\
    u_2 &= r_2 + y_1 = r_2 + C_1 x_1 + D_1 u_1,
\end{align*}
\]

- which can be rewritten as:

\[
\begin{bmatrix}
    I & -D_2 \\
    -D_1 & I
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_2
\end{bmatrix} =
\begin{bmatrix}
    0 & C_2 \\
    C_1 & 0
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix} +
\begin{bmatrix}
    I & 0 \\
    0 & I
\end{bmatrix}
\begin{bmatrix}
    r_1 \\
    r_2
\end{bmatrix}
\]

- From above, the interconnected system is \textbf{well-posed} if the following matrix is invertible:

\[
\begin{bmatrix}
    I & -D_2 \\
    -D_1 & I
\end{bmatrix}
\quad \text{Or:} \quad I - D_1 D_2 \quad \text{or equivalently} \quad I - D_2 D_1 \quad \text{is invertible.}
\]

\text{Or:} \quad \left(I - H_1(s)H_2(s)\right)^{-1} \quad \text{or equivalently} \quad \left(I - H_2(s)H_1(s)\right)^{-1} \quad \text{exists and is proper}
Chapter 3: MIMO Tools: Structural Properties

- A **sufficient** condition is that either $H_1$ or $H_2$ (or both) be *strictly* proper; that is, either $D_1 = 0$ or $D_2 = 0$.

  - The significance of well-posedness is that once we have solved (*) for $u_{1,2}$, we can eliminate $u_1$ and $u_2$ and arrive at a state-space description of the closed-loop system, with state vector $(x_1, x_2)$.

  - Without well-posedness, $u_1$ and $u_2$ would not be well-defined for arbitrary $x_1, x_2, r_1$ and $r_2$, which would in turn mean that there could not be a well-defined state-space representation of the closed-loop system.

**Summary:**
- A feedback system is well posed if: all the transfer function matrices from $w$ to $y$ and $u$ are proper.
- A feedback system is well posed if and only if:
  
  $[I - K(s)P(s)]^{-1}$ Exists and it is proper
  
  $[I - K(\infty)P(\infty)]$ Is invertible
  
  
  $\begin{bmatrix} I & -D_2 \\ -D_1 & I \end{bmatrix}$ Is invertible
Internal Stability

- A system is internally stable if for all initial conditions, and all bounded signals injected at any place in the system, all states remain bounded for all future time. For a controllable (stabilizable) and observable (detectable) system, internal and external stability coincide.
Chapter 3: MIMO Tools: Structural Properties

- Consider the 2 – Block structure below

\[
\begin{align*}
\begin{bmatrix}
    w_1 \\
e_1 \\
\end{bmatrix} & \rightarrow P \\
& \rightarrow e_2 \\
& \rightarrow \hat{K} \\
\hat{K}(s) & = -K(s) \\
& \rightarrow w_2
\end{align*}
\]

\[
\begin{align*}
e_1 & = w_1 + \hat{K}e_2 \\
e_2 & = w_2 + Pe_1
\end{align*}
\]

\[
\begin{bmatrix}
    I & -\hat{K} \\
    -P & I
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix} =
\begin{bmatrix}
w_1 \\
w_2
\end{bmatrix}
\]

- Recall from matrix theory:

\[
\begin{bmatrix}
    A & D \\
    C & B
\end{bmatrix}^{-1} =
\begin{bmatrix}
    A^{-1} + E\Delta^{-1}F & -E\Delta^{-1} \\
    -\Delta^{-1}F & \Delta^{-1}
\end{bmatrix}
\]

where:

\[
\begin{align*}
\Delta & = B - CA^{-1}D \\
E & = A^{-1}D \\
F & = CA^{-1}
\end{align*}
\]
Chapter 3: MIMO Tools: Structural Properties

**Theorem 4.6 (Skogestad, p. 146):** Assume that the components $P$ and $K$ contain no unstable hidden modes. Then the feedback system is internally stable if and only if all four closed-loop transfer matrices below are stable.

\[
\begin{bmatrix}
I & -K \\
-P & I
\end{bmatrix}^{-1} = \begin{bmatrix}
(I - KP)^{-1} & K(I - KP)^{-1} \\
P(I - KP)^{-1} & (I - PK)^{-1}
\end{bmatrix} = \begin{bmatrix}
I + K(I - PK)^{-1}P & K(I - PK)^{-1} \\
(I - KP)^{-1}P & (I - PK)^{-1}
\end{bmatrix}
\]

**NOTE:** $K$ should be $\hat{K}$

**Lemma 5.3 (Zhou, p. 122):** The system in the block diagram is internally stable if and only if the transfer matrix from $w$ to $e$:

\[
\begin{bmatrix}
I & -K \\
-P & I
\end{bmatrix}^{-1} = \begin{bmatrix}
(I - KP)^{-1} & K(I - KP)^{-1} \\
P(I - KP)^{-1} & (I - PK)^{-1}
\end{bmatrix} = \begin{bmatrix}
I + K(I - PK)^{-1}P & K(I - PK)^{-1} \\
(I - KP)^{-1}P & (I - PK)^{-1}
\end{bmatrix}
\]

Belongs to $\mathcal{RH}_\infty$.

- **Recall:** $\mathcal{RH}_\infty$ is the subspace of $\mathcal{H}_\infty$ which consists of all proper and real rational stable transfer matrices.
Chapter 3: MIMO Tools: Structural Properties

- Go back to the classical unity feedback representation \([G(s) \text{ and } P(s) \text{ are the same}]\)

- Consider the following coprime (*) polynomials: \(Z(s), N(s), Z_k(s), N_k(s)\), and

\[
\begin{align*}
P(s) &= \frac{Z}{N} \\
K(s) &= \frac{Z_k}{N_k}
\end{align*}
\]

\[L(s) = P(s)K(s)\]
\[T(s) = [1 + L(s)]^{-1}L(s)\]
\[S(s) = [1 + L(s)]^{-1}\]
\[S_u(s) = K(s)S(s)\]

\[
\begin{pmatrix}
Z \\
N \\
Z_k \\
N_k
\end{pmatrix}
\]

\[
\begin{pmatrix}
P(s) \\
K(s)
\end{pmatrix}
\]

\[
\begin{pmatrix}
l(s) \\
y(s) \\
f(s) \\
q(s) \\
n(s)
\end{pmatrix}
\]

\[\square \text{ Theorem 3.2.1. (Mackenroth, ‘Robust Control Systems’, p. 45)}\]

A feedback system is internally stable, if and only if one of the following conditions is satisfied:

1. The polynomial \(ZZ_k + NN_k\) has no zeros in \(\mathbb{C}_+\)

2. The sensitivity function \(S = (I + L)^{-1}\) is stable and there is no unstable cancellation when the loop transfer function \(L = KP\) is formed.

\[\text{NOTE: This is equivalent to requiring } S(s), KS(s), PS(s), \text{ and } T(s) \text{ to be proper, real, rational, and stable transfer matrices.}\]

\((*)\) two polynomials are coprime iff their GCD is 1
Chapter 3: MIMO Tools: Structural Properties

- **Exercise:** Check the internal stability of the following systems

\[
\begin{align*}
\begin{cases}
    P(s) = \frac{1}{1 - Ts} \\
    K(s) = k_R \frac{1 - Ts}{s}
\end{cases} & \quad \begin{cases}
    P(s) = \frac{1 - T_1s}{1 + Ts} \\
    K(s) = k_R \frac{1 + Ts}{s(1 - T_1s)}
\end{cases}
\end{align*}
\]

- Internal stability can be also be inferred through the state space representation

\[
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2
\end{bmatrix} = 
\begin{bmatrix}
    A_1 & B_1 \\
    C_1 & D_1
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix} + 
\begin{bmatrix}
    A_2 & B_2 \\
    C_2 & D_2
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_2
\end{bmatrix}
\]

- Assume the closed loop system be well posed, thus:

\[
\begin{bmatrix}
    I & -D_2 \\
    -D_1 & I
\end{bmatrix}
\] is invertible.
Let the exogenous inputs be zero, and solve for $e_1$ and $e_2$:

$$
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix} =
\begin{bmatrix}
I & -D_2^{-1} \\
-D_1 & I
\end{bmatrix}
\begin{bmatrix}
0 & C_2 \\
C_1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
$$

Write the closed loop dynamics:

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \tilde{A}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\quad \tilde{A} =
\begin{bmatrix}
A_1 & 0 \\
0 & A_2
\end{bmatrix}
+ 
\begin{bmatrix}
B_1 & 0 \\
0 & B_2
\end{bmatrix}
\begin{bmatrix}
I & -D_2^{-1} \\
-D_1 & I
\end{bmatrix}
\begin{bmatrix}
0 & C_2 \\
C_1 & 0
\end{bmatrix}
$$

Lemma 5.2. (Zhou, p. 121): Assume $P(s)$ and $K(s)$ be stabilizable and detectable realizations. Then the closed loop system is internally stable if and only if the closed loop matrix $A^\sim$ is Hurwitz.
Question: given the previous results, can we formalize requirements on a controller \( K(s) \), such that the closed loop is well posed and internally stable?

- Can we find a controller structure that is admissible and stabilizing?

\[
\begin{align*}
\dot{x} &= A x + B_1 w + B_2 u \\
z &= C_1 x + D_{11} w + D_{12} u \\
y &= C_2 x + D_{21} w + D_{22} u \\
\end{align*}
\]

\[
P(s) = \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix} = \begin{bmatrix}
P_{zw} & P_{zw} \\
P_{yw} & P_{yu}
\end{bmatrix}
\]

\[
K(s) \Rightarrow \begin{cases}
\dot{q} = Kq + Ly \\
u = Mq + Ny
\end{cases}
\]

- Review: Examples of the above are LQR and/or pole placement with state reconstruction

\[
\begin{align*}
\dot{x} &= A x + B_1 w + B_2 u \\
z &= C_1 x + D_{11} w + D_{12} u \\
y &= C_2 x + D_{21} w + D_{22} u \\
\dot{q} &= A q + B u + K_p (y - C q) \\
u &= K_c q \\
\end{align*}
\]

\[
\Rightarrow \begin{cases}
\dot{q} = Kq + Ly \\
u = Mq + Ny
\end{cases}
\]
• For zero exogenous input $w = 0$, the input–output transfer matrix is $y(s) = P_{22}(s)u(s)$.

Whereas the closed loop system is described by:

$$z = \left\{ P_{11} + P_{12}K(I-P_{22}K)^{-1}P_{21} \right\}w = T_{zw}(s)w$$

☐ Some Conclusions first...

• **Definition (Mackenroth, p163):** The feedback system is internally stable if the closed loop matrix $A_c$ is Hurwitz.

\[
A_c = \begin{bmatrix}
A & B_2 M \\
0 & K \\
\end{bmatrix}
+ \begin{bmatrix}
B_2 N \\
L \\
\end{bmatrix}(I - D_{22}N)^{-1} \begin{bmatrix}
C_2 & D_{22} M \\
\end{bmatrix}
\]

• **Note:** If the system is internally stable, the closed loop transfer matrix $T_{zw}(s)$ is stable
Theorem 6.5.1. (M.P163):
1. An admissible controller exists, if and only if \((A, B_2)\) is stabilizable and \((C_2, A)\) detectable.

2. Suppose \((A, B_2)\) is stabilizable and \((C_2, A)\) detectable, let \(F\) and \(L\) such that \(A+B_2F, A+LC_2\) stable, Then an admissible controller \(K(s)\) has the form:

\[
K(s) = \begin{bmatrix}
A + B_2F + LC_2 + LD_22F & -L \\
F & 0
\end{bmatrix}
\]

Lemma 6.5.2. (Mackenroth, p164): Let \(P(s)\) and \(K(s)\) stabilizable and detectable realizations, then the following holds:

1. The following matrix has full column rank for all \(\lambda\) with \(\Re\lambda \geq 0\)

\[
\begin{bmatrix}
A - \lambda I & B_2 \\
C_2 & D_{21}
\end{bmatrix}
\]

Constraint for controllability (stabilizability)

2. The following matrix has full row column rank for all \(\lambda\) with \(\Re\lambda \geq 0\)

\[
\begin{bmatrix}
A - \lambda I & B_2 \\
C_1 & D_{12}
\end{bmatrix}
\]

Constraint for Observability (detectability)

3. The controller \(K(s)\) is an internally stabilizing controller if and only if

\[
T_{2u}(s) \in RH_\infty \\
\{P_{2u} + P_{2u}K(I-P_{2e}K)^{-1}P_{2e}\}=T_{2u}(s)
\]
Chapter 3: MIMO Tools: Stabilizing Controllers

• Sketch of Proof of Theorem 6.5.1.

\[
\begin{align*}
\dot{x} &= Ax + B_1w + B_2u \\
z &= C_1x + D_{11}w + D_{12}u \\
y &= C_2x + D_{21}w + D_{22}u \\
&\iff \begin{bmatrix} z \\ y \end{bmatrix} = P(s) \begin{bmatrix} w \\ u \end{bmatrix} \\
&\iff \begin{bmatrix} q \\ u \end{bmatrix} = Kq + Ly \\
&\iff \begin{bmatrix} u \\ q \end{bmatrix} = Mq + Ny
\end{align*}
\]

• Separate dynamic and algebraic equations

\[
\begin{align*}
\begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w \\
\begin{bmatrix} I & -N \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} I + NR^{-1}D_{22} & NR^{-1} \\ R^{-1}D_{22} & R^{-1} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} &= \begin{bmatrix} 0 & M \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} w \\
\end{align*}
\]

• Recall Well-Posedness Requirement

\[
|R| = \det(I - D_{22}N) \neq 0
\]

\[
\begin{bmatrix} I & -N \\ -D_{22} & I \end{bmatrix}^{-1} = \begin{bmatrix} I + NR^{-1}D_{22} & NR^{-1} \\ R^{-1}D_{22} & R^{-1} \end{bmatrix} \hspace{1cm} \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} NR^{-1}C_2 & M(I + NR^{-1}D_{22}) \\ R^{-1}C_2 & R^{-1}D_{22}M \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} + \cdots
\]

\[
\begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} + \begin{bmatrix} B_2NR^{-1}C_2 & B_2M(I + NR^{-1}D_{22}) \\ LR^{-1}C_2 & K + LR^{-1}D_{22}M \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + \begin{bmatrix} B_2NR^{-1}D_{21} \\ LD_{21} \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix}
\]
Chapter 3: MIMO Tools: Stabilizing Controllers

\[ A_C = \begin{bmatrix} A & 0 \\ 0 & K \end{bmatrix} + \begin{bmatrix} B_2NR^{-1}C_2 & B_2M(I + NR^{-1}D_{22}) \\ LR^{-1}C_2 & K + LR^{-1}D_{22}M \end{bmatrix} = \begin{bmatrix} A & B_2M \\ 0 & K \end{bmatrix} + \begin{bmatrix} B_2N \\ L \end{bmatrix} R^{-1} \begin{bmatrix} C_2 & D_{22}M \end{bmatrix} \]

\[ B_C = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + \begin{bmatrix} B_2NR^{-1}D_{21} \\ LR^{-1}D_{21} \end{bmatrix} \]

\[ C_C = \begin{bmatrix} C_1 + D_{12}NR^{-1}C_2 & D_{12}M + D_{12}NR^{-1}D_{22}M \end{bmatrix} \]

\[ D_C = D_{11} + D_{12}NR^{-1}D_{21} \]

\[ z(s) = T_{zw}(s)w(s) \Rightarrow T_{zw}(s) = \begin{bmatrix} A_C & B_C \\ C_C & D_C \end{bmatrix} \]

- Assume \((A, B_2)\) stabilizable and \((C_2, A)\) detectable. From pole assignment and/or LQR, we know that we can find two matrices \(F\) and \(L\) such that \(A+B_2F\), and \(A+L_2C_2\) are stable.

- Consider an observer-based compensator:

\[
\begin{align*}
\dot{x} &= Ax + B_2u \\
y &= C_2x + D_{22}u
\end{align*}
\]

\[
\begin{align*}
\dot{q} &= Kq + Ly \\
u &= Mq + Ny
\end{align*} \Leftrightarrow \begin{align*}
\dot{q} &= Aq + B_2u + L(C_2q + D_{22}u - y) \\
u &= Fq
\end{align*}
\]
Chapter 3: MIMO Tools: Stabilizing Controllers

• The compensator transfer matrix $K(s)$ can be found as:

$$\begin{align*}
\dot{q} &= Aq + B_2Fq + L(C_2q + D_{22}Fq) - Ly \\
u &= K(s)\begin{bmatrix} q \\ y \end{bmatrix}
\end{align*}$$

$$K(s) = \begin{bmatrix} A + B_2F + LC_2 + LD_{22}F & -L \\ F \end{bmatrix}$$

• Since $N = 0$, the feedback loop problem is well-posed. Thus we can set $D_{22} = 0$

• The closed loop system matrix becomes:

$$\begin{align*}
\dot{x} &= A x + B_2Fq \\
\dot{q} &= -LC_2q + A + B_2F + LC_2qq
\end{align*}$$

$$A_C\begin{bmatrix} x \\ q \end{bmatrix} = \begin{bmatrix} A + B_2F + LC_2 + LD_{22}F & -L \\ F \end{bmatrix}$$

• Using the nonsingular similarity transformation matrix:

$$T = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}, T^{-1} = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$$

$$TA_CT^{-1} = \begin{bmatrix} A + B_2F & B_2F \\ 0 & A + LC_2 \end{bmatrix}$$

Therefore the closed loop system is internally stable
Chapter 3: MIMO Tools: Stabilizing Controllers

Equivalence of 2 – Block formats

- Rewrite the previous results (note change in nomenclature)

\[
\begin{align*}
\dot{x} &= A_x x + B_1 w + B_2 u \\
z &= C_1 x + D_{11} w + D_{12} u \\
y &= C_2 x + D_{21} w + D_{22} u
\end{align*}
\]

\[
\begin{align*}
\dot{q} &= A_k q + B_k y \\
u &= C_k q + D_k y
\end{align*}
\]

\[
G(s) = \begin{bmatrix} G_{2w} & G_{2u} \\ G_{yw} & G_{yu} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}
\]

- Controller transfer matrix:

\[
K(s) = C_k (sI - A_k)^{-1} B_k + D_k
\]

- Closed loop system transfer matrix:

\[
\begin{bmatrix} G_{11} + G_{12} K(I - G_{22} K)^{-1} G_{21} \end{bmatrix} = T_{zw}(s)
\]

- The system is well posed if and only if \((I - G_{22} K)\) is invertible
Chapter 3: MIMO Tools: Stabilizing Controllers

- The augmented closed loop system is given by:

\[
\begin{bmatrix}
\dot{x} \\
q_k
\end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_k \end{bmatrix} \begin{bmatrix} x \\ q_k \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w + \begin{bmatrix} B_2 \\ 0 \end{bmatrix} u
\]

\[
\begin{bmatrix}
I \\
-D_{k2}
\end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} 0 & C_k \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ q_k \end{bmatrix} + \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} w
\]

- Substitute the known result above:

\[
\begin{bmatrix} I & -D_k \end{bmatrix}^{-1} = \begin{bmatrix} I + D_k R^{-1} D_{k2} & D_k R^{-1} \\ R^{-1} D_{k2} & R^{-1} \end{bmatrix} \quad R = I - D_{k2}D_k
\]

\[
T_{zw}(s) = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}
\]

\[
A_c = \begin{bmatrix} A & B_2 C_k \\ 0 & A_k \end{bmatrix} + \begin{bmatrix} B_2 D_k \\ B_k \end{bmatrix} R^{-1} \begin{bmatrix} C_2 & D_{k2} C_k \end{bmatrix} \\
B_c = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + \begin{bmatrix} B_2 D_k \\ B_k \end{bmatrix} R^{-1} D_{21}
\]

\[
C_c = \begin{bmatrix} C_1 & D_{12} C_k \end{bmatrix} + D_{12} D_k R^{-1} \begin{bmatrix} C_2 & D_{k2} C_k \end{bmatrix} \\
D_c = D_{11} + D_{12} D_k R^{-1} D_{21}
\]
Chapter 3: MIMO Tools: Stabilizing Controllers

- Signals \((u, y)\) on the left diagram are the same as signals \((v_1, v_2)\) on right diagram.

- From Figure 2 (irrespective of \(d_i\)), follows:

\[
\begin{bmatrix}
I & -D_k \\
-D_{22} & I
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} 0 & C_k \\
C_2 & 0
\end{bmatrix} \begin{bmatrix} x \\ q
\end{bmatrix}
\]

- Compute the closed loop transfer matrix from disturbance to output:

\[
\begin{bmatrix} v_1 \\ v_2
\end{bmatrix}
= T(s) \begin{bmatrix} d_1 \\ d_2
\end{bmatrix} = \begin{bmatrix} A_c & B_c
\\ C_c & D_c
\end{bmatrix} \begin{bmatrix} d_1 \\ d_2
\end{bmatrix}
\]

\[
A_c = \begin{bmatrix}
A & B_2C_k \\
0 & A_k
\end{bmatrix} + \begin{bmatrix}
B_1 & 0 \\
0 & B_k
\end{bmatrix} R^{-1} \begin{bmatrix}
C_2 & D_{22}C_k
\end{bmatrix}
\]

\[
B_c = \begin{bmatrix}
B_1 & 0 \\
0 & B_k
\end{bmatrix} D_c
\]

\[
C_c = D_c \begin{bmatrix}
0 & C_k
\\ C_2 & 0
\end{bmatrix}
\]

\[
D_c = D_{11} + D_{12} D_c R^{-1} D_{21}
\]
Lemma 6.5.3. (Mackenroth, p. 166): Since the two closed loop transfer matrices are the same, the system in Figure 1 is well-posed and internally stable, if and only if the system in Figure 2 has the same properties.

- The equivalence between: $T_{zw}^w(s) \Leftrightarrow \left( G_{22}(s), K(s) \right)$ implies

\[
\begin{bmatrix}
I & -K \\
-G_{22} & I
\end{bmatrix}
\begin{bmatrix}
v_1 \\ v_2
\end{bmatrix}
=
\begin{bmatrix}
d_1 \\ d_2
\end{bmatrix}
\begin{bmatrix}
I & -K \\
-G_{22} & I
\end{bmatrix}^{-1}
\begin{bmatrix}
(I - KG_{22})^{-1} & K (I - G_{22}K)^{-1} \\
G_{22} (I - KG_{22})^{-1} & (I - G_{22}K)^{-1}
\end{bmatrix}
\]

- Use the identities:

\[
\begin{cases}
G_{22} (I - KG_{22})^{-1} = (I - KG_{22})^{-1} G_{22} \\
(I - KG_{22})^{-1} = I + K (I - G_{22}K)^{-1} G_{22}
\end{cases}
\]

- Then we can conclude that the inverse of

\[
\begin{bmatrix}
I & -K \\
-G_{22} & I
\end{bmatrix}
\]

Exists if $(I - G_{22}^{-1})$ exists. But this holds if and only if:

$I - G_{22}^{-1} K \to = I - D_{22} D_k$
• So we can verify well posedness and internal stability using Theorem 3.2.1 from Mackenroth

\[ T(s) = \begin{bmatrix} I & -K \\ -G_{22} & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - KG_{22})^{-1} & K(I - G_{22}K)^{-1} \\ G_{22}(I - KG_{22})^{-1} & (I - G_{22}K)^{-1} \end{bmatrix} = \begin{bmatrix} S_i & KS_o \\ G_{22}S_i & S_o \end{bmatrix} \]
How does Frequency shaping extend to MIMO systems?
The main difference is that in MIMO systems we deal with transfer matrices. 
- In the computation of the closed loop system we need to satisfy matrix algebra.
- In the frequency response approach we need to use different measures of size. Since we deal with matrices of rational functions and not scalar functions, the only measure of size is a matrix induced norm. In particular, the spectral norm.

- In linear optimal control we typically use quadratic functionals as standard performance indexes. If we wish to regulate the system’s error with limited amount of control energy in a finite (or infinite) time interval, we can write:

\[
J = \int_{0}^{t_f} \left[ e^T \begin{bmatrix} e \mid 0 \end{bmatrix} \right] \begin{bmatrix} Z & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} e \\ 0 \end{bmatrix} dt = \int_{0}^{t_f} \begin{bmatrix} e \\ u \end{bmatrix}^T Q \begin{bmatrix} e \\ u \end{bmatrix} dt
\]

Equation (*) is actually the square of a **weighted \(L_2\)-norm on the signal \(z(t)\)** (**)

\[
J = \left\| z(t) \right\|_{2,Q,(0,t_f)}^2
\]
Chapter 3: MIMO Tools: Frequency Shaping

- In order to compute $J$, we either use (*), which requires the knowledge of the time histories of the entire appropriate variables, or we can use (**)

$$\sqrt{J} = \sqrt{\|z(t)\|_2^2} = \sqrt{\int_0^\infty \|z(t)\|^2 dt} = \begin{cases} \|G(s)\|_2 \\ \|G(s)\|_\infty \end{cases} \|u(s)\|_2$$

- If the input is the unit impulse $\delta(t)$
- If the input is a norm bounded signal $u(t)$

- There are cases, however, when it is required by the closed loop system to achieve maximum performance, or to maintain performance below a threshold defined by uncertainties. In this case, the $L_\infty$-norm or $H_\infty$ norm can be used.

  - In the study of robust stability we encounter the following inequality

  $$\frac{1}{l_m(\omega)} > \sigma_{\max} \left[ GK \left( I + GK \right)^{-1} \right] = \sigma_{\max} [T(j\omega)]$$

  - define: $\gamma^2 = \max_\omega \frac{1}{l_m(\omega)}$

  $$\frac{1}{l_m(\omega)} > \sigma_{\max} [T(j\omega)] \iff \begin{cases} \|T(s)\|_\infty < \gamma^2 \\ \frac{1}{\gamma^2} T(s) \|_\infty < 1 \Rightarrow \|T(s)\|_\infty < 1 \quad \text{for } \gamma = 1 \end{cases}$$
Frequency shaping requirements for MIMO systems

- Consider the standard unity feedback loop in the nominal case (no uncertainties).

\[ G(s)K(s) \neq K(s)G(s) \]

**Loop opening** is critical now dealing with transfer matrices, and it depends on the particular control requirements:

1. (plant input) the TFM is \( K(s)G(s) \)
   - high frequency error modeling;
   - high frequency disturbances;
   - sensor noise.

2. (plant output) the TFM is \( G(s)K(s) \)
   - neglected dynamics;
   - actuator errors;
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- Loop signal algebra is the same, however the signals are vector-valued functions:
  \[ y(s) = T(s)[r(s) - n(s)] + S(s)d_o(s) + G(s)S(s)d_i(s) \]
  \[ e(s) = S(s)[r(s) - n(s)] - S(s)d_o(s) - G(s)S(s)d_i(s) \]

- The loop transfer matrices of interest, with the loop broken at the output (if loop is broken at the input replace \( GK \) with \( KG \)), are:
  \[ S(s) = [I + G(s)K(s)]^{-1} \]
  \[ L(s) = G(s)K(s) \]
  \[ T(s) = S(s)L(s) = [I + G(s)K(s)]^{-1}G(s)K(s) \]
  \[ S_u(s) = K(s)S(s) = K(s)[I + G(s)K(s)]^{-1} \]

- The return difference matrix \([I + L(s)]\) still plays a fundamental role in the well-posedness and stability of the closed loop system, however the loop transfer matrix \( L(s) \) changes depending on the loop opening.
  - The matrix \([I + L(s)]\) must be invertible (see earlier results)
  - \( L(s) = G(s)K(s) \) or \( L(s) = K(s)G(s) \) depending on the loop opening
  - \( T(s) \) is the same no matter where the loop is opened since it is the closed loop description
• **Command Following**

\[ d_1(s) = d_0(s) = n(s) = 0 \]

\[ e(s) = S(s)r(s) \]

• Which implies:

\[ \|e(s)\|_2 = \|S(s)r(s)\|_2 \leq \|S(s)\|_\infty \cdot \|r(s)\|_2 \]

• The error norm is therefore bounded by the minimum and maximum singular value of the Sensitivity matrix

\[ \|e\|_{2\text{MIN}} = \sigma_{\text{min}}[S(j\omega)] \leq \|e\|_2 \leq \sigma_{\text{max}}[S(j\omega)] = \|e\|_{2\text{MAX}} \]

• To have high response accuracy we require a small error to a unit norm reference signal, thus:

\[ \sigma_{\text{max}}[S(j\omega)] \ll 1; \forall \|r(s)\|_2 = 1 \]
Chapter 3: MIMO Tools: Frequency Shaping

- Recall from singular value algebra:

\[
\sigma_{\max} [A^{-1}] = \frac{1}{\sigma_{\min} [A]} \\
\sigma_{\min} [A] - 1 \leq \sigma_{\min} [I + A] \leq \sigma_{\min} [A] + 1
\]

\[
\sigma_{\max} [S(j\omega)] \ll 1 \Rightarrow \sigma_{\max} [(I + GK)^{-1}] \ll 1 \Rightarrow \frac{1}{\sigma_{\min} [(I + GK)]} \ll 1 \Rightarrow \sigma_{\min} [(I + GK)] >> 1
\]

- Therefore for a good command following we must have in the appropriate frequency band (assuming \( p(\omega) \) some design constraint):

\[
\sigma_{\max} [S(j\omega)] \ll 1 \\
\sigma_{\min} [G(j\omega)K(j\omega)] \gg 1 \Rightarrow \sigma_{\min} [G(j\omega)K(j\omega)] \gg p(\omega)
\]
Chapter 3: MIMO Tools: Frequency Shaping

- **Disturbance Rejection:**
  \[
  \begin{align*}
    &\sigma_{\text{max}} \left[ S(j\omega) \right] \ll 1 \\
    &\sigma_{\text{min}} \left[ (G(j\omega)K(j\omega)) \right] \gg 1
  \end{align*}
  \]
• A similar result can be obtained using the complementary sensitivity matrix $T(s)$. Recall:

$$\sigma_i[T(s)] = \sigma_i[I - S(s)] \Rightarrow \sigma_{\max}[I - T(s)] \ll 1$$

• Define a positive and arbitrarily small design parameter $\delta$, we obtain:

$$1 - \delta \leq \sigma_{\min}[T(j\omega)] \leq \sigma_{\max}[T(j\omega)] \leq 1 + \delta$$
• **High Frequency Sensor Noise:**

\[
\begin{align*}
\sigma_{\text{max}} [T(j\omega)] & \ll 1 \\
\sigma_{\text{max}} [(G(j\omega)K(j\omega))] & \ll 1
\end{align*}
\]

• Equivalently, define a positive and arbitrarily small design parameter \(\gamma\), we obtain:

\[
1 - \gamma \leq \sigma_{\text{min}} [S(j\omega)] \leq \sigma_{\text{max}} [S(j\omega)] \leq 1 + \gamma, \gamma \to 0
\]
• **Summary:**

- **Loop Gain:**
  - \( \sigma_i(L) \)
  - \( \bar{\sigma}(L) \)
  - \( L = KG \)
  - Loop gain crossover frequency

- **Roll off plant for robustness to noise, high frequency unmodeled dynamics**

- **Need DC gain for command tracking**

- **Sensitivity:**
  - \( \bar{\sigma}(S) \)
  - \( S(s) = (I + L)^{-1} \)
  - Want errors small at low freq for command tracking + disturbance rejection

- **Stability:**

- **Roll off plant:**

- **Complementary Sensitivity:**
  - \( \bar{\sigma}(T) \)
  - \( T(s) = (I + L)^{-1} L \)
  - Roll off plant for robustness to noise, high freq unmodeled dynamics
Chapter 3: MIMO Tools: Frequency Shaping

- **Note on stochastic inputs:**

  - When the system is subjected to random inputs (external disturbances, component noise...) the performance index becomes a random variable, and its expected value can be used as measure of performance. For stationary random processes and stable closed loop systems operating over a large enough horizon ($t_f \to \infty$), we can write:

  $$J = \int_0^{t_f \to \infty} E[Z^TQz]dt = E[Z^TQz]$$

  - Using the properties of the trace operator:

    $$J = tr \left\{ E[Z^T(t)Qz(t)] \right\} = tr \left\{ QE[Z(t)Z^T(t)] \right\} = tr \left\{ Q\Sigma_z \right\}$$

  - The output correlation matrix $\Sigma_z$ can be computed from the input correlation function $w(t)$:

    $$\Sigma_z = \int_0^\infty \int_0^\infty g_{cl}(\tau_1)R_w(\tau_1 - \tau_2)g_{cl}^T(\tau_2)d\tau_1 d\tau_2; \quad R_w(\tau) = E[Z(\tau_1)Z^T(\tau_2)]$$
Chapter 3: MIMO Tools: Frequency Shaping

- For a white noise stochastic process input, with spectral density equal to $S_w$, we also have:

$$J = tr \left\{ Q \right\} \int_{0}^{\infty} g_{cl}(\tau) S_w g_{cl}^T(\tau) d\tau$$

- Then the mean square of some output $y = Cx$ is:

$$\begin{align*}
E[y^T(t)y(t)] &= tr \left[ \Sigma_y \right] \\
\Sigma_y &= S_w \int_{0}^{\infty} g(t)g^T(t)dt \\
\sqrt{y^T(t)y(t)} &= \|G\|_2 \sqrt{S_w}
\end{align*}$$

- Using the relationship between “output” vector and state vector $y(t) = Cx(t)$ we also have:

$$J = tr \left\{ Q C \Sigma_x C^T \right\}$$

$$A_{cl} \Sigma_x + \Sigma_x A_{cl}^T + B_{cl} S_w B_{cl}^T = 0$$

and $A_{cl} < 0$

- **Conclusions:** Signals can be expressed with the use of the 2-norm, or $\mathcal{H}_2$ norm, in terms of their average value (energy or power spectrum).
Chapter 3: MIMO Tools: Extras

In mathematical analysis, a **Cauchy sequence**, named after Augustin Cauchy, is a sequence whose elements become close as the sequence progresses. To be more precise, by dropping a finite number of elements from the start of the sequence we can make the distance between any two remaining elements arbitrarily small.

Cauchy sequences require the notion of distance so they can only be defined in a metric space. Generalizations to more abstract uniform spaces exist in the form of Cauchy filter and Cauchy net.

They are of interest because in a complete space, all such sequences converge to a limit, and one can test for "Cauchiness" without knowing the value of the limit (if it exists), in contrast to the definition of convergence.

**Complete space**

From Wikipedia, the free encyclopedia.

*For Cauchy completion in category theory, see Karoubi envelope.*

In mathematical analysis, a metric space \( M \) is said to be complete (or Cauchy) if every Cauchy sequence of points in \( M \) has a limit that is also in \( M \).

Intuitively, a space is complete if it "doesn't have any holes", if there aren't any "points missing". For instance, the rational numbers are not complete, because \( \sqrt{2} \) is "missing" even though you can construct a Cauchy sequence of rational numbers that converge to it. (See the examples below.) It is always possible to "fill all the holes", leading to the completion of a given space, as will be explained below.

**Inner product space**

From Wikipedia, the free encyclopedia.

*For the scalar product or dot product of spatial vectors see dot product*

In mathematics, an **inner product space** is a vector space with additional structure, an inner product (also called scalar product or dot product), which allows us to introduce geometrical notions such as angles and lengths of vectors. Inner product spaces generalize Euclidean spaces (with the dot product as the inner product) and are studied in functional analysis.

An inner product space is sometimes also called a **pre-Hilbert space**, since its completion with respect to the metric induced by its inner product is a Hilbert space.
Chapter 3: MIMO Tools: Extras

Cauchy sequence

A sequence \( x_0, x_1, x_2, \ldots \) in a metric space \((X, d)\) is a Cauchy sequence if, for every real number \( \epsilon > 0 \), there exists a natural number \( N \) such that \( d(x_n, x_m) < \epsilon \) whenever \( n, m > N \).

Banach space

A Banach space \((X, \| \cdot \|)\) is a normed vector space such that \( X \) is complete under the metric induced by the norm \( \| \cdot \| \).

Some authors use the term Banach space only in the case where \( X \) is infinite-dimensional, although on Planetmath finite-dimensional spaces are also considered to be Banach spaces.

If \( Y \) is a Banach space and \( X \) is any normed vector space, then the set of continuous linear maps \( f : X \rightarrow Y \) forms a Banach space, with norm given by the operator norm. In particular, since \( \mathbb{R} \) and \( \mathbb{C} \) are complete, the continuous linear functionals on a normed vector space \( B \) form a Banach space, known as the dual space of \( B \).

Examples:

- Finite-dimensional normed vector spaces
- \( L^p \) spaces are by far the most common example of Banach spaces.
- \( L^p \) spaces are \( L^p \) spaces for the counting measure on \( \mathbb{N} \)
- Continuous functions on a compact set under the supremum norm
- Finite signed measures on a \( \sigma \)-algebra
Chapter 3: MIMO Tools: Extras

A real-valued function \( f \) defined on the reals \( \mathbb{R} \) is called Lebesgue integrable if there exists a sequence of step functions \( \{f_n\} \) such that the following two conditions are satisfied:

\[
\sum_{n=1}^{\infty} |f_n| < \infty \quad f(x) = \sum_{n=1}^{\infty} f_n(x)
\]

for every \( x \in \mathbb{R} \) such that

\[
\sum_{n=1}^{\infty} |f_n| < \infty
\]

Definition 1:
A real-valued function \( f \) defined on the reals \( \mathbb{R} \) is called Lebesgue integrable if there exists a sequence of step functions \( \{f_n\} \) such that the following two conditions are satisfied:

Definition 2:
A function \( f : X \rightarrow \mathbb{R} \) is measurable if, for every real number \( a \), the set \( f^{-1}(a) \) is measurable. When \( f : X \rightarrow \mathbb{R} \) with Lebesgue measure, or more generally any Borel measure, then all continuous functions are measurable. In fact, practically any function that can be described is measurable. Measurable functions are closed under addition and multiplication, but not composition.
Chapter 3: MIMO Tools: Extras

Rule 1. Speed of response to reject disturbances. We require $\omega_c > \omega_d$. More specifically, $|S(j\omega)| \leq |1/G_d(j\omega)|_{\forall \omega}$.

Rule 2. Speed of response to track reference changes. We require $|S(j\omega)| \leq 1/R$ up to the frequency $\omega_r$ where tracking is required.

Rule 3. Input constraints arising from disturbances. For acceptable control ($|e| < 1$) we require $|G(j\omega)| > |G_d(j\omega)| - 1$ at frequencies where $|G_d(j\omega)| > 1$.

Rule 4. Input constraints arising from setpoints. We require $|G(j\omega)| > R - 1$ up to the frequency $\omega_r$ where tracking is required. (See (3.41)).

Rule 5. Time delay $\theta$ in $G(s)G_m(s)$. We approximately require $\omega_c < 1/\theta$. (See (3.33)).

Rule 6. Tight control at low frequencies with a RHP-zero $z$ in $G(s)G_m(s)$. For a real RHP-zero we require $\omega_c < z/2$. (See (3.35)).

Rule 7. Phase lag constraint. We require in most practical cases (e.g. with PID control): $\omega_u < \omega_u$. Here the ultimate frequency $\omega_u$ is where $\angle GG_m(j\omega_u) = -180^\circ$.

Rule 8. Real open-loop unstable pole in $G(s)$ at $s = p$. We need high feedback gains to stabilize the system and require $\omega_c > 2p$. In addition, for unstable plants we need $|G| > |G_d|$ up to the frequency $p$ (which may be larger than $\omega_d$ where $|G_d| = 1$). Otherwise, the input may saturate when there are disturbances, and the plant cannot be stabilized.
Chapter 3: MIMO Tools: Extras