

☐ **Newtonian Mechanics:**

- Main assumptions/definitions
- Reference Systems, Rigid Body Motion

☐ **Atmospheric Flight Dynamics:**

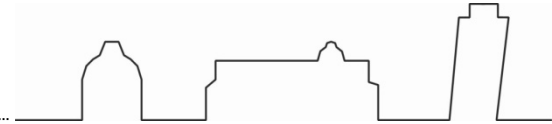
- Reference systems
- A/C Nonlinear Equations of Motion
- Applied forces and moments
- Geometry, Flight Envelope, Equilibrium Conditions
- Linearization, Natural Modes, Stability Derivatives

☐ **A/C Control** Sensors, Actuators, Primary Control Loops

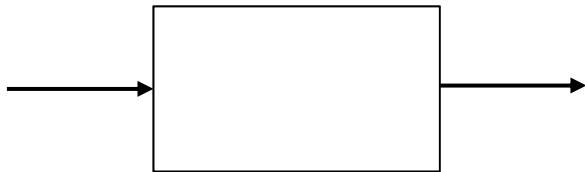
- Stability Augmentation, Autopilots
- Basic Human Operator in - the - Loop
- Operator in the Control Loop, Classical Models
- MMI, Haptics

☐ **Orbital Mechanics:**

- Law of Gravitation, 2-Body Problem
- Keplerian Orbits, Kepler Equation
- Orbit Transfer, Patched Conics
- Lambert theorem, Restricted 3-Body Problem



- ❑ **Design by Modeling:** Mathematics and Computer Science often use the concept of 'box' (white, grey, black) to indicate the level of knowledge of the process being modeled.
- **Preliminary design of GNC systems relies on simplified models**

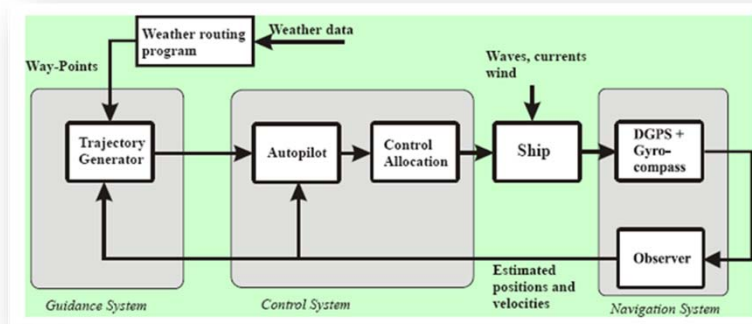


Physical Principles provide complete mathematical description of the behavior of the system.



Physical Principles provide a mathematical description of the behavior of the system, however some parameters may be unknown and/or too complex to model, thus requiring experimentation or identification.

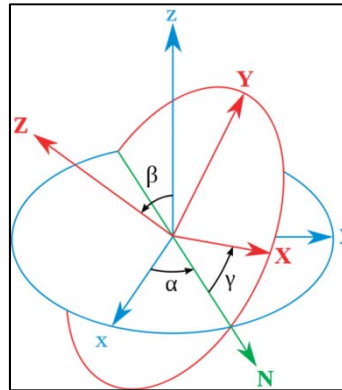
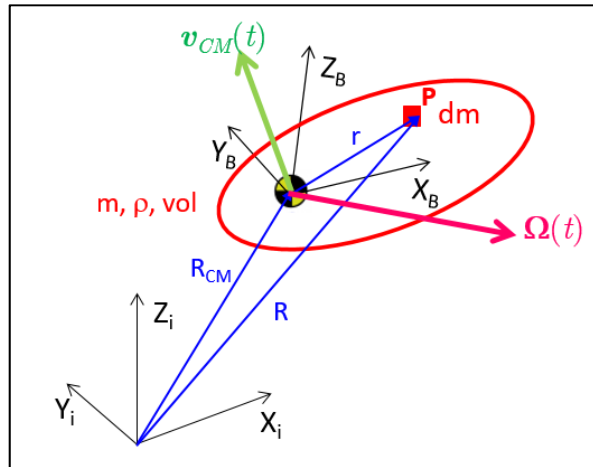
□ General multiple loop structure:



- **Stability Augmentation:** Inner loop, **high bandwidth continuous operation**. Traditional controllers may be necessary for guaranteeing adequate stability, change of stability properties, and to provide satisfactory disturbance rejection
- **Autopilots:** **lower bandwidth inner loop**. These are control systems that are responsible for achieving satisfactory behavior of specific state variables or degrees of freedom. Examples are constant speed, constant altitude (depth), specified angular velocity, etc. Autopilots constitute an integral part of guidance as well stability augmentation, depending on the application, and frequency content.
- **Guidance Controller:** A guidance law answers the following main question: **How do I go from point A to point B?** It is a lower bandwidth loop closed over the augmentation and the autopilot loops. The guidance controller may incorporate a continuous feed of desired trajectory (path planner), especially if the motion requires careful obstacle avoidance, or the possibility of dynamic interference with other vehicles nearby (traffic).
- **Navigation System:** The controller responsible for the navigation of a vehicle is usually the outermost loop of the global control system. It answers the basic questions: **Where am I? Where am I trying to go?** Navigation is necessary to provide accurate information on the vehicle current position.

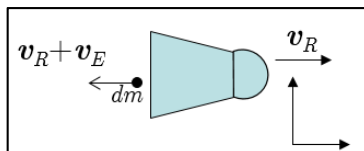
Equations of motion based on Newtonian Mechanics:

$$\begin{bmatrix} T(\Phi, \Theta, \Psi) \end{bmatrix}_{\text{INERTIAL}}^{\text{BODY}} = \begin{bmatrix} c\Theta c\Psi & c\Theta s\Psi & -s\Theta \\ -c\Phi s\Psi + s\Phi s\Theta c\Psi & c\Phi c\Psi + s\Phi s\Theta s\Psi & s\Phi c\Theta \\ s\Phi s\Psi + c\Phi s\Theta c\Psi & -s\Phi c\Psi + c\Phi s\Theta s\Psi & c\Phi c\Theta \end{bmatrix} \begin{cases} P = \dot{\Phi} - \dot{\Psi} \sin \Theta \\ Q = \dot{\Theta} \cos \Phi + \dot{\Psi} \sin \Phi \cos \Theta \\ R = \dot{\Psi} \cos \Phi \cos \Theta - \dot{\Theta} \sin \Phi \end{cases}$$



$$\begin{cases} \mathbf{F} = m\dot{\mathbf{v}} = m\dot{\mathbf{v}} = m\dot{\mathbf{v}}_{CM} + m(\boldsymbol{\Omega} \times \mathbf{v}_{CM}) \\ \mathbf{M} = I\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times (I\boldsymbol{\Omega}) = I\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times (I\boldsymbol{\Omega}) \end{cases}$$

$$\begin{cases} m \begin{bmatrix} \dot{u} - vr + wq \\ \dot{v} - wp + ur \\ \dot{w} - uq + vp \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \\ \begin{bmatrix} I_{xx} \dot{p} + (I_{zz} - I_{yy})qr - (\dot{r} + pq)I_{xz} + (r^2 - q^2)I_{yz} + (pr - \dot{q})I_{xy} \\ I_{yy} \dot{q} + (I_{xx} - I_{zz})rp - (\dot{p} + qr)I_{yx} + (p^2 - r^2)I_{zx} + (qp - \dot{r})I_{yz} \\ I_{zz} \dot{r} + (I_{yy} - I_{xx})pq - (\dot{q} + rp)I_{zy} + (q^2 - p^2)I_{xy} + (rq - \dot{p})I_{zx} \end{bmatrix} = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} \end{cases}$$

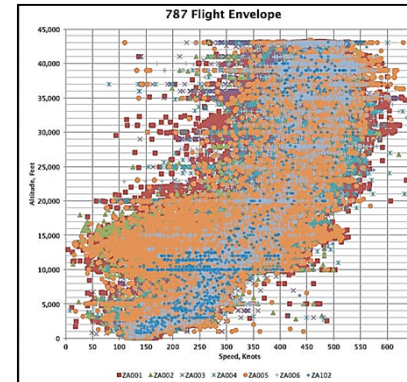
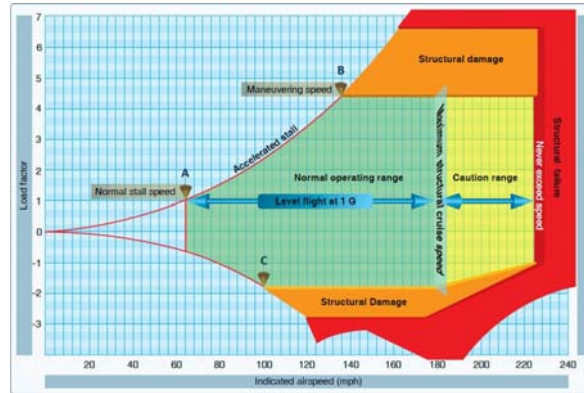


$$\frac{d\mathbf{p}_{SYS}}{dt} = -\frac{dm}{dt}(\mathbf{v}_E + \mathbf{v}_R)$$

$$\Delta \mathbf{v} = I_{SP} g_0 \ln \frac{m_R^I}{m_R^F}$$

Summary

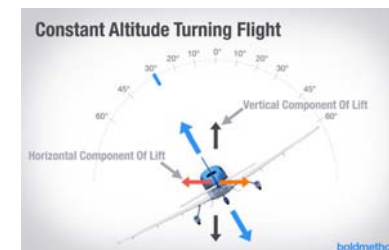
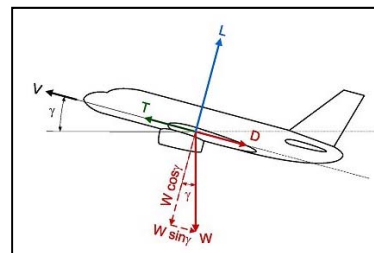
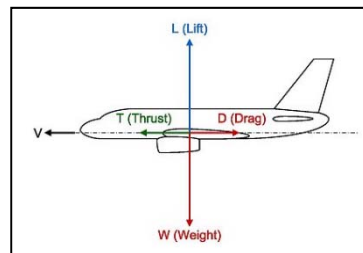
- **Flight Envelope:** range of static and dynamic conditions within which an aircraft operates



- **Main Assumptions:**

- Flat Earth Approximation
- Air Density as exponential model
- Engine dynamics separate from aircraft body motion
- Still Atmosphere

- **Trim Conditions:** Phases of motion that can be treated as reference in terms of equilibrium and levels of acceleration generated





- ❑ **Reference Systems:** Aerospace systems use specific sets of inertial and non inertial references, depending on the application, as well as for improved mathematical tractability.

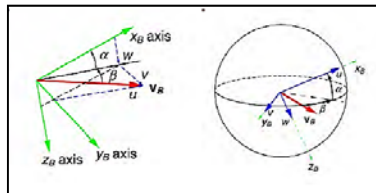
- **Generic body-fixed axes reference system** $[\cdot]_B$: Origin at center of mass of the aircraft, x-direction positive out the nose, on the plane of symmetry, z-direction positive down, on the plane of symmetry, y-direction positive to the right.

Φ = Roll (Rollio) angle

Θ = Pitch (Beccheggio) angle

Ψ = Yaw (Imbardata) angle

- **Wind and Velocity axes** refer to the velocity direction of the aircraft. They differ in that the former is relative to the **true airspeed** (inclusive of wind), whereas the latter refer to the **ground speed** (atmosphere at rest).



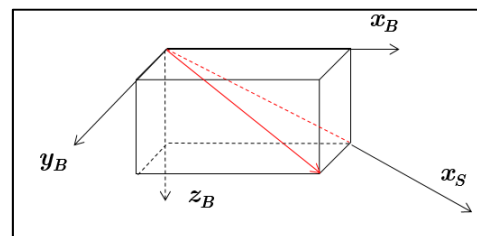
χ = Heading angle (type 3 rotation)

γ = Flight path angle (type 2 rotation)

μ = Bank angle about the velocity vector (type 1 rotation)

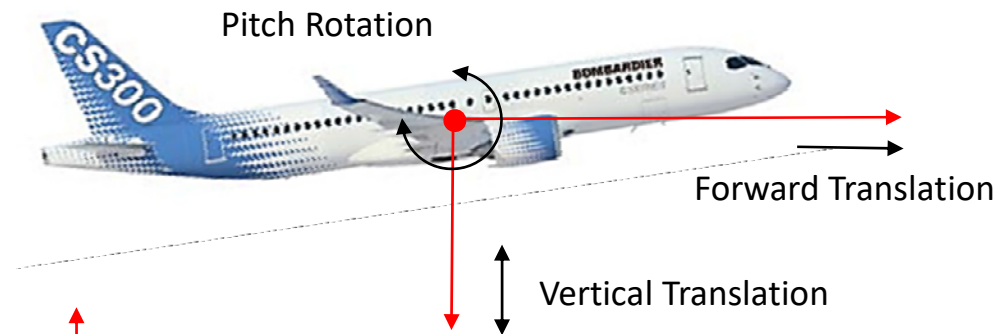
$$\text{Angle of Attack } \alpha = \tan^{-1} \left(\frac{w}{u} \right) \quad \text{Sideslip angle } \beta = \sin^{-1} \left(\frac{v}{|V|} \right)$$

- **Stability Axes:** Body reference system where the side component of the velocity vector is zero (no sideslip), and the x axis is aligned with the forward component of the velocity vector

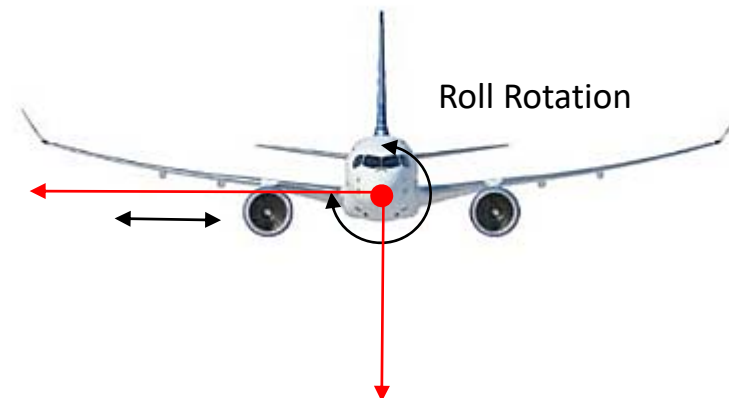
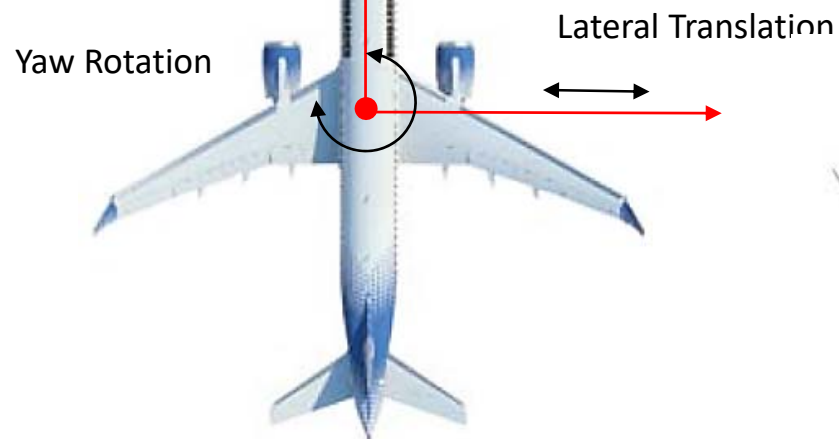


❑ Preferred Planes of Motion

▪ 3 DOF Longitudinal Motion



▪ 3 DOF Lateral - Directional Motion



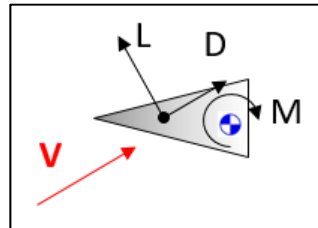
External Forces and Moments:

1. Forces and Moments due to interaction with the fluid (Aerodynamics)
2. Forces and Moments due to propulsion
3. Forces and Moments due to Gravity
4. Forces and Moments due to external disturbances (wind, turbulence, etc.)

$$F = \begin{bmatrix} F_{xB} \\ F_{yB} \\ F_{zB} \end{bmatrix} \begin{array}{l} \text{Longitudinal Force} \\ \text{Side Force} \\ \text{Normal Force} \end{array}$$

$$M = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} \begin{array}{l} \text{Rolling Moment} \\ \text{Pitching Moment} \\ \text{Yawing Moment} \end{array}$$

$$F = \begin{bmatrix} -D \\ Y \\ -L \end{bmatrix} \begin{array}{l} \text{Drag} \\ \text{Side Force (equal to zero in stability axes)} \\ \text{Lift} \end{array}$$

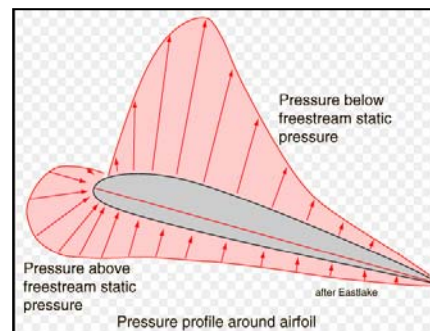


$$F_i = \frac{1}{2} \rho S V^2 C_i = \left[\frac{kg}{m^3} \right] \cdot [m^2] \cdot \left[\frac{m}{sec^2} \right] = Q S C_i$$

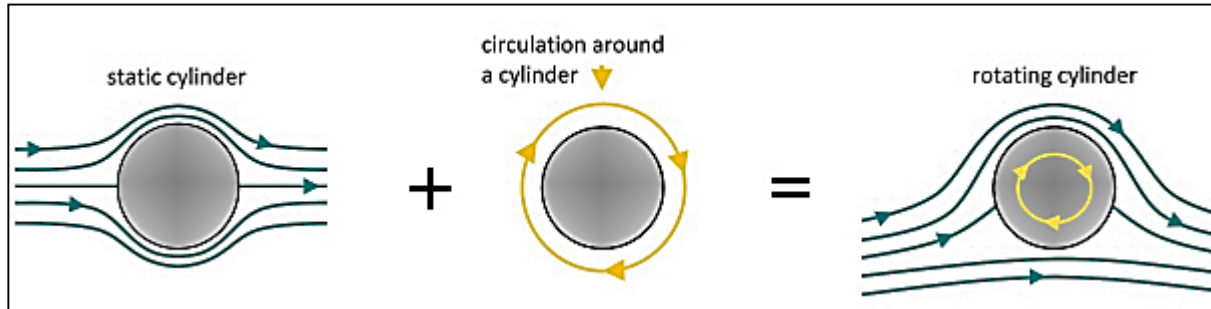
$$M_i = \frac{1}{2} \rho S V^2 C_i b = \left[\frac{kg}{m^3} \right] \cdot [m^2] \cdot \left[\frac{m}{sec^2} \right] \cdot [m] = Q S b C_i$$

LIFT

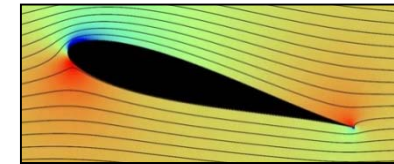
- Bernoulli's principle for incompressible (constant density), inviscid (no dissipation due to friction), irrotational flow:



Summary



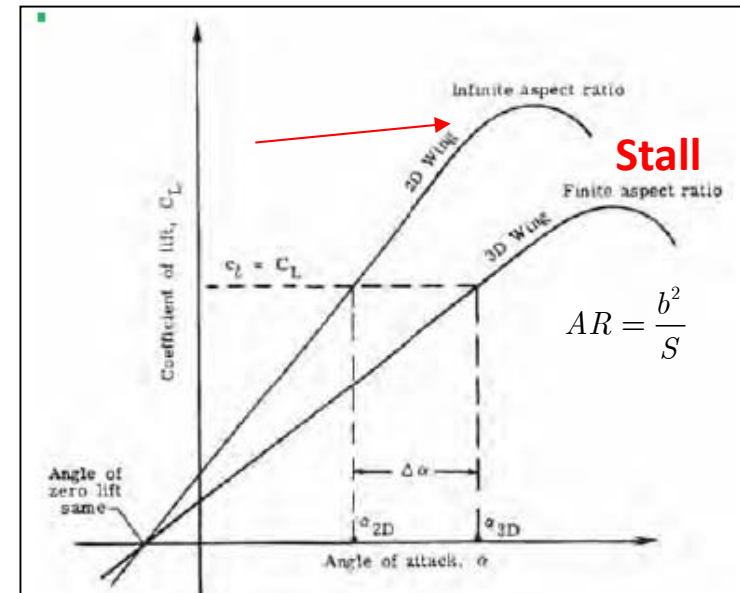
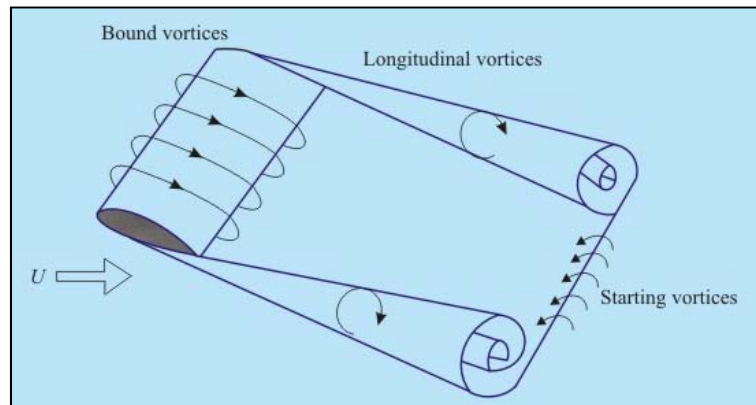
Kutta –Joukowski Condition



$$L = \frac{1}{2} \rho S V^2 C_L = Q S C_L$$

$$C_L = C_L(\text{Re}, M, \alpha)$$

$$C_L = C_{L0} + C_{L\alpha} \cdot \alpha + \dots$$



Summary

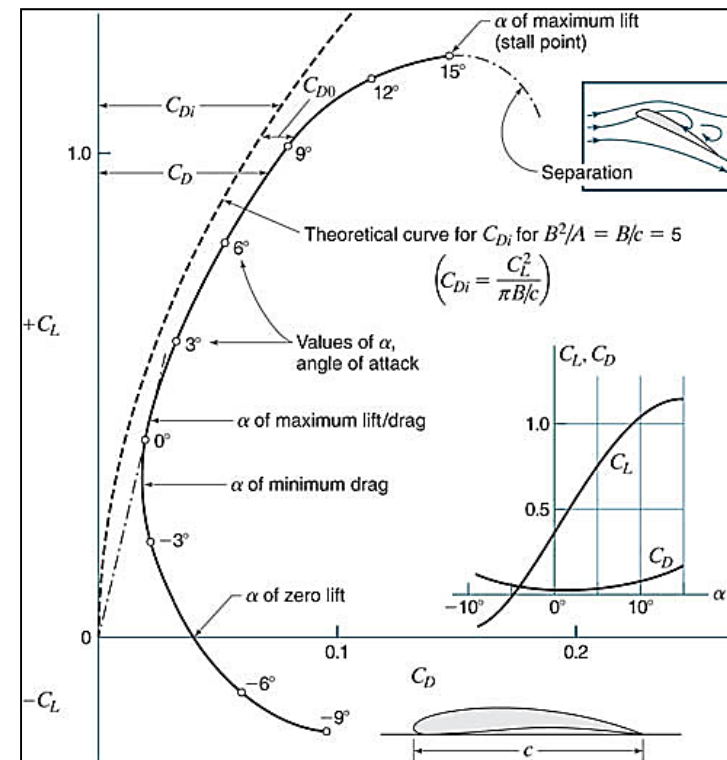
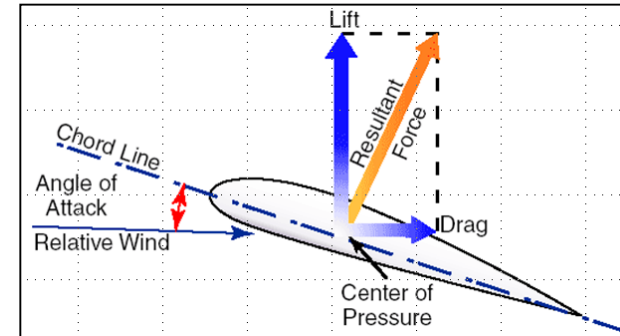
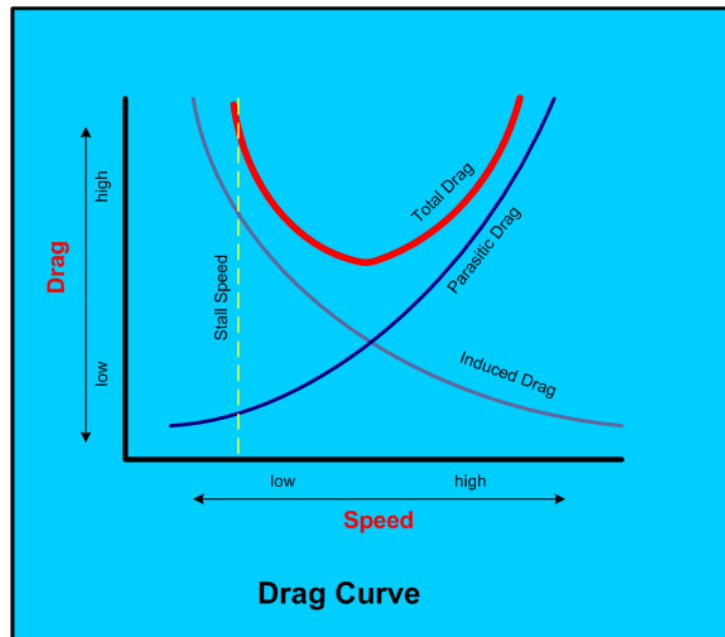
- ❑ **DRAG:** Main force opposing motion. In aerospace dynamics, drag is divided into several components

$$D = \frac{1}{2} \rho S V^2 C_D$$

$$C_D = C_D(S, Re, M, \alpha)$$

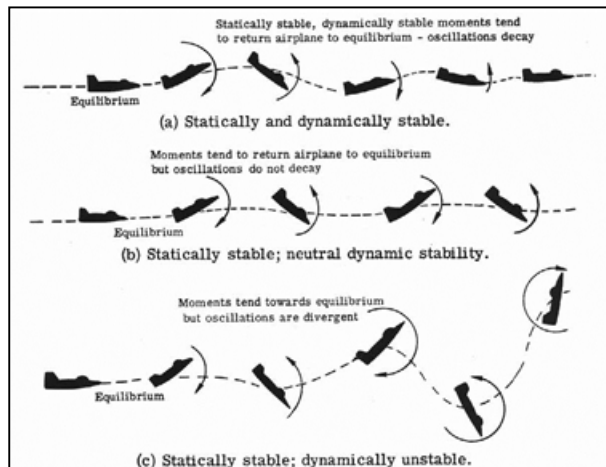
$$C_D = C_{D0} + K C_L^2 \approx C_{D0} + K [C_{L0} + C_{L\alpha} \cdot \alpha]^2$$

$$K \approx K_0 + \frac{1}{\pi A Re}$$



Summary

- ❑ **PITCHING MOMENT:** Primary moment in the longitudinal dynamic motion of an Aircraft (Negative nose down). The pitching moment is strictly connected with the longitudinal stability characteristics (static and dynamic) of a flying vehicle

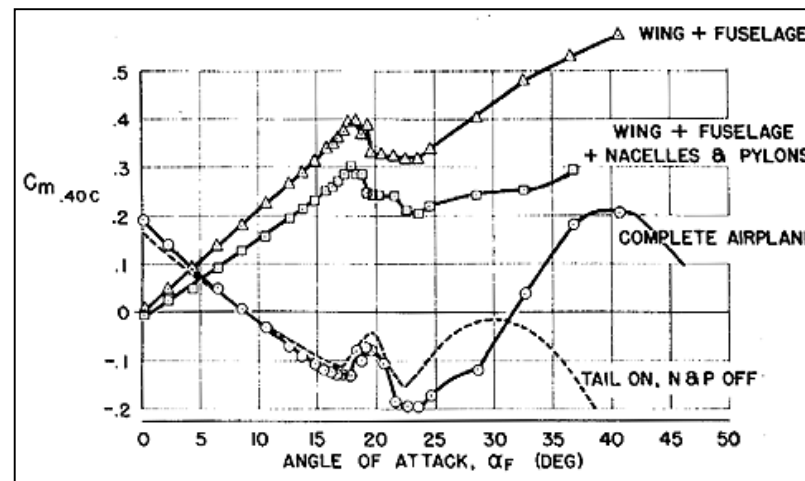


$$C_m = C_{m_0} + C_{m\alpha} \cdot \alpha$$

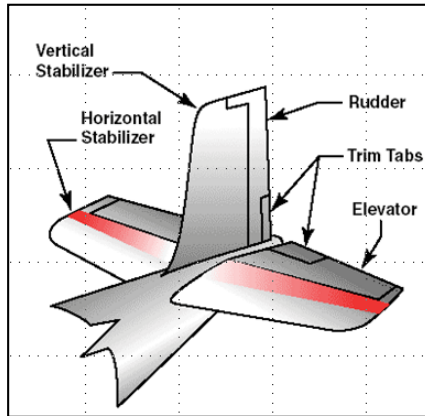
$$\begin{cases} C_{m_0} > 0 \\ C_{m\alpha} < 0 \end{cases}$$

$$C_{m\alpha} = C_{L\alpha} \left(\frac{X_{CG}}{\bar{c}} - \frac{X_{AC}}{\bar{c}} \right)$$

- **Longitudinal equilibrium (Trim Point on the XZ plane):** steady flight with no net moment about the center of mass
- **Aerodynamic Center:** point of application of aerodynamic lift and drag where the total pitching moment is independent of the angle of attack, usually constant at 25% of the mean chord
- **Influence of Horizontal Tail:** Horizontal plane, Elevator
- **Neutral Point:** location of center of mass at which the total pitching moment is zero
- **Static Margin:** distance between center of mass and neutral point



- The ELEVATOR is the main component for longitudinal control.



$$C_L = C_{L0} + C_{L\alpha} \alpha + C_{L\delta E} \delta_E + \dots$$

$$C_m = C_{m0} + C_{m\alpha} \alpha + C_{m\delta E} \delta_E + \dots$$

□ Lateral – Directional Static Stability:

$$Y = \frac{1}{2} \rho S V^2 C_y$$

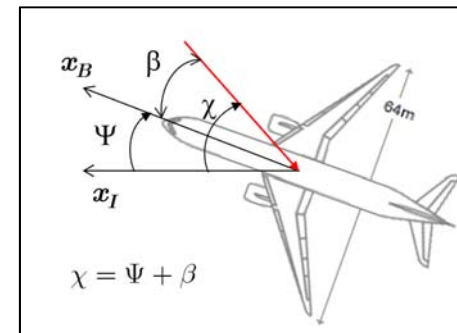
$$l = \frac{1}{2} \rho S b V^2 C_l$$

$$n = \frac{1}{2} \rho S b V^2 C_n$$

$$C_y = C_y(M, \beta, \delta_A, \delta_R, \dots)$$

$$C_l = C_l(M, \beta, \delta_A, \delta_R, \dots)$$

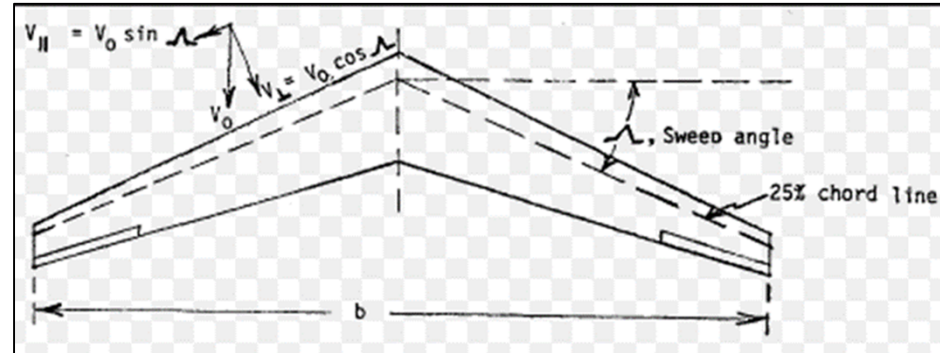
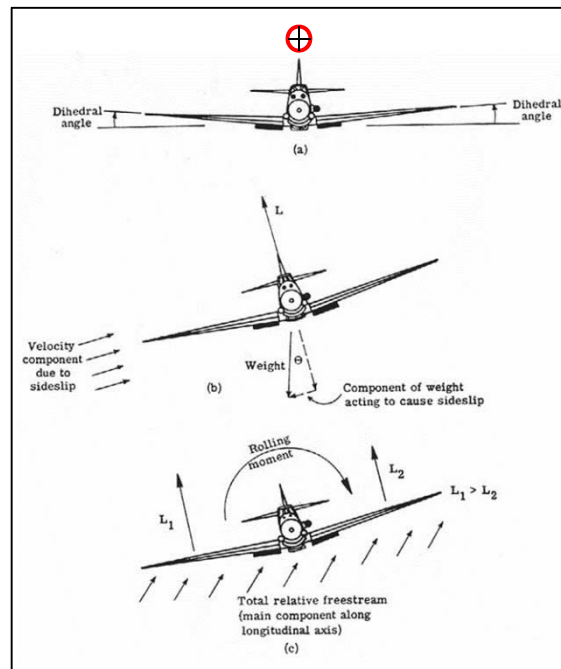
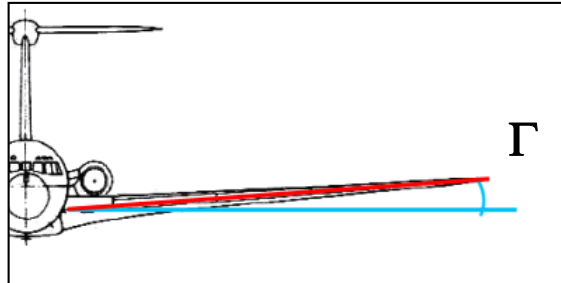
$$C_n = C_n(M, \beta, \delta_A, \delta_R, \dots)$$



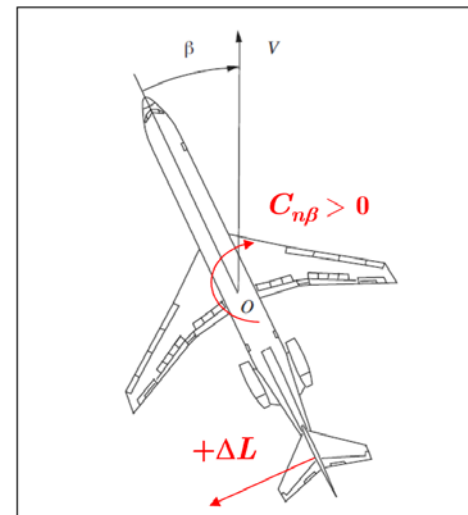
- Yawing Moment and Rolling Moment static stability provide guidelines for the sizing of the Vertical Tail, sizing of the Ailerons, the Dihedral/angle location of the wings, and their Sweep

Summary

- $C_{l\beta} < 0$ can be achieved by a swept back wing and/or positive Dihedral Γ



- A restoring yawing moment is generated if $C_{n\beta} > 0$ (Positive weathercock stability)



Linearization of the Equations of Motion

x_1	u	First six elements of the state are longitudinal variables	x_1	$\begin{bmatrix} \mathbf{X}_{Lon} \\ \mathbf{X}_{Lat-Dir} \end{bmatrix}$	u
x_2	v		x_2		w
x_3	w		x_3		x
x_4	x		x_4		z
x_5	y		x_5		q
x_6	z		x_6		θ
x_7	p	Second six elements of the state are lateral-directional variables	x_7	$\begin{bmatrix} \mathbf{X}_{Lon} \\ \mathbf{X}_{Lat-Dir} \end{bmatrix}$	v
x_8	q		x_8		y
x_9	r		x_9		p
x_{10}	ϕ		x_{10}		r
x_{11}	θ		x_{11}		ϕ
x_{12}	ψ		x_{12}		ψ

$$\mathbf{x}(t) = \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix}; \mathbf{u}(t) = \begin{bmatrix} \delta_E \\ \delta_T \end{bmatrix} \quad \alpha = w/U_0$$

$$\mathbf{x}(t) = \begin{bmatrix} v \\ p \\ \phi \\ r \end{bmatrix}; \mathbf{u}(t) = \begin{bmatrix} \delta_A \\ \delta_R \end{bmatrix} \quad \beta = v/U_0$$

$$\begin{aligned} \left[\frac{d}{dt} - X_u \right] u + g_0 \cos \Theta_0 \theta - X_w w &= X_{\delta_e} \delta_e + X_{\delta_T} \delta_T \\ -Z_u u + \left[(1 - Z_{\dot{w}}) \frac{d}{dt} - Z_w \right] w - [u_0 + Z_q] q + g_0 \sin \Theta_0 \theta &= Z_{\delta_e} \delta_e + Z_{\delta_T} \delta_T \\ -M_u u - \left[M_{\dot{w}} \frac{d}{dt} + M_w \right] w + \left[\frac{d}{dt} - M_q \right] q &= M_{\delta_e} \delta_e + M_{\delta_T} \delta_T \end{aligned}$$

$$\dot{\theta} = q$$

$$\begin{aligned} \left[\frac{d}{dt} - Y_v \right] v - Y_p p + [u_0 - Y_r] r - g_0 \cos \Theta_0 \phi &= Y_{\delta_r} \delta_r \\ -L_v v + \left[\frac{d}{dt} - L_p \right] p - \left[\frac{I_{xz}}{I_x} \frac{d}{dt} + L_r \right] r &= L_{\delta_r} \delta_r + L_{\delta_a} \delta_a \\ -N_v v - \left[\frac{I_{xz}}{I_z} \frac{d}{dt} + N_p \right] p + \left[\frac{d}{dt} - N_r \right] r &= N_{\delta_r} \delta_r + N_{\delta_a} \delta_a \end{aligned}$$

$$\begin{cases} \dot{\phi} = p + r \cos \theta_0 \\ \dot{\psi} = \frac{r}{\cos \theta_0} \end{cases}$$

Summary



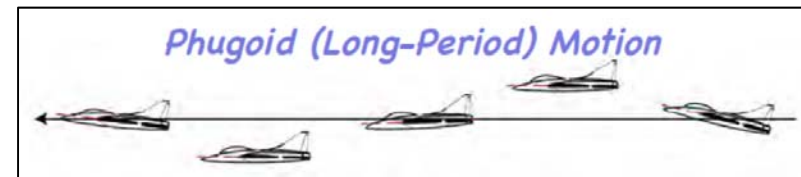
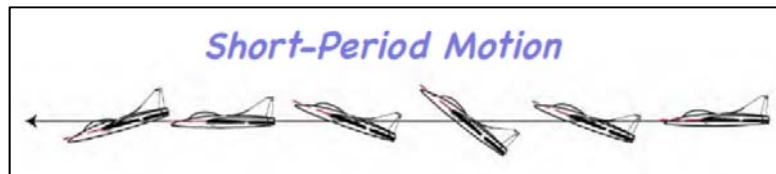
X_u	$\frac{1}{m} \frac{\partial X}{\partial u}$	$\frac{1}{\text{sec}}$	$\frac{\rho S U}{m} (-C_D - C_{D_u})^a$
X_w	$\frac{1}{m} \frac{\partial X}{\partial w}$	$\frac{1}{\text{sec}}$	$\frac{\rho S U}{2m} (C_L - C_{D_\alpha})$
X_δ	$\frac{1}{m} \frac{\partial X}{\partial \delta}$	$\frac{\text{ft}}{\text{rad} \cdot \text{sec}^2}$	$\frac{\rho S U^2}{2m} (-C_{D_\delta})$
Z_u	$\frac{1}{m} \frac{\partial Z}{\partial u}$	$\frac{1}{\text{sec}}$	$\frac{\rho S U}{m} (-C_L - C_{L_u})^c$
Z_w	$\frac{1}{m} \frac{\partial Z}{\partial w}$	$\frac{1}{\text{sec}}$	$\frac{\rho S U}{2m} (-C_{L_\alpha} - C_D)$
$Z_{\dot{w}}$	$\frac{1}{m} \frac{\partial Z}{\partial \dot{w}}$	$\frac{1}{\text{ft}}$	$\frac{\rho S c}{4m} (-C_{L_z})$
Z_q	$\frac{1}{m} \frac{\partial Z}{\partial q}$	$\frac{\text{ft}}{\text{rad} \cdot \text{sec}}$	$\frac{\rho S U c}{4m} (-C_{L_q})$
Z_δ	$\frac{1}{m} \frac{\partial Z}{\partial \delta}$	$\frac{\text{ft}}{\text{rad} \cdot \text{sec}^2}$	$\frac{\rho S U^2}{2m} (-C_{L_\delta})$
M_u	$\frac{1}{I_y} \frac{\partial M}{\partial u}$	$\frac{1}{\text{ft} \cdot \text{sec}}$	$\frac{\rho S U c}{I_y} (C_M + C_{M_u})$
M_w	$\frac{1}{I_y} \frac{\partial M}{\partial w}$	$\frac{1}{\text{ft} \cdot \text{sec}}$	$\frac{\rho S U c}{2I_y} C_{M_x}$
$M_{\dot{w}}$	$\frac{1}{I_y} \frac{\partial M}{\partial \dot{w}}$	$\frac{1}{\text{ft}}$	$\frac{\rho S c^2}{4I_y} C_{M_z}$
M_q	$\frac{1}{I_y} \frac{\partial M}{\partial q}$	$\frac{1}{\text{sec}}$	$\frac{\rho S U c^2}{4I_y} C_{M_q}$
M_δ	$\frac{1}{I_y} \frac{\partial M}{\partial \delta}$	$\frac{1}{\text{rad} \cdot \text{sec}^2}$	$\frac{\rho S U^2 c}{2I_y} C_{M_\delta}$

Y_v	$\frac{1}{m U} \frac{\partial Y}{\partial \beta}$	$\frac{1}{\text{sec}}$	$\frac{\rho S U}{2m} C_{Y_\beta}$
Y_z	$\frac{1}{m U} \frac{\partial Y}{\partial \dot{\beta}}$	$\frac{1}{\text{ft}}$	$\frac{\rho S b}{4m} C_{Y_\beta}$
Y_r^*	$\frac{1}{m U} \frac{\partial Y}{\partial r}$	$\frac{1}{\text{rad}}$	$\frac{\rho S b}{4m} C_{Y_r}$
$Y_{\dot{r}}^*$	$\frac{1}{m U} \frac{\partial Y}{\partial \dot{r}}$	$\frac{1}{\text{rad}}$	$\frac{\rho S b}{4m} C_{Y_p}$
Y_δ^*	$\frac{1}{m U} \frac{\partial Y}{\partial \delta}$	$\frac{1}{\text{rad} \cdot \text{sec}}$	$\frac{\rho S U}{2m} C_{Y_\delta}$
N_β	$\frac{1}{I_x} \frac{\partial N}{\partial \beta}$	$\frac{1}{\text{sec}^2}$	$\frac{\rho S U^2 b}{2I_x} C_{N_\beta}$
$N_{\dot{\beta}}$	$\frac{1}{I_x} \frac{\partial N}{\partial \dot{\beta}}$	$\frac{1}{\text{sec}}$	$\frac{\rho S U b^2}{4I_x} C_{N_\beta}$
N_r	$\frac{1}{I_x} \frac{\partial N}{\partial r}$	$\frac{1}{\text{sec}}$	$\frac{\rho S U b^2}{4I_x} C_{N_r}$
N_p	$\frac{1}{I_x} \frac{\partial N}{\partial p}$	$\frac{1}{\text{sec}}$	$\frac{\rho S U b^2}{4I_x} C_{N_p}$
N_δ	$\frac{1}{I_x} \frac{\partial N}{\partial \delta}$	$\frac{1}{\text{rad} \cdot \text{sec}^2}$	$\frac{\rho S U^2 b}{2I_x} C_{N_\delta}$
L_β	$\frac{1}{I_x} \frac{\partial L}{\partial \beta}$	$\frac{1}{\text{sec}^2}$	$\frac{\rho S U^2 b}{2I_x} C_{L_\beta}$
$L_{\dot{\beta}}$	$\frac{1}{I_x} \frac{\partial L}{\partial \dot{\beta}}$	$\frac{1}{\text{sec}}$	$\frac{\rho S U b^2}{4I_x} C_{L_\beta}$
L_r	$\frac{1}{I_x} \frac{\partial L}{\partial r}$	$\frac{1}{\text{sec}}$	$\frac{\rho S U b^2}{4I_x} C_{L_r}$
L_p	$\frac{1}{I_x} \frac{\partial L}{\partial p}$	$\frac{1}{\text{sec}}$	$\frac{\rho S U b^2}{4I_x} C_{L_p}$
L_δ	$\frac{1}{I_x} \frac{\partial L}{\partial \delta}$	$\frac{1}{\text{rad} \cdot \text{sec}^2}$	$\frac{\rho S U^2 b}{2I_x} C_{L_\delta}$



□ Longitudinal natural modes

- The first mode represents a high-frequency, highly-damped oscillation called the **short-period mode**, and the second mode represents a low-frequency, lightly-damped oscillation called the **phugoid mode**.



■ Short Period Approximation

$$\begin{cases} (1 - Z_{\dot{w}})\dot{w} = Z_w w + (U_0 + Z_q)q + Z_\delta \delta \\ -M_{\dot{w}}\dot{w} + \dot{q} = M_w w + M_q q + M_\delta \delta \\ w = U_0 \alpha \end{cases}$$

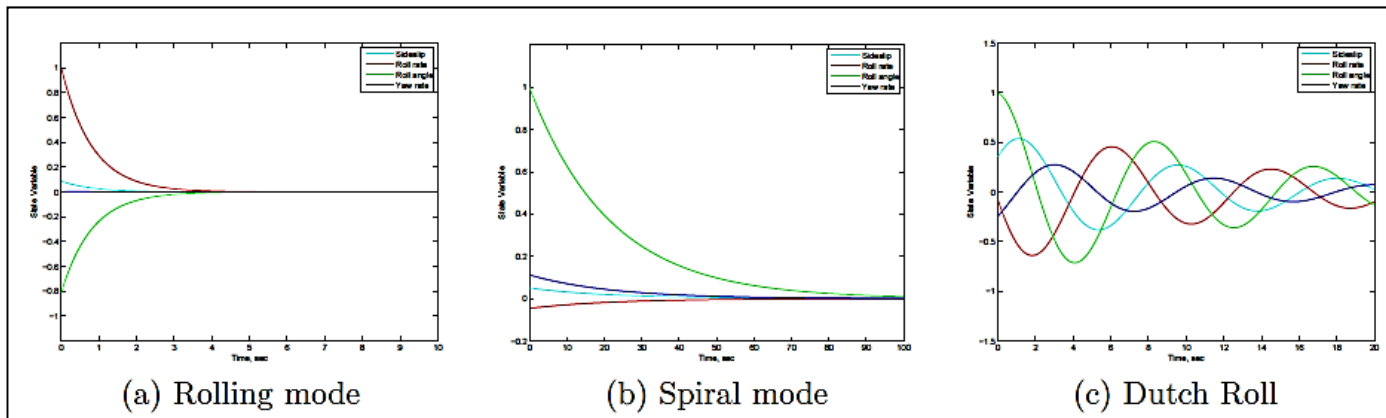
■ Phugoid Approximation

$$\begin{cases} \dot{u} = X_u u - g\theta \\ -U_0 q = Z_u u + Z_q q + Z_\delta \delta \\ q = \dot{\theta} \end{cases}$$



□ Lateral-Directional natural modes

- Roll Mode
- Spiral Mode
- Dutch Roll Mode



▪ Roll Mode Approximation

$$(1 - I_{xx} I_{zz}) \dot{p} = L_p p + I_{xx} N_p p + L_{\delta_A} \delta_A$$

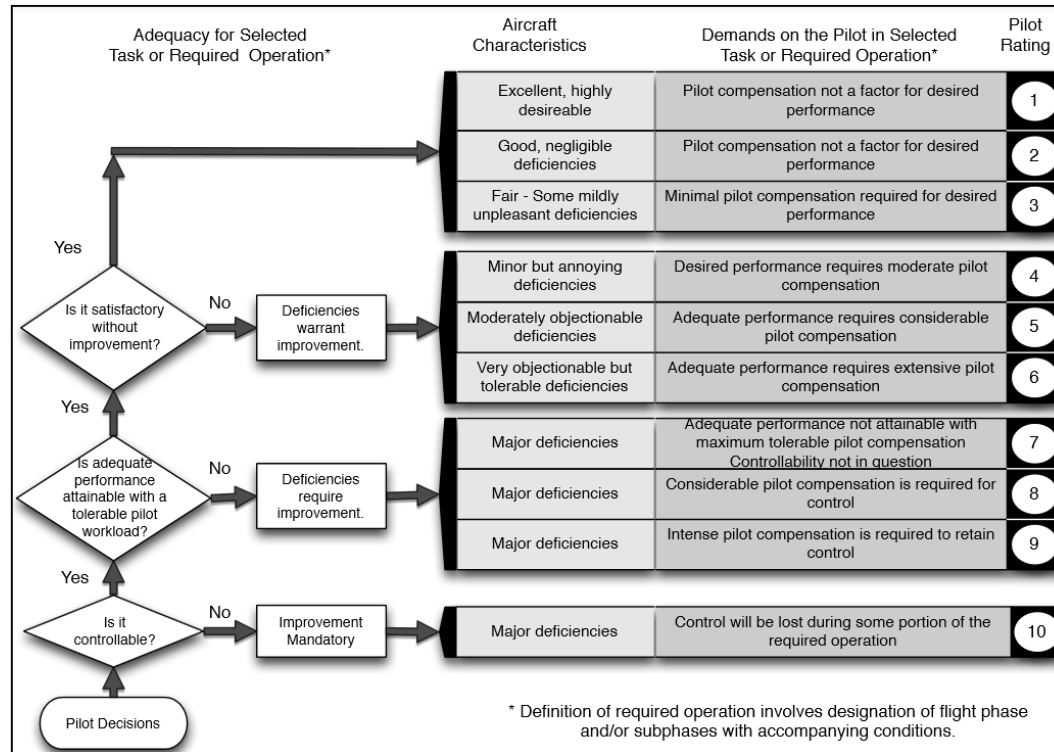
▪ Spiral Mode Approximation

$$\dot{r} = \left(N_r - \frac{L_r N_v}{L_v} \right) r + N_{\delta_A} \delta_A + N_{\delta_R} \delta_R$$

▪ Dutch Roll Mode Approximation

$$\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -U_0 \\ L_v & L_p & L_r \\ N_v & N_p & N_r \end{bmatrix} \begin{bmatrix} v \\ p \\ r \end{bmatrix} + \begin{bmatrix} Y_{\delta_R} & 0 \\ L_{\delta_R} & L_{\delta_A} \\ N_{\delta_R} & N_{\delta_A} \end{bmatrix} \begin{bmatrix} \delta_R \\ \delta_A \end{bmatrix}$$

Summary



DEPARTMENT OF DEFENSE HANDBOOK

FLYING QUALITIES OF PILOTED AIRCRAFT

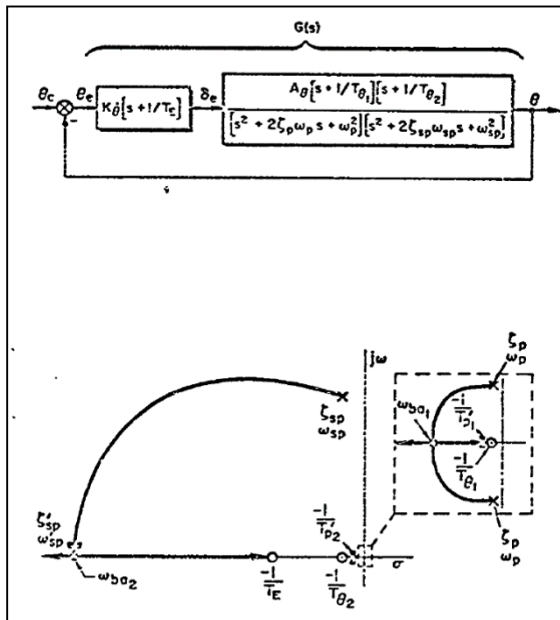


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$$Y_p = K_p e^{-\tau s} \left(\frac{T_L s + 1}{T_I s + 1} \right) \left[\frac{1}{(T_{N1} s + 1) \left(\frac{s^2}{\omega_N^2} + \frac{2\zeta_N}{\omega_N} s + 1 \right)} \right]$$

Longitudinal Feedbacks

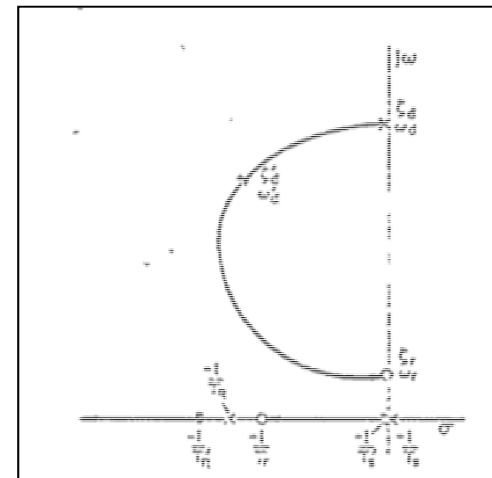
- Pitch Angle and Pitch Rate to Elevator (PH, SP)
- Angle of Attack to Elevator (SP)
- Normal Acceleration to Elevator (SP, Turn)
- Altitude to Elevator (Altitude Hold), Engine Compensation
- Forward Speed to Elevator (Mach Hold), Engine Compensation



Lateral - Directional Feedbacks

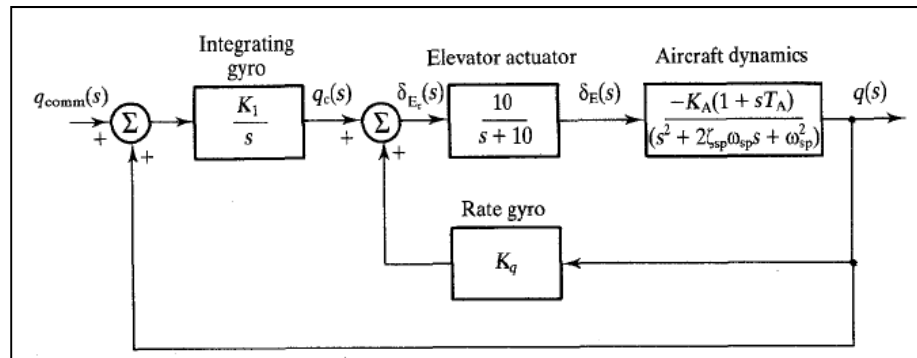
- Bank Angle and Roll Rate to Ailerons (Spiral, Roll Modes)
- Heading Angle to Rudder (Turn)
- Yaw Rate to Rudder (DR)
- Sideslip to Rudder (Coord.)
- Lateral Acceleration to Rudder (Coord.)

$$\frac{r(s)}{\delta_r(s)} = \frac{N_{\delta_r}^r}{\Delta_{lat}} = \frac{A_r \left(s + \frac{1}{T_r} \right) [s^2 + 2\zeta_{1r}\omega_{1r}s + \omega_{1r}^2]}{\left(s + \frac{1}{T_s} \right) \left(s + \frac{1}{T_R} \right) [s^2 + 2\zeta_{ar}\omega_{ar}s + \omega_{ar}^2]}$$

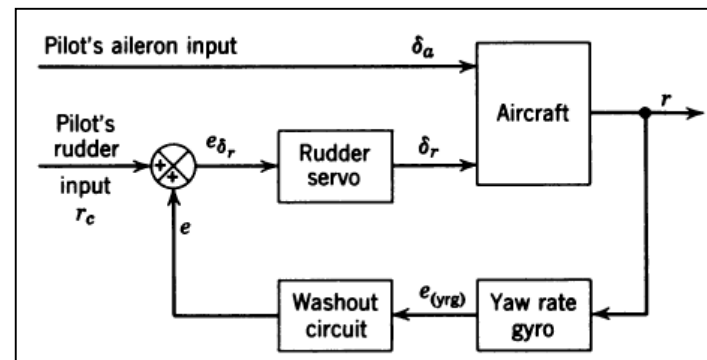


□ Stability Augmentation Systems (SAS)

- **Pitch Damper:** a pitch damper is used to compensate for poor short period damping



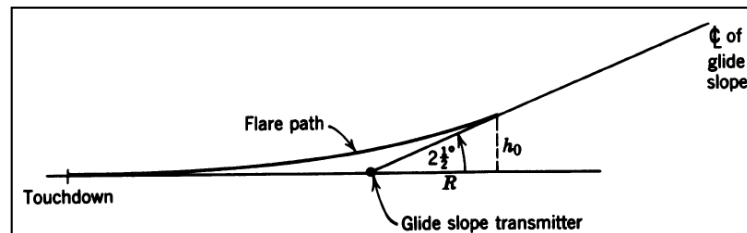
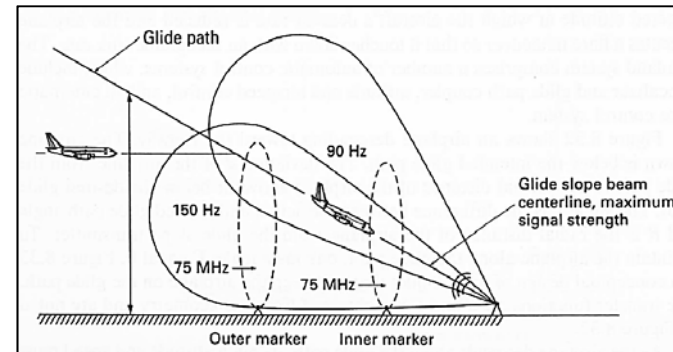
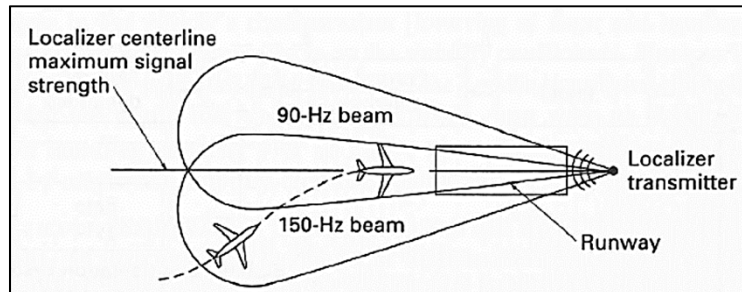
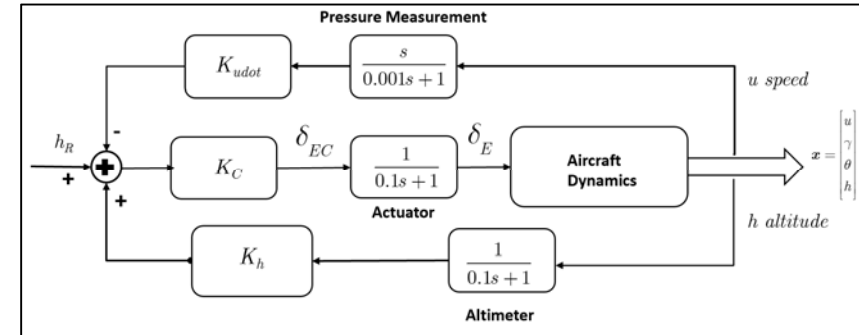
- **Yaw Damper:** a yaw damper is used to improve Dutch-Roll properties.



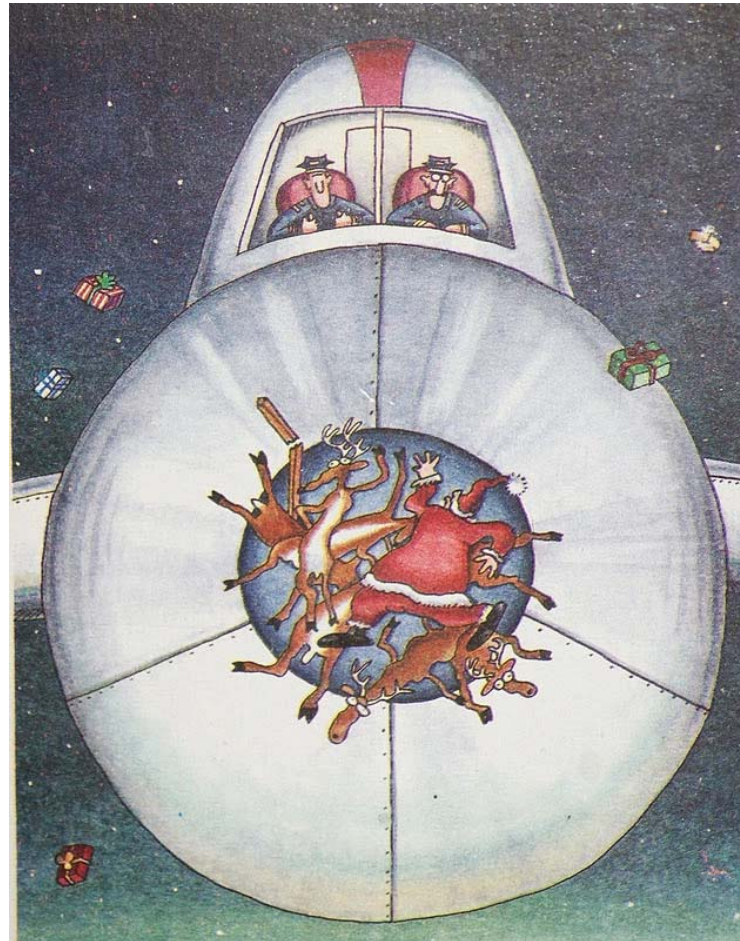
Summary

Autopilots

- Attitude hold: to maintain constant Attitude
- Auto Throttle: to maintain a specified IAS
- Altitude hold: to maintain constant altitude
- Instrument Landing System (AutoLand)
- ...



Summary



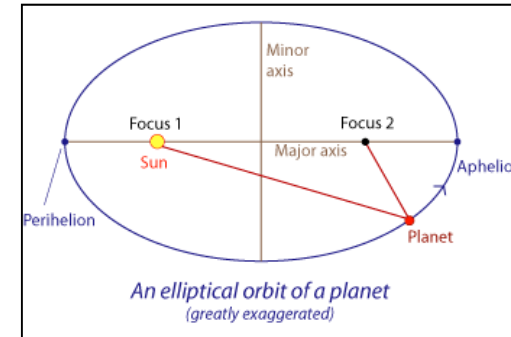


☐ Kepler's Observations (for celestial bodies):

Law 1: Planets move in elliptic orbits with one focus at the planet they orbit.

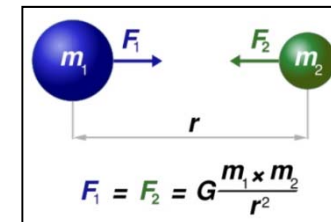
Law 2: Planets sweep out equal areas of the ellipse in equal time.

Law 3: The square of the period of the orbit is proportional to the cube of the semi-major axis.

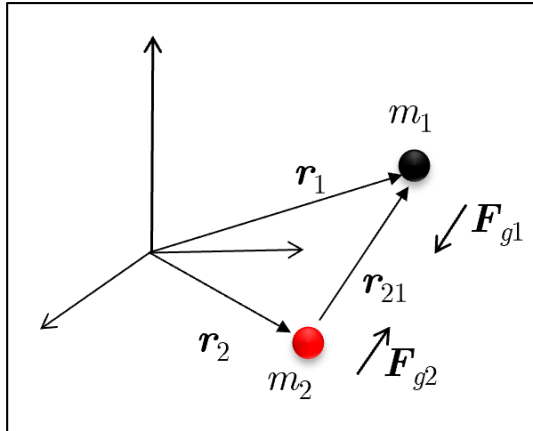


☐ Newton's Universal Law of Gravitation

- **Statement:** a particle attracts every other particle in the universe using a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.
- **Central-force problem:** to determine the motion of a particle under the influence of a single central force.
- **Application of Inverse Square Law**



❑ 2 – Body problem (relative motion of two particles)



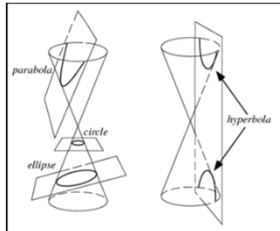
$$\frac{d^2 \mathbf{r}}{dt^2} + \frac{\mu}{r^3} \mathbf{r} = 0$$

or $\frac{d\mathbf{v}}{dt} = -\frac{\mu}{r^3} \mathbf{r}$ where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$
 $r = |\mathbf{r}| = |\mathbf{r}_2 - \mathbf{r}_1|$
 $\mu = G(m_1 + m_2)$

- The Linear Momentum of the system is an **invariant** of motion
- The total specific mechanical energy of the mass m is an **invariant** of motion
- The total specific angular momentum is an **invariant** of motion

$$r = \frac{p}{1 + e \cos f}$$

$$P^2 = (4 \frac{\pi^2}{\mu}) a^3$$



$$r(t) = \frac{h^2}{\mu (1 + e \cos f(t))}$$

$$r_p = a(1 - e)$$

$$r_a = a(1 + e)$$

$$p = a(1 - e^2) = \frac{h^2}{\mu}$$

$$b = a\sqrt{1 - e^2}$$

$$E = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$$

$$h = r_p v_p = r_a v_a \cong r^2 \dot{f}$$

$$\dot{\vec{e}} = 0$$

$$\dot{\vec{h}} = 0$$

$$\vec{h} = \vec{r} \times \dot{\vec{r}}$$

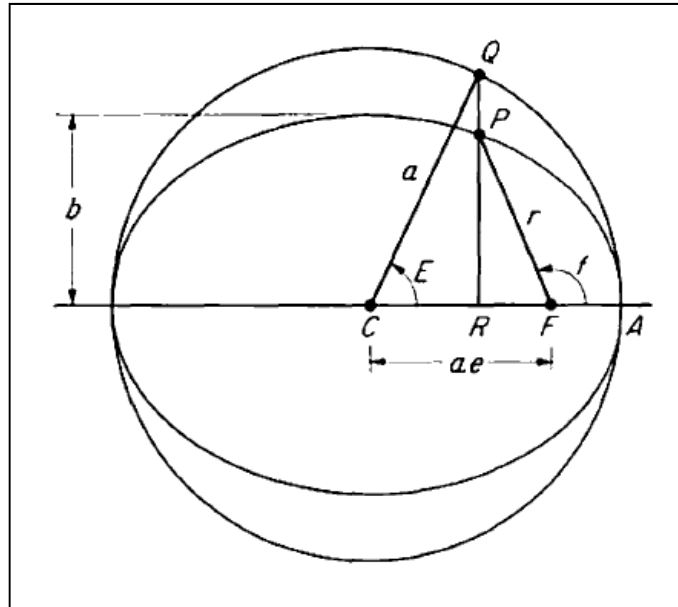
$$\vec{e} = \frac{1}{\mu} \left(\dot{\vec{r}} \times \vec{h} - \mu \frac{\vec{r}}{\|\vec{r}\|} \right)$$

$$\vec{r}(t) \times \vec{h} = 0$$

$$\vec{e} \times \vec{h} = 0$$

$$\dot{\vec{r}} = \dot{r} \hat{i} + r \dot{f} \hat{j}$$

□ Kepler's Equation



$$M(t) = E(t) - e \sin E(t) = nt = t \sqrt{\frac{\mu}{a^3}}$$

$$\tan \frac{E}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{f}{2}$$

- Computation of true anomaly knowing eccentric anomaly is straightforward
- Computation of mean motion from eccentric anomaly is straightforward
- Computation of eccentric anomaly from mean motion is a very difficult problem and it requires numerical methods

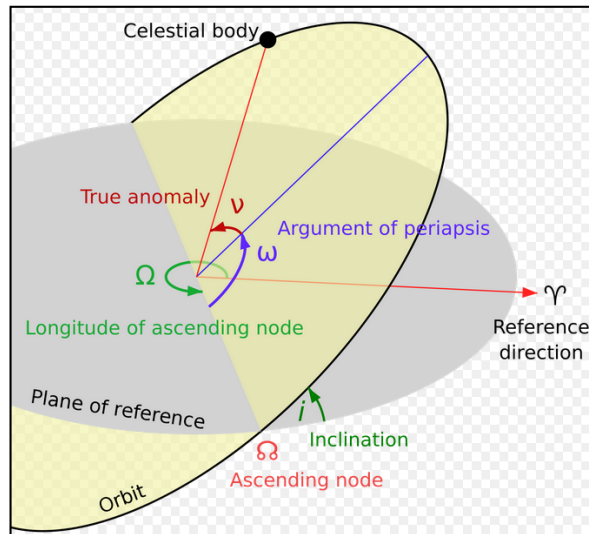
□ Coordinate Systems:

- | | |
|------------------------|---|
| ▪ Celestial Sphere: | ▪ Perifocal System: |
| ▪ Ecliptic plane: | ▪ Time and Date: |
| ▪ Geocentric System: | ▪ Distance Measures: AU, LY, Parallax, Parsec,... |
| ▪ Heliocentric System: | Standard Candles) |





- ❑ **Orbital elements:** are the parameters required to uniquely identify a specific orbit. In celestial mechanics these elements are generally considered in classical two-body systems, where a Kepler orbit is used. There are many different ways to mathematically describe the same orbit, but a set of six parameters is commonly used.



- i = Inclination of the orbit
- Ω = Right Ascension of ascending node
- ω = Argument of periapsis
- a = Semi major axis (or semilatus rectum)
- e = Eccentricity of the orbit
- f = True anomaly (or time to periapsis)

- **Orbital Elements from position and velocity** $\begin{bmatrix} \mathbf{r} & \dot{\mathbf{r}} = \mathbf{v} \end{bmatrix} \Rightarrow \begin{bmatrix} i & \Omega & \omega & a & e & f \end{bmatrix}$
- **Position and velocity from Orbital Elements** $\begin{bmatrix} i & \Omega & \omega & a & e & f \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{r} & \dot{\mathbf{r}} = \mathbf{v} \end{bmatrix}$



□ Lagrange Universal Coefficients

- A fundamental problem in interplanetary mission planning is to determine position and velocity at some specific time, given the position and velocity at some known time (usually initial time), that is:

$$\mathbf{r}(t) = \mathbf{F}\mathbf{r}(t_0) + \mathbf{G}\mathbf{v}(t_0)$$

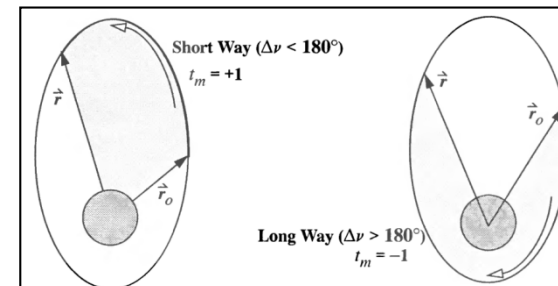
$$\mathbf{v}(t) = \dot{\mathbf{F}}\mathbf{r}(t_0) + \dot{\mathbf{G}}\mathbf{v}(t_0)$$

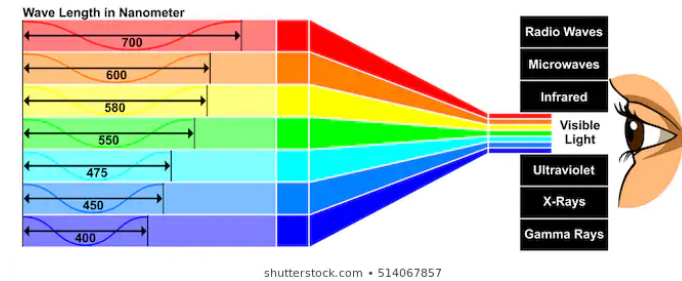
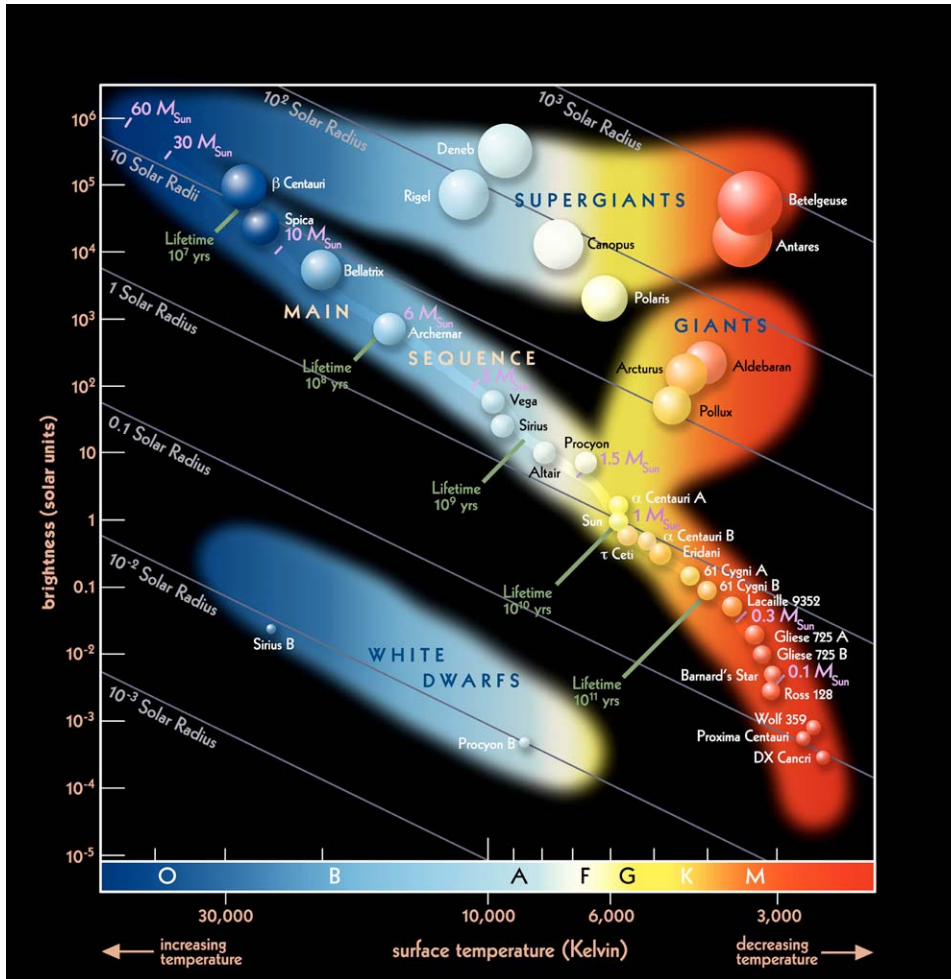
$$\begin{aligned} f &= 1 - \frac{\mu r}{h^2} (1 - \cos \Delta\theta) \\ g &= \frac{r r_0}{h} \sin \Delta\theta \\ \dot{f} &= \frac{\mu}{h} \frac{1 - \cos \Delta\theta}{\sin \Delta\theta} \left[\frac{\mu}{h^2} (1 - \cos \Delta\theta) - \frac{1}{r_0} - \frac{1}{r} \right] \\ \dot{g} &= 1 - \frac{\mu r_0}{h^2} (1 - \cos \Delta\theta) \end{aligned}$$

- **Lambert's Problem (1761):** The transfer time of a body moving between two points on a conic trajectory is a function only of the sum of the distances of the two points from the origin of the force, the linear distance between the points, and the semi major axis of the conic (Somewhat similar to Kepler's problem, but the orbit is unknown)

$$(t_2 - t_1) = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} [\alpha - \beta - (\sin \alpha - \sin \beta)]$$

$$\begin{cases} \sin \frac{\alpha}{2} = \sqrt{\frac{r_1 + r_2 + c}{4a}} \\ \sin \frac{\beta}{2} = \sqrt{\frac{r_1 + r_2 - c}{4a}} \end{cases}$$



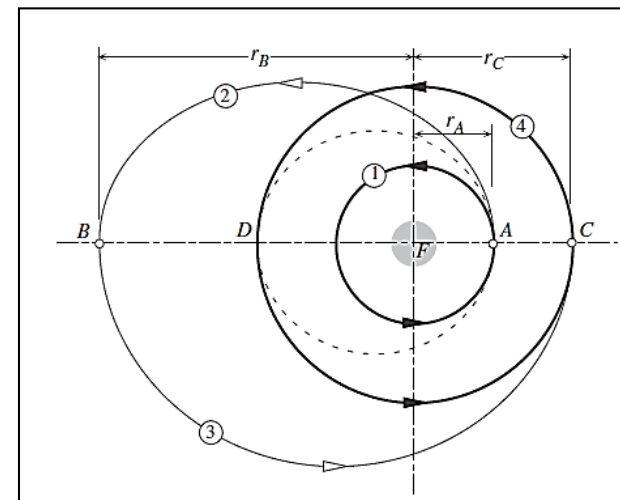
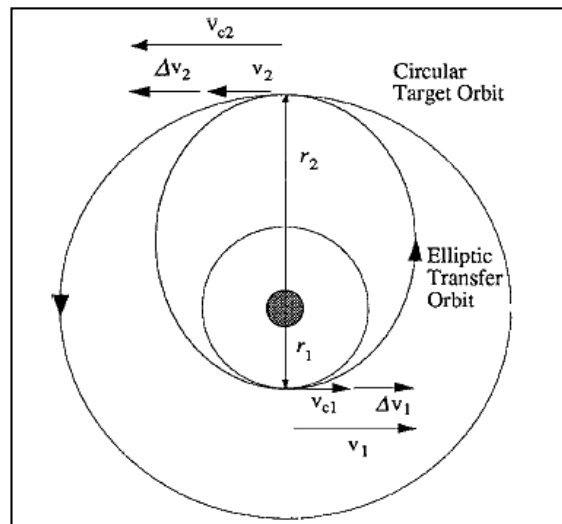
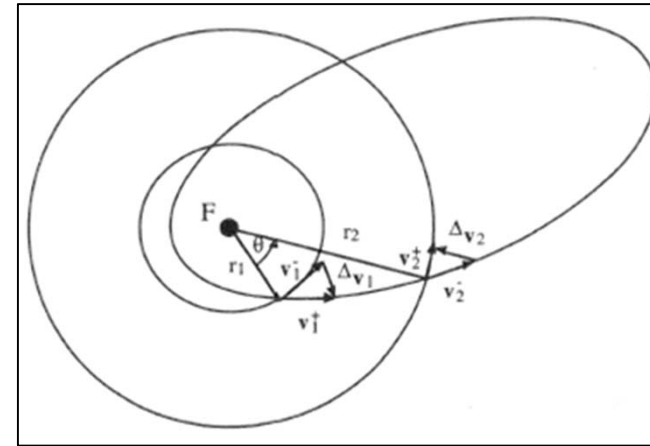
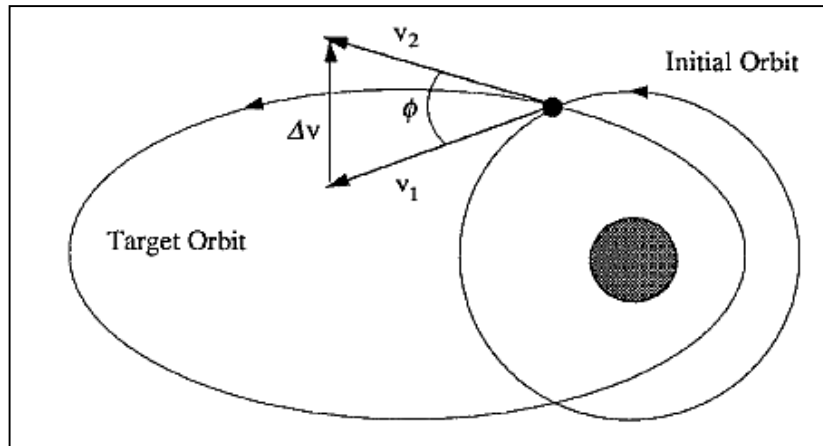


- **The apparent magnitude (m)** is the brightness of an object as it appears in the night sky from Earth. Apparent magnitude depends on an object's intrinsic luminosity, its distance, and the extinction reducing its brightness.
- **The absolute magnitude (M)** describes the intrinsic luminosity emitted by an object and is defined to be equal to the apparent magnitude that the object would have if it were placed at a certain distance from Earth, 10 parsecs for stars (32.6 LY).
- **Color**
- **Temperature**

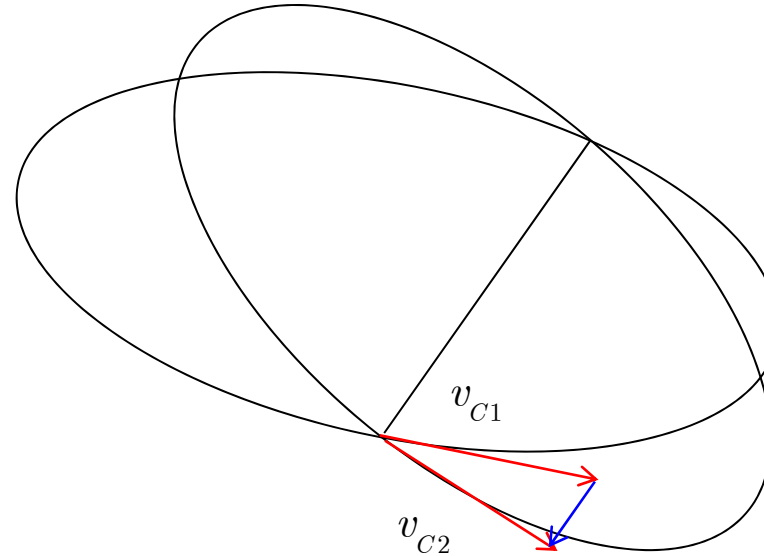
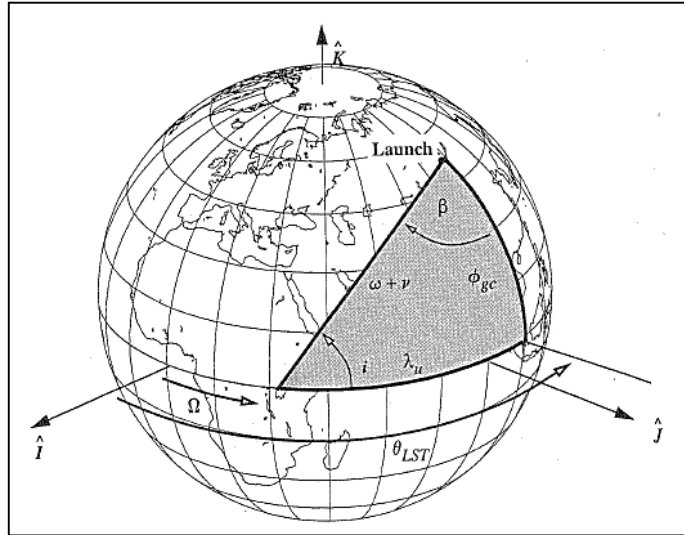
- Plank, Wien, Stefan-Boltzman

Summary

- ❑ **Orbit Transfer:** Orbit transfer is mostly based on propellant consumption, defined as Δv , that is the instantaneous change in velocity (acceleration) needed for transfer.



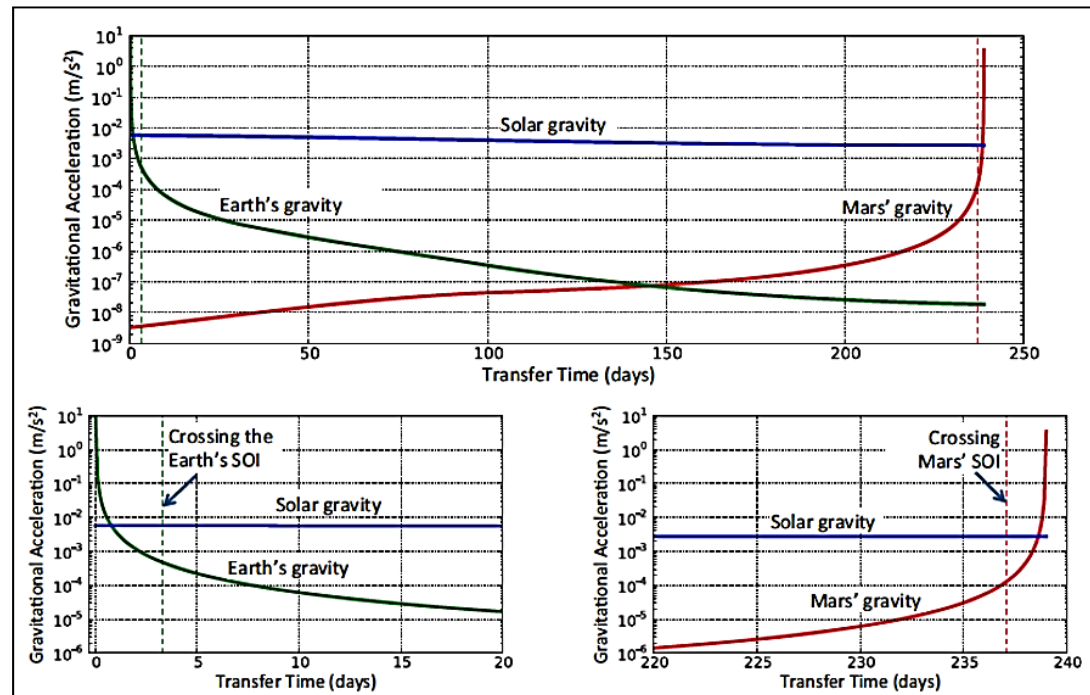
❑ Out of Plane Orbit Transfer



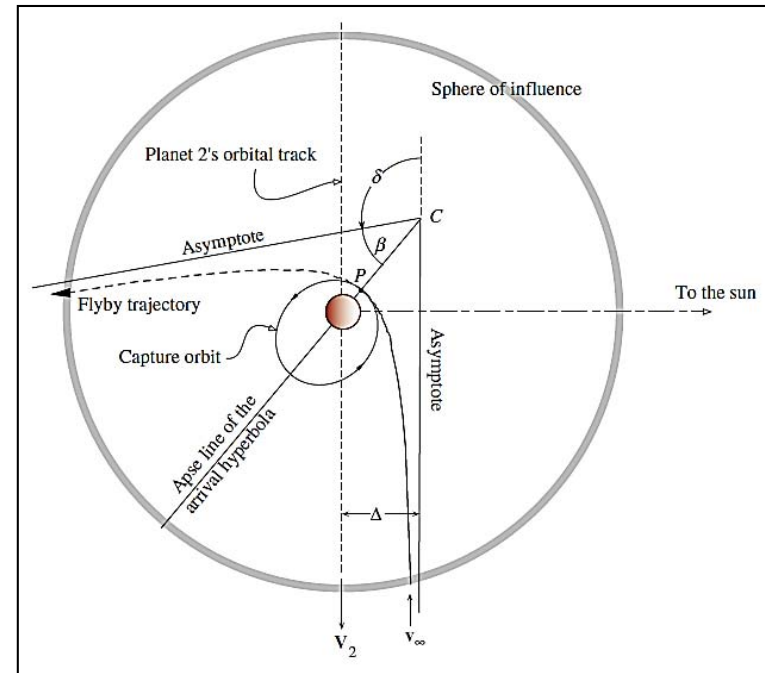
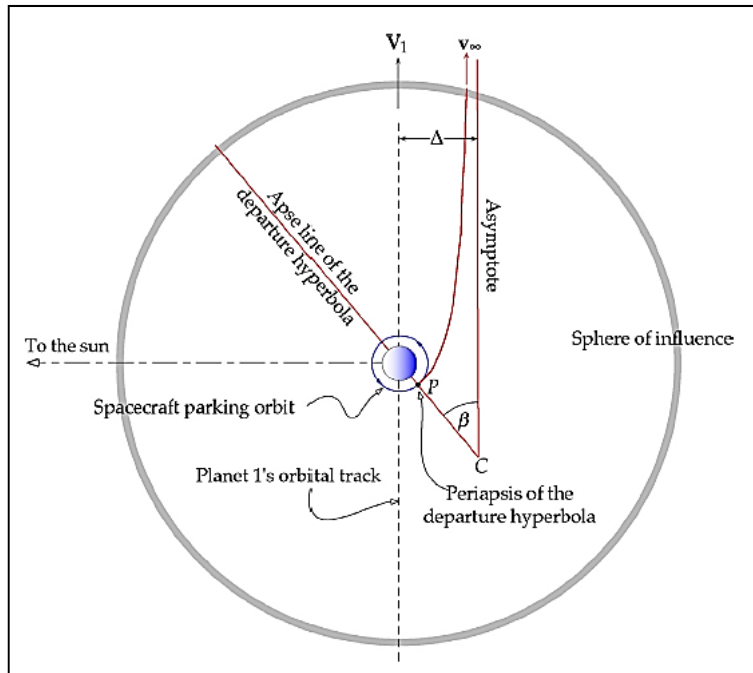
- ❑ **Preliminary interplanetary Mission planning:** The design of an interplanetary mission is a complex – multi year project, and the mission itself may last many decades. Elements such as orbit perturbations, and multiple body gravitational attraction must be taken into account

Summary

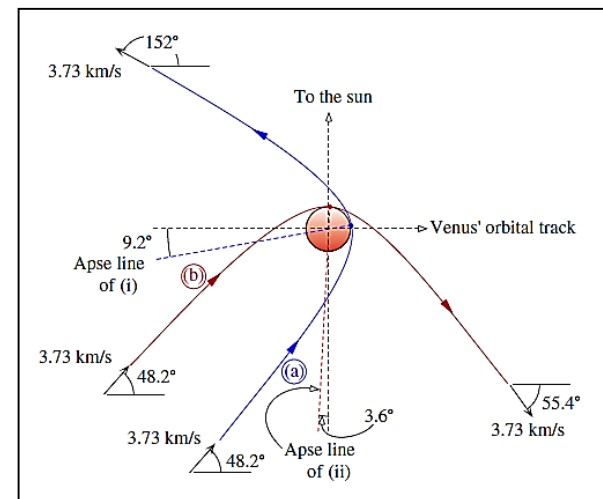
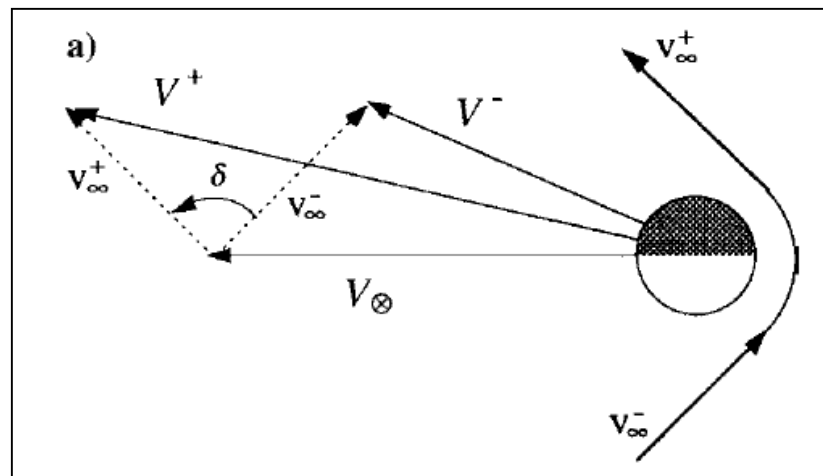
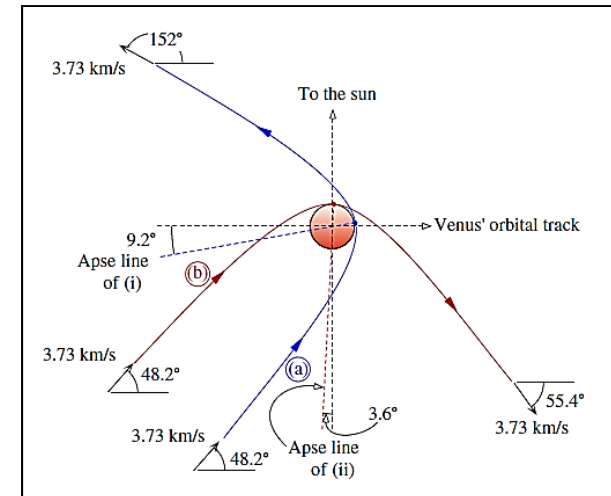
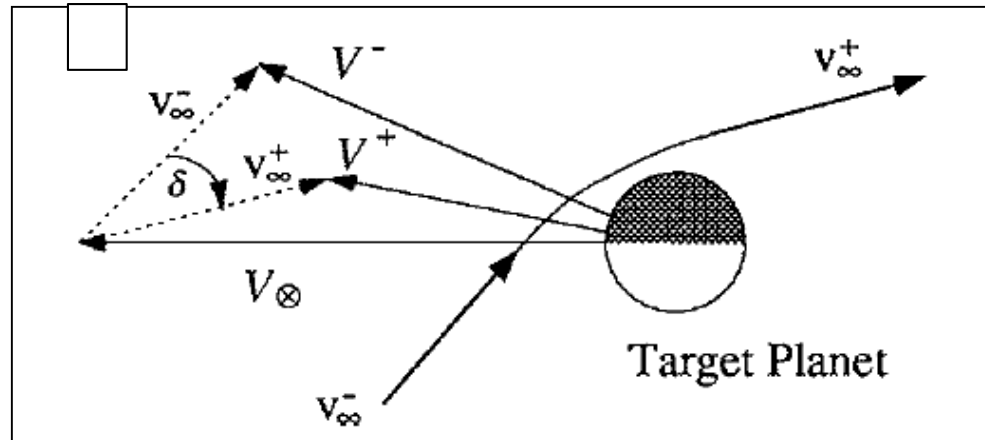
- ❑ **Patched Conics Method:** The trajectory is composed of Keplerian orbits (or orbit sections), with the main body given by the planets of interest and the Sun.
- ❑ **Sphere of Influence:** A sphere of influence (SOI) is the oblate-spheroid-shaped region around a celestial body where the primary gravitational influence on an orbiting object is that body.



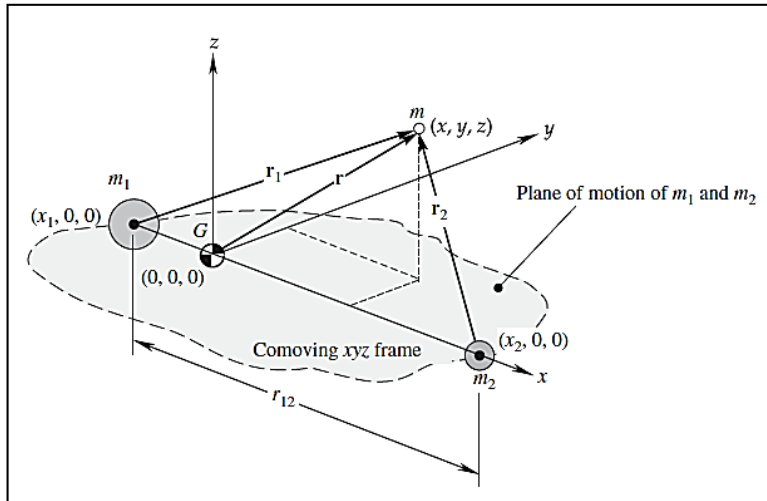
- ❑ **Escape and Capture Trajectories:** hyperbolic paths needed to leave or enter the SOI of a planet.



□ Planetary Flyby or Gravity Assist:



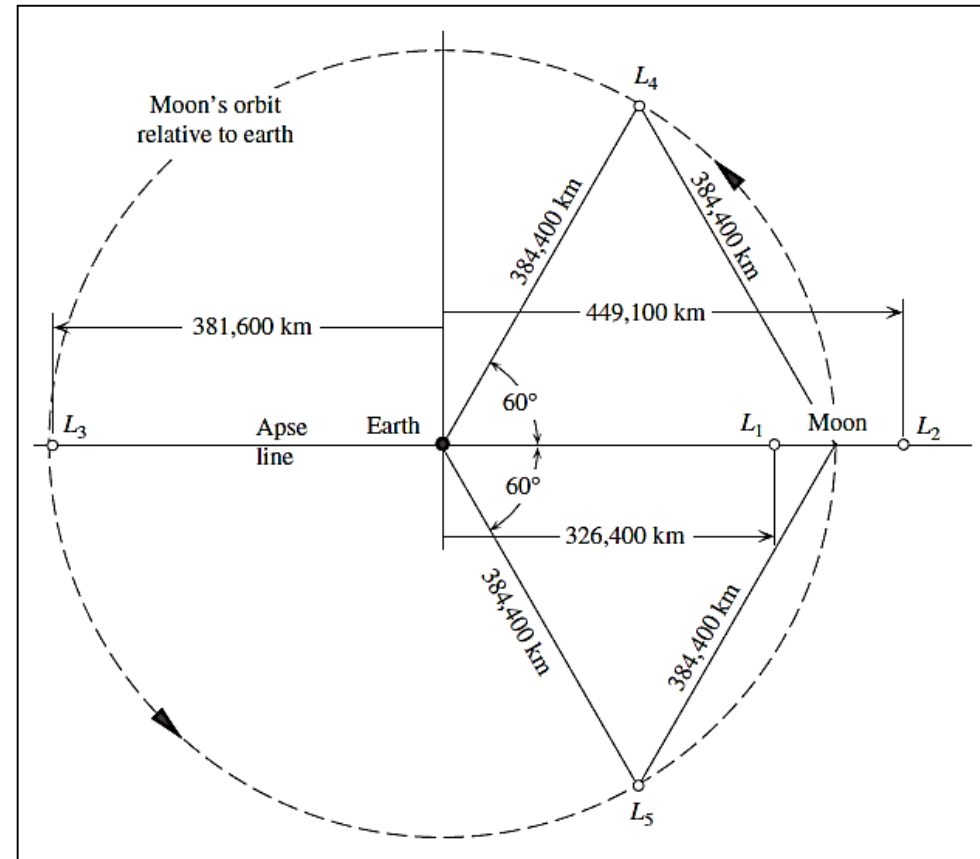
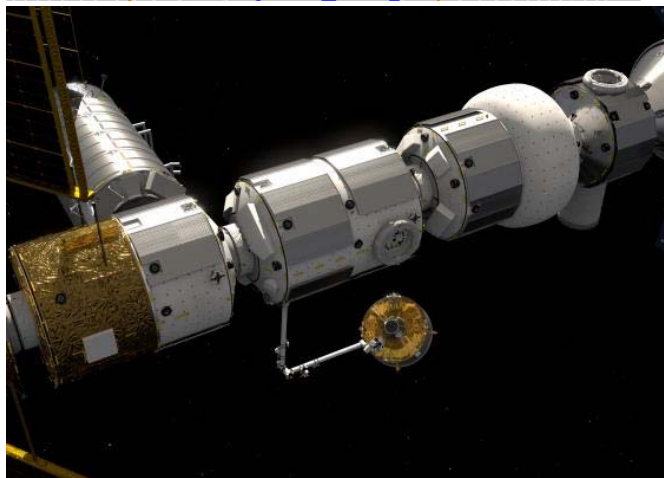
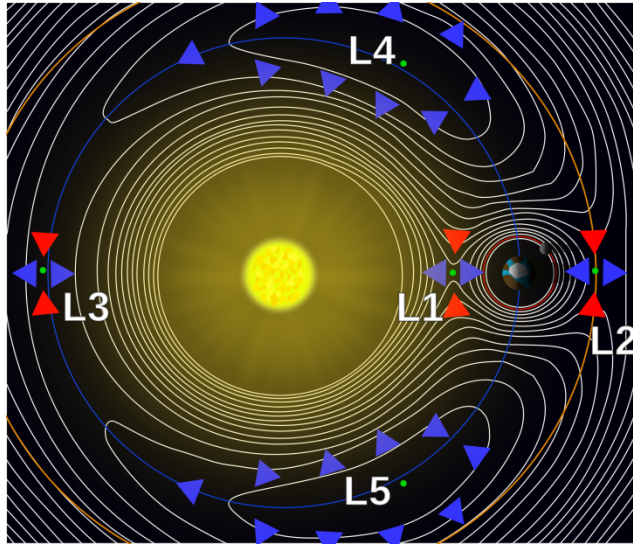
Three Body Problem:



- **Circular restricted 3 – body problem:** two primary bodies moving in circular orbits about their common center of mass, plus a third small mass with negligible gravitational influence.

$$\begin{cases} \ddot{X} - 2\dot{Y} - X = -\frac{(1-\rho)(X-\rho)}{r_1^3} - \frac{\mu_2(X+1-\rho)}{r_2^3} \\ \ddot{Y} + 2\dot{X} - Y = -\frac{(1-\rho)Y}{r_1^3} - \frac{\rho Y}{r_2^3} \\ \ddot{Z} = -\frac{(1-\rho)Z}{r_1^3} - \frac{\rho Z}{r_2^3} \end{cases} \quad \begin{cases} r_1^2 = (X-\rho)^2 + Y^2 + Z^2 \\ r_2^2 = (X+1-\rho)^2 + Y^2 + Z^2 \end{cases}$$

□ Lagrangian Points:





That's here. That's home. That's us.
A mote of dust, suspended in a sunbeam.

Think about it.



Sizes

MATLAB Scripts

D

D.1 Introduction

This appendix lists MATLAB scripts that implement all of the numbered algorithms presented throughout the text. The programs use only the most basic features of MATLAB and are liberally commented so as to make reading the code as easy as possible. To “drive” the various algorithms, one can use MATLAB to create graphical user interfaces (GUIs). However, in the interest of simplicity and keeping our focus on the algorithms rather than elegant programming techniques, GUIs were not developed. Furthermore, the scripts do not use files to import and export data. Data is defined in declaration statements within the scripts. All output is to the screen, that is, to the MATLAB Command Window. It is hoped that interested students will embellish these simple scripts or use them as a springboard toward generating their own programs.

Each algorithm is illustrated by a MATLAB coding of a related example problem in the text. The actual output of each of these examples is also listed.

It would be helpful to have MATLAB documentation at hand. There are a number of practical references on the subject, including Hahn (2002), Kermit and Davis (2002), and Magrab (2000). MATLAB documentation may also be found at The MathWorks Web site (www.mathworks.com). Should it be necessary to do so, it is a fairly simple matter to translate these programs into other software languages.

These programs are presented solely as an alternative to carrying out otherwise lengthy hand computations and are intended for academic use only. They are all based exclusively on the introductory material presented in this text.

Chapter 1

D.2 Algorithm 1.1: Numerical integration by Runge–Kutta methods RK1, RK2, RK3, or RK4

Function file rkf1_4.m

```
~~~~~
function [tout, yout] = rk1_4(ode_function, tspan, y0, h, rk)
% ~~~~~
%{
    This function uses a selected Runge-Kutta procedure to integrate
    a system of first-order differential equations  $dy/dt = f(t,y)$ .
```

e2 MATLAB Scripts

y - column vector of solutions
f - column vector of the derivatives dy/dt
t - time
rk - = 1 for RK1; = 2 for RK2; = 3 for RK3; = 4 for RK4
n_stages - the number of points within a time interval that
 the derivatives are to be computed
a - coefficients for locating the solution points within
 each time interval
b - coefficients for computing the derivatives at each
 interior point
c - coefficients for the computing solution at the end of
 the time step
ode_function - handle for user M-function in which the derivatives f
 are computed
tspan - the vector [t0 tf] giving the time interval for the
 solution
t0 - initial time
tf - final time
y0 - column vector of initial values of the vector y
tout - column vector of times at which y was evaluated
yout - a matrix, each row of which contains the components of y
 evaluated at the corresponding time in tout
h - time step
ti - time at the beginning of a time step
yi - values of y at the beginning of a time step
t_inner - time within a given time step
y_inner - values of y within a given time step

User M-function required: ode_function

```
%}  
% -----  
  
%...Determine which of the four Runge-Kutta methods is to be used:  
switch rk  
    case 1  
        n_stages = 1;  
        a = 0;  
        b = 0;  
        c = 1;  
    case 2  
        n_stages = 2;  
        a = [0 1];  
        b = [0 1]';  
        c = [1/2 1/2];  
    case 3  
        n_stages = 3;  
        a = [0 1/2 1];  
        b = [ 0 0  
              1/2 0  
              -1 2];
```

```

        c = [1/6 2/3 1/6];
    case 4
        n_stages = 4;
        a = [0 1/2 1/2 1];
        b = [ 0    0    0
              1/2  0    0
              0   1/2  0
              0    0   1];
        c = [1/6 1/3 1/3 1/6];
    otherwise
        error('The parameter rk must have the value 1, 2, 3 or 4.')
end

t0 = tspan(1);
tf = tspan(2);
t = t0;
y = y0;
tout = t;
yout = y';

while t < tf
    ti = t;
    yi = y;
    %...Evaluate the time derivative(s) at the 'n_stages' points within the
    % current interval:
    for i = 1:n_stages
        t_inner = ti + a(i)*h;
        y_inner = yi;
        for j = 1:i-1
            y_inner = y_inner + h*b(i,j)*f(:,j);
        end
        f(:,i) = feval(ode_function, t_inner, y_inner);
    end

    h = min(h, tf-t);
    t = t + h;
    y = yi + h*f*c';
    tout = [tout;t]; % adds t to the bottom of the column vector tout
    yout = [yout;y']; % adds y' to the bottom of the matrix yout
end

end
% ~~~~~

```

Function file: Example_1_18.m

```

% ~~~~~
function Example_1_18
% ~~~~~

```

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```
%{
    This function uses the RK1 through RK4 methods with two
    different time steps each to solve for and plot the response
    of a damped single degree of freedom spring-mass system to
    a sinusoidal forcing function, represented by


$$x'' + 2*z*wn*x' + wn^2*x = (Fo/m)*sin(w*t)$$


    The numerical integration is done by the external
    function 'rk1_4', which uses the subfunction 'rates'
    herein to compute the derivatives.

    This function also plots the exact solution for comparison.

    x          - displacement (m)
    '          - shorthand for d/dt
    t          - time (s)
    wn         - natural circular frequency (radians/s)
    z          - damping factor
    wd         - damped natural frequency
    Fo         - amplitude of the sinusoidal forcing function (N)
    m          - mass (kg)
    w          - forcing frequency (radians/s)
    t0         - initial time (s)
    tf         - final time (s)
    h          - uniform time step (s)
    tspan      - a row vector containing t0 and tf
    x0         - value of x at t0 (m)
    x_dot0     - value of dx/dt at t0 (m/s)
    f0         - column vector containing x0 and x_dot0
    rk         - = 1 for RK1; = 2 for RK2; = 3 for RK3; = 4 for RK4
    t          - solution times for the exact solution
    t1,...,t4  - solution times for RK1,...,RK4 for smaller
    t11,...,t41 - solution times for RK1,...,RK4 for larger h
    f1,...,f4  - solution vectors for RK1,...,RK4 for smaller h
    f11,...,f41 - solution vectors for RK1,...,RK4 for larger h

    User M-functions required: rk1_4
    User subfunctions required: rates
%}

% -----

clear all; close all; clc

%...Input data:
m      = 1;
z      = 0.03;
wn     = 1;
```

```

Fo      = 1;
w       = 0.4*wn;

x0      = 0;
x_dot0  = 0;
f0      = [x0; x_dot0];

t0      = 0;
tf      = 110;
tspan   = [t0 tf];
%...End input data

%...Solve using RK1 through RK4, using the same and a larger
%   time step for each method:
rk = 1;
h = .01; [t1, f1] = rk1_4(@rates, tspan, f0, h, rk);
h = 0.1; [t11, f11] = rk1_4(@rates, tspan, f0, h, rk);

rk = 2;
h = 0.1; [t2, f2] = rk1_4(@rates, tspan, f0, h, rk);
h = 0.5; [t21, f21] = rk1_4(@rates, tspan, f0, h, rk);

rk = 3;
h = 0.5; [t3, f3] = rk1_4(@rates, tspan, f0, h, rk);
h = 1.0; [t31, f31] = rk1_4(@rates, tspan, f0, h, rk);

rk = 4;
h = 1.0; [t4, f4] = rk1_4(@rates, tspan, f0, h, rk);
h = 2.0; [t41, f41] = rk1_4(@rates, tspan, f0, h, rk);

output

return

% ~~~~~~
function dfdt = rates(t,f)
% -----
%{
    This function calculates first and second time derivatives
    of x as governed by the equation


$$x'' + 2z*wn*x' + wn^2*x = (Fo/m)*sin(w*t)$$


    Dx   - velocity (x')
    D2x  - acceleration (x'')
    f     - column vector containing x and Dx at time t
    dfdt - column vector containing Dx and D2x at time t

```

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```
User M-functions required: none
%}
% ~~~~~

x    = f(1);
Dx   = f(2);
D2x  = Fo/m*sin(w*t) - 2*z*wn*Dx - wn^2*x;
dfdt = [Dx; D2x];
end %rates

% ~~~~~
function output
% -----
%...Exact solution:
wd = wn*sqrt(1 - z^2);
den = (wn^2 - w^2)^2 + (2*w*wn*z)^2;
C1 = (wn^2 - w^2)/den*Fo/m;
C2 = -2*w*wn*z/den*Fo/m;
A = x0*wn/wd + x_dot0/wd + (w^2 + (2*z^2 - 1)*wn^2)/den*w/wd*Fo/m;
B = x0 + 2*w*wn*z/den*Fo/m;

t = linspace(t0, tf, 5000);
x = (A*sin(wd*t) + B*cos(wd*t)).*exp(-wn*z*t) ...
    + C1*sin(w*t) + C2*cos(w*t);

%...Plot solutions
% Exact:
subplot(5,1,1)
plot(t/max(t), x/max(x), 'k', 'LineWidth',1)
grid off
axis tight
title('Exact')

% RK1:
subplot(5,1,2)
plot(t1/max(t1), f1(:,1)/max(f1(:,1)), '-r', 'LineWidth',1)
hold on
plot(t11/max(t11), f11(:,1)/max(f11(:,1)), '-k')
grid off
axis tight
title('RK1')
legend('h = 0.01', 'h = 0.1')

% RK2:
subplot(5,1,3)
plot(t2/max(t2), f2(:,1)/max(f2(:,1)), '-r', 'LineWidth',1)
hold on
plot(t21/max(t21), f21(:,1)/max(f21(:,1)), '-k')
```

```

grid off
axis tight
title('RK2')
legend('h = 0.1', 'h = 0.5')

% RK3:
subplot(5,1,4)
plot(t3/max(t3), f3(:,1)/max(f3(:,1)), '-r', 'LineWidth',1)
hold on
plot(t31/max(t31), f31(:,1)/max(f31(:,1)), '-k')
grid off
axis tight
title('RK3')
legend('h = 0.5', 'h = 1.0')

% RK4:
subplot(5,1,5)
plot(t4/max(t4), f4(:,1)/max(f4(:,1)), '-r', 'LineWidth',1)
hold on
grid off
plot(t41/max(t41), f41(:,1)/max(f41(:,1)), '-k')
axis tight
title('RK4')
legend('h = 1.0', 'h = 2.0')
end %output

end %Example_1_18
% ~~~~~

```

D.3 Algorithm 1.2: Numerical integration by Heun's predictor-corrector method

Function file: heun.m

```

% ~~~~~
function [tout, yout] = heun(ode_function, tspan, y0, h)
% ~~~~~
%{
    This function uses the predictor-corrector method to integrate a system
    of first-order differential equations  $dy/dt = f(t,y)$ .

    y          - column vector of solutions
    f          - column vector of the derivatives  $dy/dt$ 
    ode_function - handle for the user M-function in which the derivatives
                  f are computed
    t          - time
%}

```

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t0	- initial time
tf	- final time
tspan	- the vector [t0 tf] giving the time interval for the solution
h	- time step
y0	- column vector of initial values of the vector y
tout	- column vector of the times at which y was evaluated
yout	- a matrix, each row of which contains the components of y evaluated at the corresponding time in tout
feval	- a built-in MATLAB function which executes 'ode_function' at the arguments t and y
tol	- Maximum allowable relative error for determining convergence of the corrector
itermax	- maximum allowable number of iterations for corrector convergence
iter	- iteration number in the corrector convergence loop
t1	- time at the beginning of a time step
y1	- value of y at the beginning of a time step
f1	- derivative of y at the beginning of a time step
f2	- derivative of y at the end of a time step
favg	- average of f1 and f2
y2p	- predicted value of y at the end of a time step
y2	- corrected value of y at the end of a time step
err	- maximum relative error (for all components) between y2p and y2 for given iteration
eps	- unit roundoff error (the smallest number for which $1 + \text{eps} > 1$). Used to avoid a zero denominator.

User M-function required: ode_function

```
%}
```

```
% -----
```

```
tol      = 1.e-6;
```

```
itermax  = 100;
```

```
t0       = tspan(1);
```

```
tf       = tspan(2);
```

```
t        = t0;
```

```
y        = y0;
```

```
tout     = t;
```

```
yout     = y';
```

```
while t < tf
```

```
    h     = min(h, tf-t);
```

```
    t1    = t;
```

```
    y1    = y;
```

```
    f1    = feval(ode_function, t1, y1);
```

```
    y2    = y1 + f1*h;
```

```

t2 = t1 + h;
err = tol + 1;
iter = 0;
while err > tol && iter <= itermax
    y2p = y2;
    f2 = feval(ode_function, t2, y2p);
    favg = (f1 + f2)/2;
    y2 = y1 + favg*h;
    err = max(abs((y2 - y2p)./(y2 + eps)));
    iter = iter + 1;
end

if iter > itermax
    fprintf('\n Maximum no. of iterations (%g)',itermax)
    fprintf('\n exceeded at time = %g',t)
    fprintf('\n in function "heun."\n\n')
    return
end

t = t + h;
y = y2;
tout = [tout;t]; % adds t to the bottom of the column vector tout
yout = [yout;y']; % adds y' to the bottom of the matrix yout
end
% ~~~~~

```

Function file: Example_1_19.m

```

% ~~~~~
function Example_1_19
% ~~~~~
%{
    This program uses Heun's method with two different time steps to solve
    for and plot the response of a damped single degree of freedom
    spring-mass system to a sinusoidal forcing function, represented by


$$x'' + 2\zeta\omega_n x' + \omega_n^2 x = (F_0/m)\sin(\omega t)$$


    The numerical integration is done in the external function 'heun',
    which uses the subfunction 'rates' herein to compute the derivatives.

    x      - displacement (m)
    '      - shorthand for d/dt
    t      - time (s)
    wn     - natural circular frequency (radians/s)
    z      - damping factor
    Fo     - amplitude of the sinusoidal forcing function (N)
%}

```

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```
m      - mass (kg)
w      - forcing frequency (radians/s)
t0     - initial time (s)
tf     - final time (s)
h      - uniform time step (s)
tspan  - row vector containing t0 and tf
x0     - value of x at t0 (m)
Dx0    - value of dx/dt at t0 (m/s)
f0     - column vector containing x0 and Dx0
t      - column vector of times at which the solution was computed
f      - a matrix whose columns are:
          column 1: solution for x at the times in t
          column 2: solution for x' at the times in t

User M-functions required: heun
User subfunctions required: rates
%}
% -----

clear all; close all; clc

%...System properties:
m      = 1;
z      = 0.03;
wn     = 1;
Fo     = 1;
w      = 0.4*wn;

%...Time range:
t0     = 0;
tf     = 110;
tspan  = [t0 tf];

%...Initial conditions:
x0     = 0;
Dx0    = 0;
f0     = [x0; Dx0];

%...Calculate and plot the solution for h = 1.0:
h      = 1.0;
[t1, f1] = heun(@rates, tspan, f0, h);

%...Calculate and plot the solution for h = 0.1:
h      = 0.1;
[t2, f2] = heun(@rates, tspan, f0, h);

output
```

```

return

% ~~~~~
function dfdt = rates(t,f)
% ~~~~~
%
% This function calculates first and second time derivatives of x
% for the forced vibration of a damped single degree of freedom
% system represented by the 2nd order differential equation
%
%  $x'' + 2\zeta\omega_n x' + \omega_n^2 x = (F_0/m)\sin(\omega t)$ 
%
% Dx - velocity
% D2x - acceleration
% f - column vector containing x and Dx at time t
% dfdt - column vector containing Dx and D2x at time t
%
% User M-functions required: none
% -----
x = f(1);
Dx = f(2);
D2x = F0/m*sin(w*t) - 2*z*wn*Dx - wn^2*x;
dfdt = [Dx; D2x];
end %rates

% ~~~~~
function output
% ~~~~~
plot(t1, f1(:,1), '-r', 'LineWidth',0.5)
xlabel('time, s')
ylabel('x, m')
grid
axis([0 110 -2 2])
hold on
plot(t2, f2(:,1), '-k', 'LineWidth',1)
legend('h = 1.0','h = 0.1')
end %output

end %Example_1_19
% ~~~~~

```

Function file: rkf45.m

```

% ~~~~~
function [tout, yout] = rkf45(ode_function, tspan, y0, tolerance)
% ~~~~~
%{

```

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This function uses the Runge-Kutta-Fehlberg 4(5) algorithm to integrate a system of first-order differential equations $dy/dt = f(t,y)$.

y - column vector of solutions
f - column vector of the derivatives dy/dt
t - time
a - Fehlberg coefficients for locating the six solution points (nodes) within each time interval.
b - Fehlberg coupling coefficients for computing the derivatives at each interior point
c4 - Fehlberg coefficients for the fourth-order solution
c5 - Fehlberg coefficients for the fifth-order solution
tol - allowable truncation error
ode_function - handle for user M-function in which the derivatives f are computed
tspan - the vector [t0 tf] giving the time interval for the solution
t0 - initial time
tf - final time
y0 - column vector of initial values of the vector y
tout - column vector of times at which y was evaluated
yout - a matrix, each row of which contains the components of y evaluated at the corresponding time in tout
h - time step
hmin - minimum allowable time step
ti - time at the beginning of a time step
yi - values of y at the beginning of a time step
t_inner - time within a given time step
y_inner - values of y within a given time step
te - truncation error for each y at a given time step
te_allowed - allowable truncation error
te_max - maximum absolute value of the components of te
ymax - maximum absolute value of the components of y
tol - relative tolerance
delta - fractional change in step size
eps - unit roundoff error (the smallest number for which $1 + \text{eps} > 1$)
eps(x) - the smallest number such that $x + \text{eps}(x) = x$

User M-function required: ode_function

```
%}
```

```
% -----
```

```
a = [0 1/4 3/8 12/13 1 1/2];
```

```

b = [    0          0          0          0          0
      1/4          0          0          0          0
      3/32        9/32         0          0          0
    1932/2197 -7200/2197  7296/2197         0          0
    439/216      -8       3680/513  -845/4104         0
    -8/27         2    -3544/2565  1859/4104  -11/40];

c4 = [25/216  0  1408/2565   2197/4104  -1/5   0  ];
c5 = [16/135  0  6656/12825  28561/56430  -9/50  2/55];

if nargin < 4
    tol = 1.e-8;
else
    tol = tolerance;
end

t0 = tspan(1);
tf = tspan(2);
t = t0;
y = y0;
tout = t;
yout = y';
h = (tf - t0)/100; % Assumed initial time step.

while t < tf
    hmin = 16*eps(t);
    ti = t;
    yi = y;
    %...Evaluate the time derivative(s) at six points within the current
    % interval:
    for i = 1:6
        t_inner = ti + a(i)*h;
        y_inner = yi;
        for j = 1:i-1
            y_inner = y_inner + h*b(i,j)*f(:,j);
        end
        f(:,i) = feval(ode_function, t_inner, y_inner);
    end

    %...Compute the maximum truncation error:
    te = h*f*(c4' - c5'); % Difference between 4th and
                          % 5th order solutions
    te_max = max(abs(te));

    %...Compute the allowable truncation error:
    ymax = max(abs(y));
    te_allowed = tol*max(ymax,1.0);

```

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```
%...Compute the fractional change in step size:
delta = (te_allowed/(te_max + eps))^(1/5);

%...If the truncation error is in bounds, then update the solution:
if te_max <= te_allowed
    h    = min(h, tf-t);
    t    = t + h;
    y    = yi + h*f*c5';
    tout = [tout;t];
    yout = [yout;y'];
end

%...Update the time step:
h = min(delta*h, 4*h);
if h < hmin
    fprintf(['\n\n Warning: Step size fell below its minimum\n'...
            ' allowable value (%g) at time %g.\n\n'], hmin, t)
    return
end
end
% ~~~~~
```

Function file: Example_1_20.m

```
% ~~~~~
function Example_1_20
% ~~~~~
%{
    This program uses RK4(5) with adaptive step size control
    to solve the differential equation


$$x'' + \mu/x^2 = 0$$


    The numerical integration is done by the function 'rkf45' which uses
    the subfunction 'rates' herein to compute the derivatives.

    x    - displacement (km)
    '    - shorthand for d/dt
    t    - time (s)
    mu   - = go*RE^2 (km^3/s^2), where go is the sea level gravitational
           acceleration and RE is the radius of the earth.
    x0   - initial value of x
    v0   = initial value of the velocity (x')
    y0   - column vector containing x0 and v0
    t0   - initial time
    tf   - final time
    tspan - a row vector with components t0 and tf
    t    - column vector of the times at which the solution is found
%}
```

```

f      - a matrix whose columns are:
        column 1: solution for x at the times in t
        column 2: solution for x' at the times in t

User M-function required:  rkf45
User subfunction required: rates
%}
% -----

clear all; close all; clc

mu      = 398600;
minutes = 60; %Conversion from minutes to seconds

x0 = 6500;
v0 = 7.8;
y0 = [x0; v0];
t0 = 0;
tf = 70*minutes;

[t,f] = rkf45(@rates, [t0 tf], y0);
plotit
return

% ~~~~~~
function dfdt = rates(t,f)
% -----
%{
    This function calculates first and second time derivatives of x
    governed by the equation of two-body rectilinear motion


$$x'' + \mu/x^2 = 0$$


    Dx   - velocity x'
    D2x  - acceleration x''
    f     - column vector containing x and Dx at time t
    dfdt - column vector containing Dx and D2x at time t

    User M-functions required: none
%}
% ~~~~~~
x      = f(1);
Dx     = f(2);
D2x    = -mu/x^2;
dfdt   = [Dx; D2x];
end %rates

```

```

% ~~~~~
function plotit
% ~~~~~

%...Position vs time:
subplot(2,1,1)
plot(t/minutes,f(:,1), '-ok')
xlabel('time, minutes')
ylabel('position, km')
grid on
axis([-inf inf 5000 15000])

%...Velocity versus time:
subplot(2,1,2)
plot(t/minutes,f(:,2), '-ok')
xlabel('time, minutes')
ylabel('velocity, km/s')
grid on
axis([-inf inf -10 10])
end %plotit

end %Example_1_20
% ~~~~~

```

Chapter 2

D.5 Algorithm 2.1: Numerical solution of the two-body problem relative to an inertial frame

Function file: twobody3d.m

```

% ~~~~~
function twobody3d
% ~~~~~
%{
    This function solves the inertial two-body problem in three dimensions
    numerically using the RK4(5) method.

    G                - universal gravitational constant (km^3/kg/s^2)
    m1,m2            - the masses of the two bodies (kg)
    m                - the total mass (kg)
    t0               - initial time (s)
    tf               - final time (s)
    Rl_0,Vl_0        - 3 by 1 column vectors containing the components of the
                      initial position (km) and velocity (km/s) of m1
%}

```

```

R2_0,V2_0    - 3 by 1 column vectors containing the components of the
               initial position (km) and velocity (km/s) of m2
y0           - 12 by 1 column vector containing the initial values
               of the state vectors of the two bodies:
               [R1_0; R2_0; V1_0; V2_0]
t            - column vector of the times at which the solution is found
X1,Y1,Z1     - column vectors containing the X,Y and Z coordinates (km)
               of m1 at the times in t
X2,Y2,Z2     - column vectors containing the X,Y and Z coordinates (km)
               of m2 at the times in t
VX1, VY1, VZ1 - column vectors containing the X,Y and Z components
               of the velocity (km/s) of m1 at the times in t
VX2, VY2, VZ2 - column vectors containing the X,Y and Z components
               of the velocity (km/s) of m2 at the times in t
y            - a matrix whose 12 columns are, respectively,
               X1,Y1,Z1; X2,Y2,Z2; VX1,VY1,VZ1; VX2,VY2,VZ2
XG,YG,ZG     - column vectors containing the X,Y and Z coordinates (km)
               the center of mass at the times in t

```

```

User M-function required:  rkf45
User subfunctions required: rates, output
%}
% -----
clc; clear all; close all
G = 6.67259e-20;

%...Input data:
m1  = 1.e26;
m2  = 1.e26;
t0  = 0;
tf  = 480;

R1_0 = [ 0; 0; 0];
R2_0 = [3000; 0; 0];

V1_0 = [ 10; 20; 30];
V2_0 = [ 0; 40; 0];
%...End input data

y0 = [R1_0; R2_0; V1_0; V2_0];

%...Integrate the equations of motion:
[t,y] = rkf45(@rates, [t0 tf], y0);

%...Output the results:
output

return

```

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```
% ~~~~~~
function dydt = rates(t,y)
% ~~~~~~
%{
    This function calculates the accelerations in Equations 2.19.

    t      - time
    y      - column vector containing the position and velocity vectors
              of the system at time t
    R1, R2 - position vectors of m1 & m2
    V1, V2 - velocity vectors of m1 & m2
    r      - magnitude of the relative position vector
    A1, A2 - acceleration vectors of m1 & m2
    dydt   - column vector containing the velocity and acceleration
              vectors of the system at time t
%}
% -----
R1 = [y(1); y(2); y(3)];
R2 = [y(4); y(5); y(6)];

V1 = [y(7); y(8); y(9)];
V2 = [y(10); y(11); y(12)];

r = norm(R2 - R1);

A1 = G*m2*(R2 - R1)/r^3;
A2 = G*m1*(R1 - R2)/r^3;

dydt = [V1; V2; A1; A2];

end %rates
% ~~~~~~

% ~~~~~~
function output
% ~~~~~~
%{
    This function calculates the trajectory of the center of mass and
    plots
    (a) the motion of m1, m2 and G relative to the inertial frame
    (b) the motion of m2 and G relative to m1
    (c) the motion of m1 and m2 relative to G

    User sub function required: common_axis_settings
%}
% -----
```

```

%...Extract the particle trajectories:
X1 = y(:,1); Y1 = y(:,2); Z1 = y(:,3);
X2 = y(:,4); Y2 = y(:,5); Z2 = y(:,6);

%...Locate the center of mass at each time step:
XG = []; YG = []; ZG = [];
for i = 1:length(t)
    XG = [XG; (m1*X1(i) + m2*X2(i))/(m1 + m2)];
    YG = [YG; (m1*Y1(i) + m2*Y2(i))/(m1 + m2)];
    ZG = [ZG; (m1*Z1(i) + m2*Z2(i))/(m1 + m2)];
end

%...Plot the trajectories:
figure (1)
title('Figure 2.3: Motion relative to the inertial frame')
hold on
plot3(X1, Y1, Z1, '-r')
plot3(X2, Y2, Z2, '-g')
plot3(XG, YG, ZG, '-b')
common_axis_settings

figure (2)
title('Figure 2.4a: Motion of m2 and G relative to m1')
hold on
plot3(X2 - X1, Y2 - Y1, Z2 - Z1, '-g')
plot3(XG - X1, YG - Y1, ZG - Z1, '-b')
common_axis_settings

figure (3)
title('Figure 2.4b: Motion of m1 and m2 relative to G')
hold on
plot3(X1 - XG, Y1 - YG, Z1 - ZG, '-r')
plot3(X2 - XG, Y2 - YG, Z2 - ZG, '-g')
common_axis_settings

% ~~~~~~
function common_axis_settings
% ~~~~~~
%{
    This function establishes axis properties common to the several plots.
%}
% -----
text(0, 0, 0, 'o')
axis('equal')
view([2,4,1.2])
grid on
axis equal
xlabel('X (km)')

```

```

ylabel('Y (km)')
xlabel('Z (km)')
end %common_axis_settings

end %output

end %twobody3d
% ~~~~~

```

D.6 Algorithm 2.2: Numerical solution of the two-body relative motion problem

Function file: orbit.m

```

% ~~~~~
function orbit
% ~~~~~
%{
    This function computes the orbit of a spacecraft by using rkf45 to
    numerically integrate Equation 2.22.

    It also plots the orbit and computes the times at which the maximum
    and minimum radii occur and the speeds at those times.

    hours      - converts hours to seconds
    G          - universal gravitational constant (km^3/kg/s^2)
    m1         - planet mass (kg)
    m2         - spacecraft mass (kg)
    mu         - gravitational parameter (km^3/s^2)
    R          - planet radius (km)
    r0         - initial position vector (km)
    v0         - initial velocity vector (km/s)
    t0,tf      - initial and final times (s)
    y0         - column vector containing r0 and v0
    t          - column vector of the times at which the solution is found
    y          - a matrix whose columns are:
                columns 1, 2 and 3:
                    The solution for the x, y and z components of the
                    position vector r at the times in t
                columns 4, 5 and 6:
                    The solution for the x, y and z components of the
                    velocity vector v at the times in t
    r          - magnitude of the position vector at the times in t
    imax       - component of r with the largest value
    rmax       - largest value of r
    imin       - component of r with the smallest value
%}

```

```

rmin      - smallest value of r
v_at_rmax - speed where r = rmax
v_at_rmin - speed where r = rmin

User M-function required:  rkf45
User subfunctions required: rates, output
%}
% -----

clc; close all; clear all

hours = 3600;
G      = 6.6742e-20;

%...Input data:
%  Earth:
m1 = 5.974e24;
R  = 6378;
m2 = 1000;

r0 = [8000 0 6000];
v0 = [0 7 0];

t0 = 0;
tf = 4*hours;
%...End input data

%...Numerical integration:
mu  = G*(m1 + m2);
y0  = [r0 v0]';
[t,y] = rkf45(@rates, [t0 tf], y0);

%...Output the results:
output

return

% ~~~~~
function dydt = rates(t,f)
% ~~~~~
%{
    This function calculates the acceleration vector using Equation 2.22.

    t          - time
    f          - column vector containing the position vector and the
                  velocity vector at time t
    x, y, z    - components of the position vector r

```

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```
r          - the magnitude of the position vector
vx, vy, vz - components of the velocity vector v
ax, ay, az - components of the acceleration vector a
dydt       - column vector containing the velocity and acceleration
             components
%}
% -----
x  = f(1);
y  = f(2);
z  = f(3);
vx = f(4);
vy = f(5);
vz = f(6);

r  = norm([x y z]);

ax = -mu*x/r^3;
ay = -mu*y/r^3;
az = -mu*z/r^3;

dydt = [vx vy vz ax ay az]';
end %rates

% ~~~~~
function output
% ~~~~~
%{
    This function computes the maximum and minimum radii, the times they
    occur and the speed at those times. It prints those results to
    the command window and plots the orbit.

    r          - magnitude of the position vector at the times in t
    imax       - the component of r with the largest value
    rmax       - the largest value of r
    imin       - the component of r with the smallest value
    rmin       - the smallest value of r
    v_at_rmax  - the speed where r = rmax
    v_at_rmin  - the speed where r = rmin

    User subfunction required: light_gray
%}
% -----
for i = 1:length(t)
    r(i) = norm([y(i,1) y(i,2) y(i,3)]);
end

[rmax imax] = max(r);
```

```

[rmin imin] = min(r);

v_at_rmax = norm([y(imax,4) y(imax,5) y(imax,6)]);
v_at_rmin = norm([y(imin,4) y(imin,5) y(imin,6)]);

%...Output to the command window:
fprintf('\n\n-----\n')
fprintf('\n Earth Orbit\n')
fprintf(' %s\n', datestr(now))
fprintf('\n The initial position is [%g, %g, %g] (km).',...
        r0(1), r0(2), r0(3))
fprintf('\n   Magnitude = %g km\n', norm(r0))
fprintf('\n The initial velocity is [%g, %g, %g] (km/s).',...
        v0(1), v0(2), v0(3))
fprintf('\n   Magnitude = %g km/s\n', norm(v0))
fprintf('\n Initial time = %g h.\n Final time   = %g h.\n',0,tf/hours)
fprintf('\n The minimum altitude is %g km at time = %g h.',...
        rmin-R, t(imin)/hours)
fprintf('\n The speed at that point is %g km/s.\n', v_at_rmin)
fprintf('\n The maximum altitude is %g km at time = %g h.',...
        rmax-R, t(imax)/hours)
fprintf('\n The speed at that point is %g km/s\n', v_at_rmax)
fprintf('\n-----\n\n')

%...Plot the results:
%   Draw the planet
[xx, yy, zz] = sphere(100);
surf(R*xx, R*yy, R*zz)
colormap(light_gray)
caxis([-R/100 R/100])
shading interp

%   Draw and label the X, Y and Z axes
line([0 2*R], [0 0], [0 0]); text(2*R, 0, 0, 'X')
line([0 0], [0 2*R], [0 0]); text(0, 2*R, 0, 'Y')
line([0 0], [0 0], [0 2*R]); text(0, 0, 2*R, 'Z')

%   Plot the orbit, draw a radial to the starting point
%   and label the starting point (o) and the final point (f)
hold on
plot3( y(:,1), y(:,2), y(:,3),'k')
line([0 r0(1)], [0 r0(2)], [0 r0(3)])
text( y(1,1), y(1,2), y(1,3), 'o')
text( y(end,1), y(end,2), y(end,3), 'f')

%   Select a view direction (a vector directed outward from the origin)
view([1,1,.4])

```

```

% Specify some properties of the graph
grid on
axis equal
xlabel('km')
ylabel('km')
zlabel('km')

% ~~~~~~
function map = light_gray
% ~~~~~~
%{
    This function creates a color map for displaying the planet as light
    gray with a black equator.

    r - fraction of red
    g - fraction of green
    b - fraction of blue

%}
% -----
r = 0.8; g = r; b = r;
map = [r g b
       0 0 0
       r g b];
end %light_gray

end %output

end %orbit
% ~~~~~~

```

D.7 Calculation of the Lagrange f and g functions and their time derivatives in terms of change in true anomaly

Function file: f_and_g_ta.m

```

% ~~~~~~
function [f, g] = f_and_g_ta(r0, v0, dt, mu)
% ~~~~~~
%{
    This function calculates the Lagrange  $f$  and  $g$  coefficients from the
    change in true anomaly since time  $t_0$ .

    mu - gravitational parameter ( $\text{km}^3/\text{s}^2$ )
    dt - change in true anomaly (degrees)
    r0 - position vector at time  $t_0$  (km)

```

```

v0 - velocity vector at time t0 (km/s)
h  - angular momentum (km^2/s)
vr0 - radial component of v0 (km/s)
r  - radial position after the change in true anomaly
f  - the Lagrange f coefficient (dimensionless)
g  - the Lagrange g coefficient (s)

User M-functions required:  None
%}
% -----

h  = norm(cross(r0,v0));
vr0 = dot(v0,r0)/norm(r0);
r0  = norm(r0);
s  = sind(dt);
c  = cosd(dt);

%...Equation 2.152:
r  = h^2/mu/(1 + (h^2/mu/r0 - 1)*c - h*vr0*s/mu);

%...Equations 2.158a & b:
f  = 1 - mu*r*(1 - c)/h^2;
g  = r*r0*s/h;

end
% ~~~~~

```

Function file: fDot_and_gDot_ta.m

```

% ~~~~~
function [fdot, gdot] = fDot_and_gDot_ta(r0, v0, dt, mu)
% ~~~~~
%{
This function calculates the time derivatives of the Lagrange
f and g coefficients from the change in true anomaly since time t0.

mu  - gravitational parameter (km^3/s^2)
dt  - change in true anomaly (degrees)
r0  - position vector at time t0 (km)
v0  - velocity vector at time t0 (km/s)
h   - angular momentum (km^2/s)
vr0 - radial component of v0 (km/s)
fdot - time derivative of the Lagrange f coefficient (1/s)
gdot - time derivative of the Lagrange g coefficient (dimensionless)

User M-functions required:  None
%}

```

```
% -----

h = norm(cross(r0,v0));
vr0 = dot(v0,r0)/norm(r0);
r0 = norm(r0);
c = cosd(dt);
s = sind(dt);

%...Equations 2.158c & d:
fdot = mu/h*(vr0/h*(1 - c) - s/r0);
gdot = 1 - mu*r0/h^2*(1 - c);

end
% ~~~~~
```

D.8 Algorithm 2.3: Calculate the state vector from the initial state vector and the change in true anomaly

Function file: rv_from_r0v0_ta.m

```
% ~~~~~
function [r,v] = rv_from_r0v0_ta(r0, v0, dt, mu)
% ~~~~~
%{
    This function computes the state vector (r,v) from the
    initial state vector (r0,v0) and the change in true anomaly.

    mu - gravitational parameter (km^3/s^2)
    r0 - initial position vector (km)
    v0 - initial velocity vector (km/s)
    dt - change in true anomaly (degrees)
    r - final position vector (km)
    v - final velocity vector (km/s)

    User M-functions required: f_and_g_ta, fDot_and_gDot_ta
%}
% -----

%global mu

%...Compute the f and g functions and their derivatives:
[f, g] = f_and_g_ta(r0, v0, dt, mu);
[fdot, gdot] = fDot_and_gDot_ta(r0, v0, dt, mu);

%...Compute the final position and velocity vectors:
r = f*r0 + g*v0;
```

```
v = fdot*r0 + gdot*v0;
```

```
end
```

```
% ~~~~~
```

Script file: Example_2_13.m

```
% ~~~~~
```

```
% Example_2_13
```

```
% ~~~~~
```

```
%{
```

This program computes the state vector [R,V] from the initial state vector [R0,V0] and the change in true anomaly, using the data in Example 2.13.

mu - gravitational parameter (km³/s²)

R0 - the initial position vector (km)

V0 - the initial velocity vector (km/s)

r0 - magnitude of R0

v0 - magnitude of V0

R - final position vector (km)

V - final velocity vector (km/s)

r - magnitude of R

v - magnitude of V

dt - change in true anomaly (degrees)

User M-functions required: rv_from_r0v0_ta

```
%}
```

```
% -----
```

```
clear all; clc
```

```
mu = 398600;
```

```
%...Input data:
```

```
R0 = [8182.4 -6865.9 0];
```

```
V0 = [0.47572 8.8116 0];
```

```
dt = 120;
```

```
%...End input data
```

```
%...Algorithm 2.3:
```

```
[R,V] = rv_from_r0v0_ta(R0, V0, dt, mu);
```

```
r = norm(R);
```

```
v = norm(V);
```

```
r0 = norm(R0);
```

```
v0 = norm(V0);
```

```

fprintf('-----')
fprintf('\n Example 2.9 \n')
fprintf('\n Initial state vector:\n')
fprintf('\n   r = [%g, %g, %g] (km)', R0(1), R0(2), R0(3))
fprintf('\n       magnitude = %g\n', norm(R0))

fprintf('\n   v = [%g, %g, %g] (km/s)', V0(1), V0(2), V0(3))
fprintf('\n       magnitude = %g', norm(V0))

fprintf('\n\n State vector after %g degree change in true anomaly:\n', dt)
fprintf('\n   r = [%g, %g, %g] (km)', R(1), R(2), R(3))
fprintf('\n       magnitude = %g\n', norm(R))
fprintf('\n   v = [%g, %g, %g] (km/s)', V(1), V(2), V(3))
fprintf('\n       magnitude = %g', norm(V))
fprintf('\n-----\n')
% ~~~~~

```

***Output from* Example_2_13.m**

Example 2.9

Initial state vector:

```

r = [8182.4, -6865.9, 0] (km)
    magnitude = 10681.4

```

```

v = [0.47572, 8.8116, 0] (km/s)
    magnitude = 8.82443

```

State vector after 120 degree change in true anomaly:

```

r = [1454.99, 8251.47, 0] (km)
    magnitude = 8378.77

```

```

v = [-8.13238, 5.67854, -0] (km/s)
    magnitude = 9.91874

```

D.9 Algorithm 2.4: Find the root of a function using the bisection method

Function file: bisect.m

```

% ~~~~~
function root = bisect(fun, xl, xu)
% ~~~~~
%{

```

This function evaluates a root of a function using the bisection method.

```
tol - error to within which the root is computed
n   - number of iterations
xl  - low end of the interval containing the root
xu  - upper end of the interval containing the root
i   - loop index
xm  - mid-point of the interval from xl to xu
fun - name of the function whose root is being found
fxl - value of fun at xl
fxm - value of fun at xm
root - the computed root
```

User M-functions required: none

```
%}
% -----

tol = 1.e-6;
n   = ceil(log(abs(xu - xl)/tol)/log(2));

for i = 1:n
    xm = (xl + xu)/2;
    fxl = feval(fun, xl);
    fxm = feval(fun, xm);
    if fxl*fxm > 0
        xl = xm;
    else
        xu = xm;
    end
end

root = xm;

end
% ~~~~~
```

Function file: Example_2_16.m

```
% ~~~~~
function Example_2_16
% ~~~~~
%{
    This program uses the bisection method to find the three roots of
    Equation 2.204 for the earth-moon system.

    m1 - mass of the earth (kg)
```

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```
m2 - mass of the moon (kg)
r12 - distance from the earth to the moon (km)
p - ratio of moon mass to total mass
x1 - vector containing the low-side estimates of the three roots
xu - vector containing the high-side estimates of the three roots
x - vector containing the three computed roots

User M-function required: bisect
User subfunction required: fun
%}
% -----

clear all; clc

%...Input data:
m1 = 5.974e24;
m2 = 7.348e22;
r12 = 3.844e5;

x1 = [-1.1 0.5 1.0];
xu = [-0.9 1.0 1.5];
%...End input data

p = m2/(m1 + m2);

for i = 1:3
    x(i) = bisect(@fun, x1(i), xu(i));
end

%...Output the results
output

return

% ~~~~~
function f = fun(z)
% -----
%{
    This subroutine evaluates the function in Equation 2.204.

    z - the dimensionless x-coordinate
    p - defined above
    f - the value of the function

%}
% ~~~~~
f = (1 - p)*(z + p)/abs(z + p)^3 + p*(z + p - 1)/abs(z + p - 1)^3 - z;
end %fun
```

```
% ~~~~~~
function output
% ~~~~~~
%{
    This function prints out the x-coordinates of L1, L2 and L3
    relative to the center of mass.
%}
%...Output to the command window:
fprintf('\n\n-----\n')
fprintf('\n For\n')
fprintf('\n    m1 = %g kg', m1)
fprintf('\n    m2 = %g kg', m2)
fprintf('\n    r12 = %g km\n', r12)
fprintf('\n the 3 colinear Lagrange points (the roots of\n')
fprintf(' Equation 2.204) are:\n')
fprintf('\n L3: x = %10g km    (f(x3) = %g)', x(1)*r12, fun(x(1)))
fprintf('\n L1: x = %10g km    (f(x1) = %g)', x(2)*r12, fun(x(2)))
fprintf('\n L2: x = %10g km    (f(x2) = %g)', x(3)*r12, fun(x(3)))
fprintf('\n\n-----\n')

end %output

end %Example_2_16
% ~~~~~~
```

Output from Example_2_16.m

For

```
    m1 = 5.974e+24 kg
    m2 = 7.348e+22 kg
    r12 = 384400 km
```

the 3 colinear Lagrange points (the roots of
Equation 2.204) are:

```
L3: x =    -386346 km    (f(x3) = -1.55107e-06)
L1: x =     321710 km    (f(x1) = 5.12967e-06)
L2: x =     444244 km    (f(x2) = -4.92782e-06)
```

D.10 MATLAB solution of Example 2.18

Function file: Example_2_18.m

```
% ~~~~~~
function Example_2_18
% ~~~~~~
%{
    This program uses the Runge-Kutta-Fehlberg 4(5) method to solve the
    earth-moon restricted three-body problem (Equations 2.192a and 2.192b)
    for the trajectory of a spacecraft having the initial conditions
    specified in Example 2.18.

    The numerical integration is done in the external function 'rkf45',
    which uses the subfunction 'rates' herein to compute the derivatives.

    days      - converts days to seconds
    G          - universal gravitational constant (km^3/kg/s^2)
    rmoon      - radius of the moon (km)
    rearth     - radius of the earth (km)
    r12        - distance from center of earth to center of moon (km)
    m1,m2      - masses of the earth and of the moon, respectively (kg)
    M          - total mass of the restricted 3-body system (kg)
    mu         - gravitational parameter of earth-moon system (km^3/s^2)
    mu1,mu2    - gravitational parameters of the earth and of the moon,
                 respectively (km^3/s^2)
    pi_1,pi_2  - ratios of the earth mass and the moon mass, respectively,
                 to the total earth-moon mass
    W          - angular velocity of moon around the earth (rad/s)
    x1,x2      - x-coordinates of the earth and of the moon, respectively,
                 relative to the earth-moon barycenter (km)
    d0         - initial altitude of spacecraft (km)
    phi        - polar azimuth coordinate (degrees) of the spacecraft
                 measured positive counterclockwise from the earth-moon line
    v0         - initial speed of spacecraft relative to rotating earth-moon
                 system (km/s)
    gamma      - initial flight path angle (degrees)
    r0         - initial radial distance of spacecraft from the earth (km)
    x,y        - x and y coordinates of spacecraft in rotating earth-moon
                 system (km)
    vx,vy      - x and y components of spacecraft velocity relative to
                 rotating earth-moon system (km/s)
    f0         - column vector containing the initial values of x, y, vx and vy
    t0,tf      - initial time and final times (s)
    t          - column vector of times at which the solution was computed
    f          - a matrix whose columns are:
                 column 1: solution for x at the times in t
```

```

        column 2: solution for y  at the times in t
        column 3: solution for vx at the times in t
        column 4: solution for vy at the times in t
    xf,yf      - x and y coordinates of spacecraft in rotating earth-moon
                  system at tf
    vxf, vyf   - x and y components of spacecraft velocity relative to
                  rotating earth-moon system at tf
    df         - distance from surface of the moon at tf
    vf         - relative speed at tf

    User M-functions required: rkf45
    User subfunctions required: rates, circle
%}
% -----

clear all; close all; clc

days    = 24*3600;
G        = 6.6742e-20;
rmoon    = 1737;
rearth   = 6378;
r12      = 384400;
m1       = 5974e21;
m2       = 7348e19;

M        = m1 + m2;;
pi_1     = m1/M;
pi_2     = m2/M;

mu1      = 398600;
mu2      = 4903.02;
mu       = mu1 + mu2;

W        = sqrt(mu/r12^3);
x1       = -pi_2*r12;
x2       = pi_1*r12;

%...Input data:
d0       = 200;
phi      = -90;
v0       = 10.9148;
gamma    = 20;
t0       = 0;
tf       = 3.16689*days;

r0       = rearth + d0;
x        = r0*cosd(phi) + x1;

```

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```
y      = r0*sind(phi);

vx      = v0*(sind(gamma)*cosd(phi) - cosd(gamma)*sind(phi));
vy      = v0*(sind(gamma)*sind(phi) + cosd(gamma)*cosd(phi));
f0      = [x; y; vx; vy];

%...Compute the trajectory:
[t,f] = rkf45(@rates, [t0 tf], f0);
x      = f(:,1);
y      = f(:,2);
vx      = f(:,3);
vy      = f(:,4);

xf      = x(end);
yf      = y(end);

vxf      = vx(end);
vyf      = vy(end);

df      = norm([xf - x2, yf - 0]) - rmoon;
vf      = norm([vxf, vyf]);

%...Output the results:
output
return

% ~~~~~~
function dfdt = rates(t,f)
% ~~~~~~
%{
    This subfunction calculates the components of the relative acceleration
    for the restricted 3-body problem, using Equations 2.192a and 2.192b.

    ax,ay - x and y components of relative acceleration (km/s^2)
    r1     - spacecraft distance from the earth (km)
    r2     - spacecraft distance from the moon (km)
    f      - column vector containing x, y, vx and vy at time t
    dfdt   - column vector containing vx, vy, ax and ay at time t

    All other variables are defined above.

    User M-functions required: none
%}
% -----
x      = f(1);
y      = f(2);
vx      = f(3);
vy      = f(4);
```

```

r1    = norm([x + pi_2*r12, y]);
r2    = norm([x - pi_1*r12, y]);

ax    = 2*W*vy + W^2*x - mu1*(x - x1)/r1^3 - mu2*(x - x2)/r2^3;
ay    = -2*W*vx + W^2*y - (mu1/r1^3 + mu2/r2^3)*y;

dfdt  = [vx; vy; ax; ay];
end %rates

% ~~~~~~
function output
% ~~~~~~
%{
    This subfunction echoes the input data and prints the results to the
    command window. It also plots the trajectory.

    User M-functions required: none
    User subfunction required: circle
%}
% -----

fprintf('-----')
fprintf('\n Example 2.18: Lunar trajectory using the restricted')
fprintf('\n three body equations.\n')
fprintf('\n Initial Earth altitude (km)           = %g', d0)
fprintf('\n Initial angle between radial')
fprintf('\n    and earth-moon line (degrees)           = %g', phi)
fprintf('\n Initial flight path angle (degrees) = %g', gamma)
fprintf('\n Flight time (days)                       = %g', tf/days)
fprintf('\n Final distance from the moon (km)         = %g', df)
fprintf('\n Final relative speed (km/s)               = %g', vf)
fprintf('\n-----\n')

%...Plot the trajectory and place filled circles representing the earth
%    and moon on the plot:
plot(x, y)
%    Set plot display parameters
xmin = -20.e3; xmax = 4.e5;
ymin = -20.e3; ymax = 1.e5;
axis([xmin xmax ymin ymax])
axis equal
xlabel('x, km'); ylabel('y, km')
grid on
hold on

%...Plot the earth (blue) and moon (green) to scale
earth = circle(x1, 0, rearth);
moon  = circle(x2, 0, rmoon);

```

```

fill(earth(:,1), earth(:,2),'b')
fill( moon(:,1),  moon(:,2),'g')

% ~~~~~~
function xy = circle(xc, yc, radius)
% ~~~~~~
%{
    This subfunction calculates the coordinates of points spaced
    0.1 degree apart around the circumference of a circle.

    x,y      - x and y coordinates of a point on the circumference
    xc,yc    - x and y coordinates of the center of the circle
    radius    - radius of the circle
    xy       - an array containing the x coordinates in column 1 and the
               y coordinates in column 2

    User M-functions required: none
%}
% -----
x      = xc + radius*cosd(0:0.1:360);
y      = yc + radius*sind(0:0.1:360);
xy     = [x', y'];

end %circle

end %output

end %Example_2_18
% ~~~~~~

```

Output from Example_2_18.m

Example 2.18: Lunar trajectory using the restricted
three body equations.

Initial Earth altitude (km)	= 200
Initial angle between radial and earth-moon line (degrees)	= -90
Initial flight path angle (degrees)	= 20
Flight time (days)	= 3.16689
Final distance from the moon (km)	= 255.812
Final relative speed (km/s)	= 2.41494

Chapter 3

D.11 Algorithm 3.1: Solution of Kepler's equation by Newton's method

Function file: kepler_E.m

```
% ~~~~~~
function E = kepler_E(e, M)
% ~~~~~~
%{
    This function uses Newton's method to solve Kepler's
    equation  $E - e \sin(E) = M$  for the eccentric anomaly,
    given the eccentricity and the mean anomaly.

    E - eccentric anomaly (radians)
    e - eccentricity, passed from the calling program
    M - mean anomaly (radians), passed from the calling program
    pi - 3.1415926...

    User m-functions required: none
%}
% -----

%...Set an error tolerance:
error = 1.e-8;

%...Select a starting value for E:
if M < pi
    E = M + e/2;
else
    E = M - e/2;
end

%...Iterate on Equation 3.17 until E is determined to within
%...the error tolerance:
ratio = 1;
while abs(ratio) > error
    ratio = (E - e*sin(E) - M)/(1 - e*cos(E));
    E = E - ratio;
end

end %kepler_E
% ~~~~~~
```

Script file: Example_3_02.m

```
% ~~~~~
% Example_3_02
% ~~~~~
%{
    This program uses Algorithm 3.1 and the data of Example 3.2 to solve
    Kepler's equation.

    e - eccentricity
    M - mean anomaly (rad)
    E - eccentric anomaly (rad)

    User M-function required: kepler_E
%}
% -----

clear all; clc

%...Data declaration for Example 3.2:
e = 0.37255;
M = 3.6029;
%...

%...Pass the input data to the function kepler_E, which returns E:
E = kepler_E(e, M);

%...Echo the input data and output to the command window:
fprintf('-----\n')
fprintf('\n Example 3.2\n')
fprintf('\n Eccentricity                = %g',e)
fprintf('\n Mean anomaly (radians)          = %g\n',M)
fprintf('\n Eccentric anomaly (radians) = %g',E)
fprintf('\n-----\n')

% ~~~~~
```

Output from Example_3_02.m

```
-----
Example 3.2
```

```
Eccentricity                = 0.37255
Mean anomaly (radians)      = 3.6029
```

```
Eccentric anomaly (radians) = 3.47942
-----
```

D.12 Algorithm 3.2: Solution of Kepler's equation for the hyperbola using Newton's method

Function file: kepler_H.m

```
% ~~~~~~
function F = kepler_H(e, M)
% ~~~~~~
%{
    This function uses Newton's method to solve Kepler's equation
    for the hyperbola  $e \sinh(F) - F = M$  for the hyperbolic
    eccentric anomaly, given the eccentricity and the hyperbolic
    mean anomaly.

    F - hyperbolic eccentric anomaly (radians)
    e - eccentricity, passed from the calling program
    M - hyperbolic mean anomaly (radians), passed from the
        calling program

    User M-functions required: none
%}
% -----

%...Set an error tolerance:
error = 1.e-8;

%...Starting value for F:
F = M;

%...Iterate on Equation 3.45 until F is determined to within
%...the error tolerance:
ratio = 1;
while abs(ratio) > error
    ratio = (e*sinh(F) - F - M)/(e*cosh(F) - 1);
    F = F - ratio;
end

end %kepler_H
% ~~~~~~
```

Script file: : Example_3_05.m

```
% ~~~~~~
% Example_3_05
% ~~~~~~
%{
    This program uses Algorithm 3.2 and the data of
```

Example 3.5 to solve Kepler's equation for the hyperbola.

e - eccentricity
M - hyperbolic mean anomaly (dimensionless)
F - hyperbolic eccentric anomaly (dimensionless)

User M-function required: kepler_H

```
%}
% -----
clear

%...Data declaration for Example 3.5:
e = 2.7696;
M = 40.69;
%...

%...Pass the input data to the function kepler_H, which returns F:
F = kepler_H(e, M);

%...Echo the input data and output to the command window:
fprintf('-----\n')
fprintf('\n Example 3.5\n')
fprintf('\n Eccentricity           = %g',e)
fprintf('\n Hyperbolic mean anomaly      = %g\n',M)
fprintf('\n Hyperbolic eccentric anomaly = %g',F)
fprintf('\n-----\n')

% ~~~~~
```

Output from Example_3_05.m

```
-----
Example 3.5

Eccentricity           = 2.7696
Hyperbolic mean anomaly = 40.69

Hyperbolic eccentric anomaly = 3.46309
-----
```

D.13 Calculation of the Stumpff functions S(z) and C(z)

The following scripts implement Equations 3.52 and 3.53 for use in other programs.

Function file: stumpS.m

```
% ~~~~~
function s = stumpS(z)
```

```
% ~~~~~
%{
    This function evaluates the Stumpff function S(z) according
    to Equation 3.52.

    z - input argument
    s - value of S(z)

    User M-functions required: none
%}
% -----

if z > 0
    s = (sqrt(z) - sin(sqrt(z)))/(sqrt(z))^3;
elseif z < 0
    s = (sinh(sqrt(-z)) - sqrt(-z))/(sqrt(-z))^3;
else
    s = 1/6;
end
% ~~~~~
```

Function file: stumpC.m

```
% ~~~~~
function c = stumpC(z)
% ~~~~~
%{
    This function evaluates the Stumpff function C(z) according
    to Equation 3.53.

    z - input argument
    c - value of C(z)

    User M-functions required: none
%}
% -----

if z > 0
    c = (1 - cos(sqrt(z)))/z;
elseif z < 0
    c = (cosh(sqrt(-z)) - 1)/(-z);
else
    c = 1/2;
end
% ~~~~~
```

D.14 Algorithm 3.3: Solution of the universal Kepler's equation using Newton's method

Function file: kepler_U.m

```
% ~~~~~~
function x = kepler_U(dt, ro, vro, a)
% ~~~~~~
%{
    This function uses Newton's method to solve the universal
    Kepler equation for the universal anomaly.

    mu    - gravitational parameter (km^3/s^2)
    x      - the universal anomaly (km^0.5)
    dt     - time since x = 0 (s)
    ro     - radial position (km) when x = 0
    vro    - radial velocity (km/s) when x = 0
    a      - reciprocal of the semimajor axis (1/km)
    z      - auxiliary variable (z = a*x^2)
    C      - value of Stumpff function C(z)
    S      - value of Stumpff function S(z)
    n      - number of iterations for convergence
    nMax   - maximum allowable number of iterations

    User M-functions required: stumpC, stumpS
%}
% -----
global mu

%...Set an error tolerance and a limit on the number of iterations:
error = 1.e-8;
nMax = 1000;

%...Starting value for x:
x = sqrt(mu)*abs(a)*dt;

%...Iterate on Equation 3.65 until convergence occurs within
%...the error tolerance:
n = 0;
ratio = 1;
while abs(ratio) > error && n <= nMax
    n = n + 1;
    C = stumpC(a*x^2);
    S = stumpS(a*x^2);
    F = ro*vro/sqrt(mu)*x^2*C + (1 - a*ro)*x^3*S + ro*x - sqrt(mu)*dt;
    dFdx = ro*vro/sqrt(mu)*x*(1 - a*x^2*S) + (1 - a*ro)*x^2*C + ro;
    ratio = F/dFdx;
```

```

        x      = x - ratio;
    end

    %...Deliver a value for x, but report that nMax was reached:
    if n > nMax
        fprintf('\n **No. iterations of Kepler's equation = %g', n)
        fprintf('\n      F/dFdx                      = %g\n', F/dFdx)
    end
    % ~~~~~

```

Script file: Example_3_06.m

```

% ~~~~~
% Example_3_06
% ~~~~~
%{
    This program uses Algorithm 3.3 and the data of Example 3.6
    to solve the universal Kepler's equation.

    mu - gravitational parameter (km^3/s^2)
    x  - the universal anomaly (km^0.5)
    dt - time since x = 0 (s)
    ro - radial position when x = 0 (km)
    vro - radial velocity when x = 0 (km/s)
    a  - semimajor axis (km)

    User M-function required: kepler_U
%}
% -----

clear all; clc
global mu
mu = 398600;

%...Data declaration for Example 3.6:
ro = 10000;
vro = 3.0752;
dt = 3600;
a = -19655;
%...

%...Pass the input data to the function kepler_U, which returns x
%...(Universal Kepler's requires the reciprocal of semimajor axis):
x = kepler_U(dt, ro, vro, 1/a);

%...Echo the input data and output the results to the command window:
fprintf('-----')

```

```

fprintf('\n Example 3.6\n')
fprintf('\n Initial radial coordinate (km) = %g',ro)
fprintf('\n Initial radial velocity (km/s) = %g',vro)
fprintf('\n Elapsed time (seconds)          = %g',dt)
fprintf('\n Semimajor axis (km)                = %g\n',a)
fprintf('\n Universal anomaly (km^0.5)         = %g',x)
fprintf('\n-----\n')
% ~~~~~

```

Output from Example_3_06.m

```

-----
Example 3.6

Initial radial coordinate (km) = 10000
Initial radial velocity (km/s) = 3.0752
Elapsed time (seconds)        = 3600
Semimajor axis (km)           = -19655

Universal anomaly (km^0.5)    = 128.511
-----

```

D.15 Calculation of the Lagrange coefficients f and g and their time derivatives in terms of change in universal anomaly

The following scripts implement Equations 3.69 for use in other programs.

Function file: f_and_g.m

```

% ~~~~~
function [f, g] = f_and_g(x, t, ro, a)
% ~~~~~
%{
    This function calculates the Lagrange f and g coefficients.

    mu - the gravitational parameter (km^3/s^2)
    a   - reciprocal of the semimajor axis (1/km)
    ro  - the radial position at time to (km)
    t   - the time elapsed since ro (s)
    x   - the universal anomaly after time t (km^0.5)
    f   - the Lagrange f coefficient (dimensionless)
    g   - the Lagrange g coefficient (s)

    User M-functions required: stumpC, stumpS
%}
% -----

```

```

global mu

z = a*x^2;

%...Equation 3.69a:
f = 1 - x^2/ro*stumpC(z);

%...Equation 3.69b:
g = t - 1/sqrt(mu)*x^3*stumpS(z);

end
% ~~~~~

```

Function file: fDot_and_gDot.m

```

% ~~~~~
function [fdot, gdot] = fDot_and_gDot(x, r, ro, a)
% ~~~~~
%{
    This function calculates the time derivatives of the
    Lagrange f and g coefficients.

    mu    - the gravitational parameter (km^3/s^2)
    a      - reciprocal of the semimajor axis (1/km)
    ro     - the radial position at time to (km)
    t      - the time elapsed since initial state vector (s)
    r      - the radial position after time t (km)
    x      - the universal anomaly after time t (km^0.5)
    fdot   - time derivative of the Lagrange f coefficient (1/s)
    gdot   - time derivative of the Lagrange g coefficient (dimensionless)

    User M-functions required:  stumpC, stumpS
%}
% -----

global mu

z = a*x^2;

%...Equation 3.69c:
fdot = sqrt(mu)/r/ro*(z*stumpS(z) - 1)*x;

%...Equation 3.69d:
gdot = 1 - x^2/r*stumpC(z);
% ~~~~~

```

D.16 Algorithm 3.4: Calculation of the state vector given the initial state vector and the time lapse Δt

Function file: rv_from_r0v0.m

```
% ~~~~~
function [R,V] = rv_from_r0v0(R0, V0, t)
% ~~~~~
%{
    This function computes the state vector (R,V) from the
    initial state vector (R0,V0) and the elapsed time.

    mu - gravitational parameter (km^3/s^2)
    R0 - initial position vector (km)
    V0 - initial velocity vector (km/s)
    t   - elapsed time (s)
    R   - final position vector (km)
    V   - final velocity vector (km/s)

% User M-functions required: kepler_U, f_and_g, fDot_and_gDot
%}
% -----

global mu

%...Magnitudes of R0 and V0:
r0 = norm(R0);
v0 = norm(V0);

%...Initial radial velocity:
vr0 = dot(R0, V0)/r0;

%...Reciprocal of the semimajor axis (from the energy equation):
alpha = 2/r0 - v0^2/mu;

%...Compute the universal anomaly:
x = kepler_U(t, r0, vr0, alpha);

%...Compute the f and g functions:
[f, g] = f_and_g(x, t, r0, alpha);

%...Compute the final position vector:
R = f*R0 + g*V0;

%...Compute the magnitude of R:
r = norm(R);
```

```
%...Compute the derivatives of f and g:
[fdot, gdot] = fDot_and_gDot(x, r, r0, alpha);

%...Compute the final velocity:
V          = fdot*R0 + gdot*V0;
% ~~~~~
```

Script file: Example_3_07.m

```
% ~~~~~
% Example_3_07
% ~~~~~
%
% This program computes the state vector (R,V) from the initial
% state vector (R0,V0) and the elapsed time using the data in
% Example 3.7.
%
% mu - gravitational parameter (km^3/s^2)
% R0 - the initial position vector (km)
% V0 - the initial velocity vector (km/s)
% R - the final position vector (km)
% V - the final velocity vector (km/s)
% t - elapsed time (s)
%
% User m-functions required: rv_from_r0v0
% -----

clear all; clc
global mu
mu = 398600;

%...Data declaration for Example 3.7:
R0 = [ 7000 -12124 0];
V0 = [2.6679 4.6210 0];
t = 3600;
%...

%...Algorithm 3.4:
[R V] = rv_from_r0v0(R0, V0, t);

%...Echo the input data and output the results to the command window:
fprintf('-----')
fprintf('\n Example 3.7\n')
fprintf('\n Initial position vector (km):')
fprintf('\n   r0 = (%g, %g, %g)\n', R0(1), R0(2), R0(3))
fprintf('\n Initial velocity vector (km/s):')
fprintf('\n   v0 = (%g, %g, %g)', V0(1), V0(2), V0(3))
```

```

fprintf('\n\n Elapsed time = %g s\n',t)
fprintf('\n Final position vector (km):')
fprintf('\n   r = (%g, %g, %g)\n', R(1), R(2), R(3))
fprintf('\n Final velocity vector (km/s):')
fprintf('\n   v = (%g, %g, %g)', V(1), V(2), V(3))
fprintf('\n-----\n')
% ~~~~~

```

Output from Example_3_07

Example 3.7

Initial position vector (km):
 r0 = (7000, -12124, 0)

Initial velocity vector (km/s):
 v0 = (2.6679, 4.621, 0)

Elapsed time = 3600 s

Final position vector (km):
 r = (-3297.77, 7413.4, 0)

Final velocity vector (km/s):
 v = (-8.2976, -0.964045, -0)

Chapter 4

D.17 Algorithm 4.1: Obtain the right ascension and declination from the position vector

Function file: ra_and_dec_from_r.m

```

% ~~~~~
function [ra dec] = ra_and_dec_from_r(r)
% ~~~~~
%{
    This function calculates the right ascension and the
    declination from the geocentric equatorial position vector.

    r          - position vector
    l, m, n    - direction cosines of r
    ra         - right ascension (degrees)

```

```

    dec      - declination (degrees)
%}
% -----
l = r(1)/norm(r);
m = r(2)/norm(r);
n = r(3)/norm(r);

dec = asind(n);

if m > 0
    ra = acosd(l/cosd(dec));
else
    ra = 360 - acosd(l/cosd(dec));
end
% ~~~~~

```

Script file: Example_4_01.m

```

% ~~~~~
% Example 4.1
% ~~~~~
%{
    This program calculates the right ascension and declination
    from the geocentric equatorial position vector using the data
    in Example 4.1.

    r      - position vector r (km)
    ra     - right ascension (deg)
    dec    - declination (deg)

    User M-functions required: ra_and_dec_from_r

%}
% -----

clear all; clc

r      = [-5368 -1784 3691];
[ra dec] = ra_and_dec_from_r(r);

fprintf('\n ----- \n')
fprintf('\n Example 4.1 \n')
fprintf('\n r      = [%g %g %g] (km)', r(1), r(2), r(3))
fprintf('\n right ascension = %g deg', ra)
fprintf('\n declination     = %g deg', dec)
fprintf('\n ----- \n')

% ~~~~~

```

Output from Example_4_01.m-----
Example 4.1

```

r           = [-5368  -1784  3691] (km)
right ascension = 198.384 deg
declination   = 33.1245 deg

```

D.18 Algorithm 4.2: Calculation of the orbital elements from the state vector**Function file:** coe_from_sv.m

```

% ~~~~~
function coe = coe_from_sv(R,V,mu)
% ~~~~~
%{
% This function computes the classical orbital elements (coe)
% from the state vector (R,V) using Algorithm 4.1.
%
mu    - gravitational parameter (km^3/s^2)
R     - position vector in the geocentric equatorial frame (km)
V     - velocity vector in the geocentric equatorial frame (km)
r, v  - the magnitudes of R and V
vr    - radial velocity component (km/s)
H     - the angular momentum vector (km^2/s)
h     - the magnitude of H (km^2/s)
incl  - inclination of the orbit (rad)
N     - the node line vector (km^2/s)
n     - the magnitude of N
cp    - cross product of N and R
RA    - right ascension of the ascending node (rad)
E     - eccentricity vector
e     - eccentricity (magnitude of E)
eps   - a small number below which the eccentricity is considered
        to be zero
w     - argument of perigee (rad)
TA    - true anomaly (rad)
a     - semimajor axis (km)
pi    - 3.1415926...
coe   - vector of orbital elements [h e RA incl w TA a]

User M-functions required: None
%}

```

```

% -----

eps = 1.e-10;

r   = norm(R);
v   = norm(V);

vr  = dot(R,V)/r;

H   = cross(R,V);
h   = norm(H);

%...Equation 4.7:
incl = acos(H(3)/h);

%...Equation 4.8:
N    = cross([0 0 1],H);
n    = norm(N);

%...Equation 4.9:
if n ~= 0
    RA = acos(N(1)/n);
    if N(2) < 0
        RA = 2*pi - RA;
    end
else
    RA = 0;
end

%...Equation 4.10:
E = 1/mu*((v^2 - mu/r)*R - r*vr*V);
e = norm(E);

%...Equation 4.12 (incorporating the case e = 0):
if n ~= 0
    if e > eps
        w = acos(dot(N,E)/n/e);
        if E(3) < 0
            w = 2*pi - w;
        end
    else
        w = 0;
    end
else
    w = 0;
end

```

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```
%...Equation 4.13a (incorporating the case e = 0):
if e > eps
    TA = acos(dot(E,R)/e/r);
    if vr < 0
        TA = 2*pi - TA;
    end
else
    cp = cross(N,R);
    if cp(3) >= 0
        TA = acos(dot(N,R)/n/r);
    else
        TA = 2*pi - acos(dot(N,R)/n/r);
    end
end

%...Equation 4.62 (a < 0 for a hyperbola):
a = h^2/mu/(1 - e^2);

coe = [h e RA incl w TA a];

end %coe_from_sv
% ~~~~~
```

Script file: Example_4_03.m

```
% ~~~~~
% Example_4_03
% ~~~~~
%{
    This program uses Algorithm 4.2 to obtain the orbital
    elements from the state vector provided in Example 4.3.

    pi - 3.1415926...
    deg - factor for converting between degrees and radians
    mu - gravitational parameter (km^3/s^2)
    r - position vector (km) in the geocentric equatorial frame
    v - velocity vector (km/s) in the geocentric equatorial frame
    coe - orbital elements [h e RA incl w TA a]
        where h = angular momentum (km^2/s)
              e = eccentricity
              RA = right ascension of the ascending node (rad)
              incl = orbit inclination (rad)
              w = argument of perigee (rad)
              TA = true anomaly (rad)
              a = semimajor axis (km)
    T - Period of an elliptic orbit (s)
%}
```

```

    User M-function required: coe_from_sv
%}
% -----
clear all; clc
deg = pi/180;
mu = 398600;

%...Data declaration for Example 4.3:
r = [ -6045  -3490   2500];
v = [-3.457   6.618   2.533];
%...

%...Algorithm 4.2:
coe = coe_from_sv(r,v,mu);

%...Echo the input data and output results to the command window:
fprintf('-----\n')
fprintf('\n Example 4.3\n')
fprintf('\n Gravitational parameter (km^3/s^2) = %g\n', mu)
fprintf('\n State vector:\n')
fprintf('\n r (km)                                = [%g  %g  %g]', ...
                                                r(1), r(2), r(3))
fprintf('\n v (km/s)                             = [%g  %g  %g]', ...
                                                v(1), v(2), v(3))

disp(' ')
fprintf('\n Angular momentum (km^2/s)            = %g', coe(1))
fprintf('\n Eccentricity                          = %g', coe(2))
fprintf('\n Right ascension (deg)                   = %g', coe(3)/deg)
fprintf('\n Inclination (deg)                       = %g', coe(4)/deg)
fprintf('\n Argument of perigee (deg)                = %g', coe(5)/deg)
fprintf('\n True anomaly (deg)                      = %g', coe(6)/deg)
fprintf('\n Semimajor axis (km):                     = %g', coe(7))

%...if the orbit is an ellipse, output its period (Equation 2.73):
if coe(2)<1
    T = 2*pi/sqrt(mu)*coe(7)^1.5;
    fprintf('\n Period:')
    fprintf('\n   Seconds                                = %g', T)
    fprintf('\n   Minutes                                = %g', T/60)
    fprintf('\n   Hours                                = %g', T/3600)
    fprintf('\n   Days                                = %g', T/24/3600)
end
fprintf('\n-----\n')

% ~~~~~

```

Output from Example_4_03-----
Example 4.3Gravitational parameter (km^3/s^2) = 398600

State vector:

r (km)	= [-6045 -3490 2500]
v (km/s)	= [-3.457 6.618 2.533]

Angular momentum (km^2/s) = 58311.7

Eccentricity = 0.171212

Right ascension (deg) = 255.279

Inclination (deg) = 153.249

Argument of perigee (deg) = 20.0683

True anomaly (deg) = 28.4456

Semimajor axis (km): = 8788.1

Period:

Seconds = 8198.86

Minutes = 136.648

Hours = 2.27746

Days = 0.0948942

D.19 Calculation of $\tan^{-1}(y/x)$ to lie in the range 0 to 360° **Function file: atan2d_360.m**

```
% ~~~~~
function t = atan2d_0_360(y,x)
% ~~~~~
%{
    This function calculates the arc tangent of y/x in degrees
    and places the result in the range [0, 360].

    t - angle in degrees

%}
% -----

if x == 0
    if y == 0
        t = 0;
    elseif y > 0
        t = 90;
```

```

        else
            t = 270;
        end
    elseif x > 0
        if y >= 0
            t = atand(y/x);
        else
            t = atand(y/x) + 360;
        end
    elseif x < 0
        if y == 0
            t = 180;
        else
            t = atand(y/x) + 180;
        end
    end
end

end

% ~~~~~

```

D.20 Algorithm 4.3: Obtain the classical Euler angle sequence from a direction cosine matrix

Function file: dcm_to_euler.m

```

% ~~~~~
function [alpha beta gamma] = dcm_to_euler(Q)
% ~~~~~
%{
    This function finds the angles of the classical Euler sequence
    R3(gamma)*R1(beta)*R3(alpha) from the direction cosine matrix.

    Q      - direction cosine matrix
    alpha  - first angle of the sequence (deg)
    beta   - second angle of the sequence (deg)
    gamma  - third angle of the sequence (deg)

    User M-function required: atan2d_0_360
%}
% -----

alpha = atan2d_0_360(Q(3,1), -Q(3,2));
beta  = acosd(Q(3,3));
gamma = atan2d_0_360(Q(1,3), Q(2,3));

```

```
end
% ~~~~~
```

D.21 Algorithm 4.4: Obtain the yaw, pitch, and roll angles from a direction cosine matrix

Function file: dcm_to_ypr.m

```
% ~~~~~
function [yaw pitch roll] = dcm_to_ypr(Q)
% ~~~~~
%{
    This function finds the angles of the yaw-pitch-roll sequence
    R1(gamma)*R2(beta)*R3(alpha) from the direction cosine matrix.

    Q      - direction cosine matrix
    yaw    - yaw angle (deg)
    pitch  - pitch angle (deg)
    roll   - roll angle (deg)

    User M-function required: atan2d_0_360
%}
% -----

yaw  = atan2d_0_360(Q(1,2), Q(1,1));
pitch = asind(-Q(1,3));
roll  = atan2d_0_360(Q(2,3), Q(3,3));
end
% ~~~~~
```

D.22 Algorithm 4.5: Calculation of the state vector from the orbital elements

Function file: sv_from_coe.m

```
% ~~~~~
function [r, v] = sv_from_coe(coe,mu)
% ~~~~~
%{
    This function computes the state vector (r,v) from the
    classical orbital elements (coe).

    mu    - gravitational parameter (km^3/s^2)
    coe   - orbital elements [h e RA incl w TA]
%}
```

```

where
    h    = angular momentum (km^2/s)
    e    = eccentricity
    RA   = right ascension of the ascending node (rad)
    incl = inclination of the orbit (rad)
    w    = argument of perigee (rad)
    TA   = true anomaly (rad)
R3_w - Rotation matrix about the z-axis through the angle w
R1_i - Rotation matrix about the x-axis through the angle i
R3_W - Rotation matrix about the z-axis through the angle RA
Q_pX - Matrix of the transformation from perifocal to geocentric
       equatorial frame
rp    - position vector in the perifocal frame (km)
vp    - velocity vector in the perifocal frame (km/s)
r     - position vector in the geocentric equatorial frame (km)
v     - velocity vector in the geocentric equatorial frame (km/s)

User M-functions required: none
%}
% -----

h    = coe(1);
e    = coe(2);
RA   = coe(3);
incl = coe(4);
w    = coe(5);
TA   = coe(6);

%...Equations 4.45 and 4.46 (rp and vp are column vectors):
rp = (h^2/mu) * (1/(1 + e*cos(TA))) * (cos(TA)*[1;0;0] + sin(TA)*[0;1;0]);
vp = (mu/h) * (-sin(TA)*[1;0;0] + (e + cos(TA))*[0;1;0]);

%...Equation 4.34:
R3_W = [ cos(RA)  sin(RA)  0
        -sin(RA)  cos(RA)  0
           0       0       1];

%...Equation 4.32:
R1_i = [1      0      0
        0  cos(incl) sin(incl)
        0 -sin(incl) cos(incl)];

%...Equation 4.34:
R3_w = [ cos(w)  sin(w)  0
        -sin(w)  cos(w)  0
           0       0     1];

%...Equation 4.49:

```

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```
Q_pX = (R3_w*R1_i*R3_W)';

%...Equations 4.51 (r and v are column vectors):
r = Q_pX*rp;
v = Q_pX*vp;

%...Convert r and v into row vectors:
r = r';
v = v';

end
% ~~~~~
```

Script file: Example_4_07.m

```
% ~~~~~
% Example_4_07
% ~~~~~
%{
    This program uses Algorithm 4.5 to obtain the state vector from
    the orbital elements provided in Example 4.7.

    pi - 3.1415926...
    deg - factor for converting between degrees and radians
    mu - gravitational parameter (km^3/s^2)
    coe - orbital elements [h e RA incl w TA a]
           where h = angular momentum (km^2/s)
                  e = eccentricity
                  RA = right ascension of the ascending node (rad)
                  incl = orbit inclination (rad)
                  w = argument of perigee (rad)
                  TA = true anomaly (rad)
                  a = semimajor axis (km)
    r - position vector (km) in geocentric equatorial frame
    v - velocity vector (km) in geocentric equatorial frame

    User M-function required: sv_from_coe
%}
% -----
clear all; clc
deg = pi/180;
mu = 398600;

%...Data declaration for Example 4.5 (angles in degrees):
h = 80000;
e = 1.4;
RA = 40;
incl = 30;
```

```

w      = 60;
TA     = 30;
%...

coe = [h, e, RA*deg, incl*deg, w*deg, TA*deg];

%...Algorithm 4.5 (requires angular elements be in radians):
[r, v] = sv_from_coe(coe, mu);

%...Echo the input data and output the results to the command window:
fprintf('-----')
fprintf('\n Example 4.7\n')
fprintf('\n Gravitational parameter (km^3/s^2) = %g\n', mu)
fprintf('\n Angular momentum (km^2/s)          = %g', h)
fprintf('\n Eccentricity                          = %g', e)
fprintf('\n Right ascension (deg)                   = %g', RA)
fprintf('\n Argument of perigee (deg)                 = %g', w)
fprintf('\n True anomaly (deg)                       = %g', TA)
fprintf('\n\n State vector:')
fprintf('\n   r (km)    = [%g %g %g]', r(1), r(2), r(3))
fprintf('\n   v (km/s) = [%g %g %g]', v(1), v(2), v(3))
fprintf('\n-----\n')
% ~~~~~

```

Output from Example_4_05

Example 4.7

Gravitational parameter (km³/s²) = 398600

Angular momentum (km²/s) = 80000

Eccentricity = 1.4

Right ascension (deg) = 40

Argument of perigee (deg) = 60

True anomaly (deg) = 30

State vector:

r (km) = [-4039.9 4814.56 3628.62]

v (km/s) = [-10.386 -4.77192 1.74388]

D.23 Algorithm 4.6 Calculate the ground track of a satellite from its orbital elements

Function file: ground_track.m

```
% ~~~~~~
function ground_track
% ~~~~~~
%{
    This program plots the ground track of an earth satellite
    for which the orbital elements are specified.

    mu          - gravitational parameter (km^3/s^2)
    deg         - factor that converts degrees to radians
    J2          - second zonal harmonic
    Re         - earth's radius (km)
    we         - earth's angular velocity (rad/s)
    rP         - perigee of orbit (km)
    rA         - apogee of orbit (km)
    TA, TAo    - true anomaly, initial true anomaly of satellite (rad)
    RA, RAo    - right ascension, initial right ascension of the node (rad)
    incl       - orbit inclination (rad)
    wp, wpo    - argument of perigee, initial argument of perigee (rad)
    n_periods  - number of periods for which ground track is to be plotted
    a          - semimajor axis of orbit (km)
    T          - period of orbit (s)
    e          - eccentricity of orbit
    h          - angular momentum of orbit (km^2/s)
    E, Eo      - eccentric anomaly, initial eccentric anomaly (rad)
    M, Mo      - mean anomaly, initial mean anomaly (rad)
    to, tf     - initial and final times for the ground track (s)
    fac        - common factor in Equations 4.53 and 4.53
    RAdot      - rate of regression of the node (rad/s)
    wpdot      - rate of advance of perigee (rad/s)
    times      - times at which ground track is plotted (s)
    ra         - vector of right ascensions of the spacecraft (deg)
    dec        - vector of declinations of the spacecraft (deg)
    TA         - true anomaly (rad)
    r          - perifocal position vector of satellite (km)
    R          - geocentric equatorial position vector (km)
    R1         - DCM for rotation about z through RA
    R2         - DCM for rotation about x through incl
    R3         - DCM for rotation about z through wp
    QxX        - DCM for rotation from perifocal to geocentric equatorial
    Q          - DCM for rotation from geocentric equatorial
                into earth-fixed frame
    r_rel      - position vector in earth-fixed frame (km)
```

```

alpha    - satellite right ascension (deg)
delta    - satellite declination (deg)
n_curves - number of curves comprising the ground track plot
RA       - cell array containing the right ascensions for each of
           the curves comprising the ground track plot
Dec      - cell array containing the declinations for each of
           the curves comprising the ground track plot

User M-functions required: sv_from_coe, kepler_E, ra_and_dec_from_r
%}
% ~~~~~
clear all; close all; clc
global ra dec n_curves RA Dec

%...Constants
deg      = pi/180;
mu       = 398600;
J2       = 0.00108263;
Re       = 6378;
we       = (2*pi + 2*pi/365.26)/(24*3600);

%...Data declaration for Example 4.12:
rP       = 6700;
rA       = 10000;
TAo      = 230*deg;
Wo       = 270*deg;
incl     = 60*deg;
wpo      = 45*deg;
n_periods = 3.25;
%...End data declaration

%...Compute the initial time (since perigee) and
% the rates of node regression and perigee advance
a        = (rA + rP)/2;
T        = 2*pi/sqrt(mu)*a^(3/2);
e        = (rA - rP)/(rA + rP);
h        = sqrt(mu*a*(1 - e^2));
Eo       = 2*atan(tan(TAo/2)*sqrt((1-e)/(1+e)));
Mo       = Eo - e*sin(Eo);
to       = Mo*(T/2/pi);
tf       = to + n_periods*T;
fac      = -3/2*sqrt(mu)*J2*Re^2/(1-e^2)^2/a^(7/2);
Wdot     = fac*cos(incl);
wpdot    = fac*(5/2*sin(incl)^2 - 2);

find_ra_and_dec
form_separate_curves
plot_ground_track

```

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```
print_orbital_data

return

% ~~~~~~
function find_ra_and_dec
% ~~~~~~
% Propagates the orbit over the specified time interval, transforming
% the position vector into the earth-fixed frame and, from that,
% computing the right ascension and declination histories.
% -----
%
times = linspace(to,tf,1000);
ra    = [];
dec   = [];
theta = 0;
for i = 1:length(times)
    t      = times(i);
    M      = 2*pi/T*t;
    E      = kepler_E(e, M);
    TA     = 2*atan(tan(E/2)*sqrt((1+e)/(1-e)));
    r      = h^2/mu/(1 + e*cos(TA))*[cos(TA) sin(TA) 0]';

    W      = Wo + Wdot*t;
    wp     = wpo + wpdot*t;

    R1      = [ cos(W)  sin(W)  0
                -sin(W)  cos(W)  0
                  0      0    1];

    R2      = [ 1      0      0
                  0  cos(incl) sin(incl)
                  0 -sin(incl) cos(incl)];

    R3      = [ cos(wp)  sin(wp)  0
                -sin(wp)  cos(wp)  0
                  0      0    1];

    QxX     = (R3*R2*R1)';
    R        = QxX*r;

    theta   = we*(t - to);
    Q        = [ cos(theta)  sin(theta)  0
                  -sin(theta)  cos(theta)  0
                    0      0    1];

    r_rel    = Q*R;

    [alpha delta] = ra_and_dec_from_r(r_rel);
```

```

        ra          = [ra; alpha];
        dec          = [dec; delta];
    end

    end %find_ra_and_dec

% ~~~~~~
function form_separate_curves
% ~~~~~~
% Breaks the ground track up into separate curves which start
% and terminate at right ascensions in the range [0,360 deg].
% -----
tol = 100;
curve_no = 1;
n_curves = 1;
k        = 0;
ra_prev  = ra(1);
for i = 1:length(ra)
    if abs(ra(i) - ra_prev) > tol
        curve_no = curve_no + 1;
        n_curves = n_curves + 1;
        k = 0;
    end
    k          = k + 1;
    RA{curve_no}(k) = ra(i);
    Dec{curve_no}(k) = dec(i);
    ra_prev    = ra(i);
end
end %form_separate_curves

% ~~~~~~
function plot_ground_track
% ~~~~~~
hold on
xlabel('East longitude (degrees)')
ylabel('Latitude (degrees)')
axis equal
grid on
for i = 1:n_curves
    plot(RA{i}, Dec{i})
end

axis ([0 360 -90 90])
text( ra(1),    dec(1), 'o Start')
text(ra(end), dec(end), 'o Finish')
line([min(ra) max(ra)], [0 0], 'Color','k') %the equator
end %plot_ground_track

```

```

% ~~~~~
function print_orbital_data
% ~~~~~
coe      = [h e Wo incl wpo T Ao];
[ro, vo] = sv_from_coe(coe, mu);
fprintf('\n -----\n')
fprintf('\n Angular momentum      = %g km^2/s' , h)
fprintf('\n Eccentricity              = %g'       , e)
fprintf('\n Semimajor axis                = %g km'     , a)
fprintf('\n Perigee radius                = %g km'     , rP)
fprintf('\n Apogee radius                 = %g km'     , rA)
fprintf('\n Period                        = %g hours'    , T/3600)
fprintf('\n Inclination                    = %g deg'     , incl/deg)
fprintf('\n Initial true anomaly          = %g deg'     , T Ao/deg)
fprintf('\n Time since perigee            = %g hours'    , to/3600)
fprintf('\n Initial RA                    = %g deg'     , Wo/deg)
fprintf('\n RA_dot                         = %g deg/period' , Wdot/deg*T)
fprintf('\n Initial wp                    = %g deg'     , wpo/deg)
fprintf('\n wp_dot                        = %g deg/period' , wpdot/deg*T)
fprintf('\n')
fprintf('\n r0 = [%12g, %12g, %12g] (km)', ro(1), ro(2), ro(3))
fprintf('\n magnitude = %g km\n', norm(ro))
fprintf('\n v0 = [%12g, %12g, %12g] (km)', vo(1), vo(2), vo(3))
fprintf('\n magnitude = %g km\n', norm(vo))
fprintf('\n -----\n')

end %print_orbital_data

end %ground_track
% ~~~~~

```

Chapter 5

D.24 Algorithm 5.1: Gibbs method of preliminary orbit determination

Function file: gibbs.m

```

% ~~~~~
function [V2, ierr] = gibbs(R1, R2, R3)
% ~~~~~
%{
    This function uses the Gibbs method of orbit determination to
    to compute the velocity corresponding to the second of three

```

supplied position vectors.

mu - gravitational parameter (km^3/s^2)
 R1, R2, R3 - three coplanar geocentric position vectors (km)
 r1, r2, r3 - the magnitudes of R1, R2 and R3 (km)
 c12, c23, c31 - three independent cross products among
 R1, R2 and R3
 N, D, S - vectors formed from R1, R2 and R3 during
 the Gibbs' procedure
 tol - tolerance for determining if R1, R2 and R3
 are coplanar
 ierr - = 0 if R1, R2, R3 are found to be coplanar
 = 1 otherwise
 V2 - the velocity corresponding to R2 (km/s)

User M-functions required: none

```
%}
% -----
```

```
global mu
tol = 1e-4;
ierr = 0;
```

```
%...Magnitudes of R1, R2 and R3:
r1 = norm(R1);
r2 = norm(R2);
r3 = norm(R3);
```

```
%...Cross products among R1, R2 and R3:
c12 = cross(R1,R2);
c23 = cross(R2,R3);
c31 = cross(R3,R1);
```

```
%...Check that R1, R2 and R3 are coplanar; if not set error flag:
if abs(dot(R1,c23)/r1/norm(c23)) > tol
    ierr = 1;
end
```

```
%...Equation 5.13:
N = r1*c23 + r2*c31 + r3*c12;
```

```
%...Equation 5.14:
D = c12 + c23 + c31;
```

```
%...Equation 5.21:
S = R1*(r2 - r3) + R2*(r3 - r1) + R3*(r1 - r2);
```

```
%...Equation 5.22:
```

```
V2 = sqrt(mu/norm(N)/norm(D))*(cross(D,R2)/r2 + S);
% ~~~~~
end %gibbs
```

Script file: Example_5_01.m

```
% ~~~~~
% Example_5_01
% ~~~~~
%{
    This program uses Algorithm 5.1 (Gibbs method) and Algorithm 4.2
    to obtain the orbital elements from the data provided in Example 5.1.

    deg          - factor for converting between degrees and radians
    pi            - 3.1415926...
    mu            - gravitational parameter (km^3/s^2)
    r1, r2, r3    - three coplanar geocentric position vectors (km)
    ierr          - 0 if r1, r2, r3 are found to be coplanar
                  1 otherwise
    v2            - the velocity corresponding to r2 (km/s)
    coe            - orbital elements [h e RA incl w TA a]
                   where h    = angular momentum (km^2/s)
                         e     = eccentricity
                         RA    = right ascension of the ascending node (rad)
                         incl  = orbit inclination (rad)
                         w     = argument of perigee (rad)
                         TA    = true anomaly (rad)
                         a     = semimajor axis (km)
    T              - period of elliptic orbit (s)

    User M-functions required: gibbs, coe_from_sv
%}
% -----

clear all; clc
deg = pi/180;
global mu

%...Data declaration for Example 5.1:
mu = 398600;
r1 = [-294.32 4265.1 5986.7];
r2 = [-1365.5 3637.6 6346.8];
r3 = [-2940.3 2473.7 6555.8];
%...

%...Echo the input data to the command window:
fprintf('-----')
```

```

fprintf('\n Example 5.1: Gibbs Method\n')
fprintf('\n\n Input data:\n')
fprintf('\n Gravitational parameter (km^3/s^2) = %g\n', mu)
fprintf('\n r1 (km) = [%g %g %g]', r1(1), r1(2), r1(3))
fprintf('\n r2 (km) = [%g %g %g]', r2(1), r2(2), r2(3))
fprintf('\n r3 (km) = [%g %g %g]', r3(1), r3(2), r3(3))
fprintf('\n\n');

%...Algorithm 5.1:
[v2, ierr] = gibbs(r1, r2, r3);

%...If the vectors r1, r2, r3, are not coplanar, abort:
if ierr == 1
    fprintf('\n These vectors are not coplanar.\n\n')
    return
end

%...Algorithm 4.2:
coe = coe_from_sv(r2,v2,mu);

h = coe(1);
e = coe(2);
RA = coe(3);
incl = coe(4);
w = coe(5);
TA = coe(6);
a = coe(7);

%...Output the results to the command window:
fprintf(' Solution:')
fprintf('\n');
fprintf('\n v2 (km/s) = [%g %g %g]', v2(1), v2(2), v2(3))
fprintf('\n\n Orbital elements:');
fprintf('\n Angular momentum (km^2/s) = %g', h)
fprintf('\n Eccentricity = %g', e)
fprintf('\n Inclination (deg) = %g', incl/deg)
fprintf('\n RA of ascending node (deg) = %g', RA/deg)
fprintf('\n Argument of perigee (deg) = %g', w/deg)
fprintf('\n True anomaly (deg) = %g', TA/deg)
fprintf('\n Semimajor axis (km) = %g', a)
%...If the orbit is an ellipse, output the period:
if e < 1
    T = 2*pi/sqrt(mu)*coe(7)^1.5;
    fprintf('\n Period (s) = %g', T)
end
fprintf('\n-----\n')
% ~~~~~~

```

Output from Example_5_01

Example 5.1: Gibbs Method

Input data:

Gravitational parameter (km^3/s^2) = 398600

r_1 (km) = [-294.32 4265.1 5986.7]

r_2 (km) = [-1365.4 3637.6 6346.8]

r_3 (km) = [-2940.3 2473.7 6555.8]

Solution:

v_2 (km/s) = [-6.2176 -4.01237 1.59915]

Orbital elements:

Angular momentum (km^2/s) = 56193

Eccentricity = 0.100159

Inclination (deg) = 60.001

RA of ascending node (deg) = 40.0023

Argument of perigee (deg) = 30.1093

True anomaly (deg) = 49.8894

Semimajor axis (km) = 8002.14

Period (s) = 7123.94

D.25 Algorithm 5.2: Solution of Lambert's problem

Function file: lambert.m

```
% ~~~~~
function [V1, V2] = lambert(R1, R2, t, string)
% ~~~~~
%{
This function solves Lambert's problem.

mu          - gravitational parameter ( $\text{km}^3/\text{s}^2$ )
R1, R2      - initial and final position vectors (km)
r1, r2      - magnitudes of R1 and R2
t           - the time of flight from R1 to R2 (a constant) (s)
V1, V2      - initial and final velocity vectors (km/s)
c12         - cross product of R1 into R2
theta       - angle between R1 and R2
string      - 'pro' if the orbit is prograde
```

```

        'retro' if the orbit is retrograde
A      - a constant given by Equation 5.35
z      -  $\alpha x^2$ , where  $\alpha$  is the reciprocal of the
        semimajor axis and  $x$  is the universal anomaly
y(z)   - a function of  $z$  given by Equation 5.38
F(z,t) - a function of the variable  $z$  and constant  $t$ ,
        - given by Equation 5.40
dFdz(z) - the derivative of  $F(z,t)$ , given by Equation 5.43
ratio  -  $F/dFdz$ 
tol    - tolerance on precision of convergence
nmax   - maximum number of iterations of Newton's procedure
f, g   - Lagrange coefficients
gdot   - time derivative of  $g$ 
C(z), S(z) - Stumpff functions
dum    - a dummy variable

```

```

User M-functions required: stumpC and stumpS

```

```

%}

```

```

% -----

```

```

global mu

```

```

%...Magnitudes of R1 and R2:

```

```

r1 = norm(R1);

```

```

r2 = norm(R2);

```

```

c12 = cross(R1, R2);

```

```

theta = acos(dot(R1,R2)/r1/r2);

```

```

%...Determine whether the orbit is prograde or retrograde:

```

```

if nargin < 4 || (~strcmp(string,'retro') & (~strcmp(string,'pro')))

```

```

    string = 'pro';

```

```

    fprintf('\n ** Prograde trajectory assumed.\n')

```

```

end

```

```

if strcmp(string,'pro')

```

```

    if c12(3) <= 0

```

```

        theta = 2*pi - theta;

```

```

    end

```

```

elseif strcmp(string,'retro')

```

```

    if c12(3) >= 0

```

```

        theta = 2*pi - theta;

```

```

    end

```

```

end

```

```

%...Equation 5.35:

```

```

A = sin(theta)*sqrt(r1*r2/(1 - cos(theta)));

```

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```
%...Determine approximately where F(z,t) changes sign, and
%...use that value of z as the starting value for Equation 5.45:
z = -100;
while F(z,t) < 0
    z = z + 0.1;
end

%...Set an error tolerance and a limit on the number of iterations:
tol = 1.e-8;
nmax = 5000;

%...Iterate on Equation 5.45 until z is determined to within the
%...error tolerance:
ratio = 1;
n = 0;
while (abs(ratio) > tol) & (n <= nmax)
    n = n + 1;
    ratio = F(z,t)/dFdZ(z);
    z = z - ratio;
end

%...Report if the maximum number of iterations is exceeded:
if n >= nmax
    fprintf('\n\n **Number of iterations exceeds %g \n\n ',nmax)
end

%...Equation 5.46a:
f = 1 - y(z)/r1;

%...Equation 5.46b:
g = A*sqrt(y(z)/mu);

%...Equation 5.46d:
gdot = 1 - y(z)/r2;

%...Equation 5.28:
V1 = 1/g*(R2 - f*R1);

%...Equation 5.29:
V2 = 1/g*(gdot*R2 - R1);

return

% ~~~~~
% Subfunctions used in the main body:
% ~~~~~

%...Equation 5.38:
```

```

function dum = y(z)
    dum = r1 + r2 + A*(z*S(z) - 1)/sqrt(C(z));
end

%...Equation 5.40:
function dum = F(z,t)
    dum = (y(z)/C(z))^1.5*S(z) + A*sqrt(y(z)) - sqrt(mu)*t;
end

%...Equation 5.43:
function dum = dFdZ(z)
    if z == 0
        dum = sqrt(2)/40*y(0)^1.5 + A/8*(sqrt(y(0)) + A*sqrt(1/2/y(0)));
    else
        dum = (y(z)/C(z))^1.5*(1/2/z*(C(z) - 3*S(z)/2/C(z)) ...
            + 3*S(z)^2/4/C(z)) + A/8*(3*S(z)/C(z)*sqrt(y(z)) ...
            + A*sqrt(C(z)/y(z)));
    end
end

%...Stumpff functions:
function dum = C(z)
    dum = stumpC(z);
end

function dum = S(z)
    dum = stumpS(z);
end

end %lambert

% ~~~~~

```

Script file: Example_5_02.m

```

% ~~~~~
% Example_5_02
% ~~~~~
%{
    This program uses Algorithm 5.2 to solve Lambert's problem for the
    data provided in Example 5.2.

    deg    - factor for converting between degrees and radians
    pi      - 3.1415926...
    mu      - gravitational parameter (km^3/s^2)
    r1, r2  - initial and final position vectors (km)
    dt      - time between r1 and r2 (s)

```

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```
string - = 'pro' if the orbit is prograde
        = 'retro' if the orbit is retrograde
v1, v2 - initial and final velocity vectors (km/s)
coe     - orbital elements [h e RA incl w TA a]
        where h    = angular momentum (km^2/s)
                e    = eccentricity
                RA    = right ascension of the ascending node (rad)
                incl  = orbit inclination (rad)
                w     = argument of perigee (rad)
                TA    = true anomaly (rad)
                a     = semimajor axis (km)
TA1     - Initial true anomaly (rad)
TA2     - Final true anomaly (rad)
T       - period of an elliptic orbit (s)

User M-functions required: lambert, coe_from_sv
%}
% -----

clear all; clc
global mu
deg = pi/180;

%...Data declaration for Example 5.2:
mu    = 398600;
r1    = [ 5000 10000 2100];
r2    = [-14600 2500 7000];
dt    = 3600;
string = 'pro';
%...

%...Algorithm 5.2:
[v1, v2] = lambert(r1, r2, dt, string);

%...Algorithm 4.1 (using r1 and v1):
coe      = coe_from_sv(r1, v1, mu);
%...Save the initial true anomaly:
TA1      = coe(6);

%...Algorithm 4.1 (using r2 and v2):
coe      = coe_from_sv(r2, v2, mu);
%...Save the final true anomaly:
TA2      = coe(6);

%...Echo the input data and output the results to the command window:
fprintf('-----')
fprintf('\n Example 5.2: Lambert"s Problem\n')
fprintf('\n\n Input data:\n');
```

```

fprintf('\n   Gravitational parameter (km^3/s^2) = %g\n', mu);
fprintf('\n   r1 (km)                                = [%g   %g   %g]', ...
                                                r1(1), r1(2), r1(3))

fprintf('\n   r2 (km)                                = [%g   %g   %g]', ...
                                                r2(1), r2(2), r2(3))

fprintf('\n   Elapsed time (s)                        = %g', dt);
fprintf('\n\n Solution:\n')

fprintf('\n   v1 (km/s)                                = [%g   %g   %g]', ...
                                                v1(1), v1(2), v1(3))
fprintf('\n   v2 (km/s)                                = [%g   %g   %g]', ...
                                                v2(1), v2(2), v2(3))

fprintf('\n\n Orbital elements:')
fprintf('\n   Angular momentum (km^2/s)                    = %g', coe(1))
fprintf('\n   Eccentricity                                     = %g', coe(2))
fprintf('\n   Inclination (deg)                               = %g', coe(4)/deg)
fprintf('\n   RA of ascending node (deg)                      = %g', coe(3)/deg)
fprintf('\n   Argument of perigee (deg)                       = %g', coe(5)/deg)
fprintf('\n   True anomaly initial (deg)                     = %g', TA1/deg)
fprintf('\n   True anomaly final   (deg)                     = %g', TA2/deg)
fprintf('\n   Semimajor axis (km)                             = %g', coe(7))
fprintf('\n   Periapse radius (km)                           = %g', coe(1)^2/mu/(1 + coe(2)))
%...If the orbit is an ellipse, output its period:
if coe(2)<1
    T = 2*pi/sqrt(mu)*coe(7)^1.5;
    fprintf('\n   Period:')
    fprintf('\n       Seconds                                = %g', T)
    fprintf('\n       Minutes                                = %g', T/60)
    fprintf('\n       Hours                                  = %g', T/3600)
    fprintf('\n       Days                                  = %g', T/24/3600)
end
fprintf('\n-----\n')
% ~~~~~~

```

Output from Example_5_02

Example 5.2: Lambert's Problem

Input data:

Gravitational parameter (km³/s²) = 398600

r1 (km) = [5000 10000 2100]
r2 (km) = [-14600 2500 7000]
Elapsed time (s) = 3600

Solution:

```
v1 (km/s)           = [-5.99249  1.92536  3.24564]
v2 (km/s)           = [-3.31246  -4.19662  -0.385288]
```

Orbital elements:

```
Angular momentum (km^2/s) = 80466.8
Eccentricity              = 0.433488
Inclination (deg)         = 30.191
RA of ascending node (deg) = 44.6002
Argument of perigee (deg) = 30.7062
True anomaly initial (deg) = 350.83
True anomaly final  (deg) = 91.1223
Semimajor axis (km)       = 20002.9
Periapse radius (km)      = 11331.9
Period:
  Seconds                = 28154.7
  Minutes                 = 469.245
  Hours                   = 7.82075
  Days                    = 0.325865
```

D.26 Calculation of Julian day number at 0 hr UT

The following script implements Equation 5.48 for use in other programs.

Function file: J0.m

```
% ~~~~~
function j0 = J0(year, month, day)
% ~~~~~
%{
    This function computes the Julian day number at 0 UT for any year
    between 1900 and 2100 using Equation 5.48.

    j0    - Julian day at 0 hr UT (Universal Time)
    year  - range: 1901 - 2099
    month - range: 1 - 12
    day   - range: 1 - 31

    User m-functions required: none
%}
% -----

j0 = 367*year - fix(7*(year + fix((month + 9)/12))/4) ...
```

```

    + fix(275*month/9) + day + 1721013.5;

% ~~~~~
end %J0

```

Script file: Example_5_04.m

```

% ~~~~~
% Example_5_04
% ~~~~~
%{
    This program computes J0 and the Julian day number using the data
    in Example 5.4.

    year   - range: 1901 - 2099
    month  - range: 1 - 12
    day    - range: 1 - 31
    hour   - range: 0 - 23 (Universal Time)
    minute - range: 0 - 60
    second - range: 0 - 60
    ut     - universal time (hr)
    j0     - Julian day number at 0 hr UT
    jd     - Julian day number at specified UT

    User M-function required: J0
%}
% -----

clear all; clc

%...Data declaration for Example 5.4:
year = 2004;
month = 5;
day = 12;

hour = 14;
minute = 45;
second = 30;
%...

ut = hour + minute/60 + second/3600;

%...Equation 5.46:
j0 = J0(year, month, day);

%...Equation 5.47:
jd = j0 + ut/24;

```

```
%...Echo the input data and output the results to the command window:
fprintf('-----')
fprintf('\n Example 5.4: Julian day calculation\n')
fprintf('\n Input data:\n');
fprintf('\n   Year           = %g',   year)
fprintf('\n   Month          = %g',   month)
fprintf('\n   Day            = %g',   day)
fprintf('\n   Hour           = %g',   hour)
fprintf('\n   Minute         = %g',   minute)
fprintf('\n   Second         = %g\n', second)

fprintf('\n Julian day number = %11.3f', jd);
fprintf('\n-----\n')
% ~~~~~
```

Output from Example_5_04

Example 5.4: Julian day calculation

Input data:

```
Year           = 2004
Month          = 5
Day            = 12
Hour           = 14
Minute         = 45
Second         = 30
```

```
Julian day number = 2453138.115
-----
```

D.27 Algorithm 5.3: Calculation of local sidereal time

Function file: LST.m

```
% ~~~~~
function lst = LST(y, m, d, ut, EL)
% ~~~~~
%{
    This function calculates the local sidereal time.

    lst - local sidereal time (degrees)
    y   - year
    m   - month
    d   - day
    ut  - Universal Time (hours)
    EL  - east longitude (degrees)
```

```

j0 - Julian day number at 0 hr UT
j - number of centuries since J2000
g0 - Greenwich sidereal time (degrees) at 0 hr UT
gst - Greenwich sidereal time (degrees) at the specified UT

User M-function required: J0
User subfunction required: zeroTo360
%}
% -----

%...Equation 5.48;
j0 = J0(y, m, d);

%...Equation 5.49;
j = (j0 - 2451545)/36525;

%...Equation 5.50;
g0 = 100.4606184 + 36000.77004*j + 0.000387933*j^2 - 2.583e-8*j^3;

%...Reduce g0 so it lies in the range 0 - 360 degrees
g0 = zeroTo360(g0);

%...Equation 5.51;
gst = g0 + 360.98564724*ut/24;

%...Equation 5.52;
lst = gst + EL;

%...Reduce lst to the range 0 - 360 degrees:
lst = lst - 360*fix(lst/360);

return

% ~~~~~
function y = zeroTo360(x)
% ~~~~~
%{
    This subfunction reduces an angle to the range 0 - 360 degrees.

    x - The angle (degrees) to be reduced
    y - The reduced value
%}
% -----
if (x >= 360)
    x = x - fix(x/360)*360;
elseif (x < 0)
    x = x - (fix(x/360) - 1)*360;
end

```

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```
y = x;
end %zeroTo360

end %LST
% ~~~~~
```

Script file: Example_5_06.m

```
% ~~~~~
% Example_5_06
% ~~~~~
%{
    This program uses Algorithm 5.3 to obtain the local sidereal
    time from the data provided in Example 5.6.

    lst    - local sidereal time (degrees)
    EL     - east longitude of the site (west longitude is negative):
              degrees (0 - 360)
              minutes (0 - 60)
              seconds (0 - 60)
    WL     - west longitude
    year   - range: 1901 - 2099
    month  - range: 1 - 12
    day    - range: 1 - 31
    ut     - universal time
              hour (0 - 23)
              minute (0 - 60)
              second (0 - 60)

    User m-function required: LST
%}
% -----

clear all; clc

%...Data declaration for Example 5.6:
%   East longitude:
degrees = 139;
minutes = 47;
seconds = 0;

%   Date:
year    = 2004;
month   = 3;
day     = 3;
```

```

% Universal time:
hour    = 4;
minute  = 30;
second  = 0;
%...

%...Convert negative (west) longitude to east longitude:
if degrees < 0
    degrees = degrees + 360;
end

%...Express the longitudes as decimal numbers:
EL = degrees + minutes/60 + seconds/3600;
WL = 360 - EL;

%...Express universal time as a decimal number:
ut = hour + minute/60 + second/3600;

%...Algorithm 5.3:
lst = LST(year, month, day, ut, EL);

%...Echo the input data and output the results to the command window:
fprintf('-----')
fprintf('\n Example 5.6: Local sidereal time calculation\n')
fprintf('\n Input data:\n');
fprintf('\n   Year                = %g', year)
fprintf('\n   Month               = %g', month)
fprintf('\n   Day                 = %g', day)
fprintf('\n   UT (hr)             = %g', ut)
fprintf('\n   West Longitude (deg) = %g', WL)
fprintf('\n   East Longitude (deg) = %g', EL)
fprintf('\n\n');

fprintf(' Solution:')

fprintf('\n');
fprintf('\n   Local Sidereal Time (deg) = %g', lst)
fprintf('\n   Local Sidereal Time (hr)  = %g', lst/15)

fprintf('\n-----\n')
% ~~~~~

```

Output from Example_5_06

```
-----
Example 5.6: Local sidereal time calculation
```

```
Input data:
```

```

Year                = 2004
Month               = 3
Day                 = 3
UT (hr)             = 4.5
West Longitude (deg) = 220.217
East Longitude (deg) = 139.783

```

Solution:

```

Local Sidereal Time (deg) = 8.57688
Local Sidereal Time (hr)  = 0.571792
-----

```

D.28 Algorithm 5.4: Calculation of the state vector from measurements of range, angular position, and their rates

Function file: `rv_from_observe.m`

```

% ~~~~~
function [r,v] = rv_from_observe(rho, rhodot, A, Adot, a, ...
                                adot, theta, phi, H)
% ~~~~~
%{
This function calculates the geocentric equatorial position and
velocity vectors of an object from radar observations of range,
azimuth, elevation angle and their rates.

deg    - conversion factor between degrees and radians
pi     - 3.1415926...

Re     - equatorial radius of the earth (km)
f      - earth's flattening factor
wE     - angular velocity of the earth (rad/s)
omega  - earth's angular velocity vector (rad/s) in the
         geocentric equatorial frame

theta  - local sidereal time (degrees) of tracking site
phi    - geodetic latitude (degrees) of site
H      - elevation of site (km)
R      - geocentric equatorial position vector (km) of tracking site
Rdot   - inertial velocity (km/s) of site

rho    - slant range of object (km)
rhodot - range rate (km/s)
A      - azimuth (degrees) of object relative to observation site
Adot   - time rate of change of azimuth (degrees/s)

```

```

a      - elevation angle (degrees) of object relative to observation site
adot   - time rate of change of elevation angle (degrees/s)
dec     - topocentric equatorial declination of object (rad)
decdot  - declination rate (rad/s)
h       - hour angle of object (rad)
RA      - topocentric equatorial right ascension of object (rad)
RAdot   - right ascension rate (rad/s)

Rho     - unit vector from site to object
Rhodot  - time rate of change of Rho (1/s)
r        - geocentric equatorial position vector of object (km)
v        - geocentric equatorial velocity vector of object (km)

User M-functions required: none
%}
% -----

global f Re wE
deg  = pi/180;
omega = [0 0 wE];

%...Convert angular quantities from degrees to radians:
A     = A     *deg;
Adot  = Adot  *deg;
a     = a     *deg;
adot  = adot  *deg;
theta = theta*deg;
phi   = phi   *deg;

%...Equation 5.56:
R      = [(Re/sqrt(1-(2*f - f*f)*sin(phi)^2) + H)*cos(phi)*cos(theta), ...
          (Re/sqrt(1-(2*f - f*f)*sin(phi)^2) + H)*cos(phi)*sin(theta), ...
          (Re*(1 - f)^2/sqrt(1-(2*f - f*f)*sin(phi)^2) + H)*sin(phi)];

%...Equation 5.66:
Rdot   = cross(omega, R);

%...Equation 5.83a:
dec     = asin(cos(phi)*cos(A)*cos(a) + sin(phi)*sin(a));

%...Equation 5.83b:
h = acos((cos(phi)*sin(a) - sin(phi)*cos(A)*cos(a))/cos(dec));
if (A > 0) & (A < pi)
    h = 2*pi - h;
end

%...Equation 5.83c:

```

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```
RA = theta - h;

%...Equations 5.57:
Rho = [cos(RA)*cos(dec) sin(RA)*cos(dec) sin(dec)];

%...Equation 5.63:
r = R + rho*Rho;

%...Equation 5.84:
decdot = (-Adot*cos(phi)*sin(A)*cos(a) + adot*(sin(phi)*cos(a) ...
          - cos(phi)*cos(A)*sin(a))/cos(dec);

%...Equation 5.85:
RAdot = wE ...
        + (Adot*cos(A)*cos(a) - adot*sin(A)*sin(a) ...
          + decdot*sin(A)*cos(a)*tan(dec)) ...
        /(cos(phi)*sin(a) - sin(phi)*cos(A)*cos(a));

%...Equations 5.69 and 5.72:
Rhodot = [-RAdot*sin(RA)*cos(dec) - decdot*cos(RA)*sin(dec),...
          RAdot*cos(RA)*cos(dec) - decdot*sin(RA)*sin(dec),...
          decdot*cos(dec)];

%...Equation 5.64:
v = Rdot + rhodot*Rho + rho*Rhodot;

end %rv_from_observe
% ~~~~~
```

Script file: Example_5_10.m

```
% ~~~~~
% Example_5_10
% ~~~~~
%
% This program uses Algorithms 5.4 and 4.2 to obtain the orbital
% elements from the observational data provided in Example 5.10.
%
% deg - conversion factor between degrees and radians
% pi - 3.1415926...
% mu - gravitational parameter (km^3/s^2)
%
% Re - equatorial radius of the earth (km)
% f - earth's flattening factor
% wE - angular velocity of the earth (rad/s)
% omega - earth's angular velocity vector (rad/s) in the
% geocentric equatorial frame
```

```

% rho      - slant range of object (km)
% rhodot   - range rate (km/s)
% A        - azimuth (deg) of object relative to observation site
% Adot     - time rate of change of azimuth (deg/s)
% a        - elevation angle (deg) of object relative to observation site
% adot     - time rate of change of elevation angle (degrees/s)

% theta    - local sidereal time (deg) of tracking site
% phi      - geodetic latitude (deg) of site
% H        - elevation of site (km)

% r        - geocentric equatorial position vector of object (km)
% v        - geocentric equatorial velocity vector of object (km)

% coe      - orbital elements [h e RA incl w TA a]
%           where
%           h   = angular momentum (km^2/s)
%           e   = eccentricity
%           RA  = right ascension of the ascending node (rad)
%           incl = inclination of the orbit (rad)
%           w   = argument of perigee (rad)
%           TA  = true anomaly (rad)
%           a   = semimajor axis (km)
% rp       - perigee radius (km)
% T        - period of elliptical orbit (s)
%
% User M-functions required: rv_from_observe, coe_from_sv
% -----

clear all; clc
global f Re wE

deg      = pi/180;
f        = 1/298.256421867;
Re       = 6378.13655;
wE       = 7.292115e-5;
mu       = 398600.4418;

%...Data declaration for Example 5.10:
rho      = 2551;
rhodot   = 0;
A        = 90;
Adot     = 0.1130;
a        = 30;
adot     = 0.05651;
theta    = 300;
phi      = 60;
H        = 0;

```

```

%...

%...Algorithm 5.4:
[r,v] = rv_from_observe(rho, rhodot, A, Adot, a, adot, theta, phi, H);

%...Algorithm 4.2:
coe = coe_from_sv(r,v,mu);

h = coe(1);
e = coe(2);
RA = coe(3);
incl = coe(4);
w = coe(5);
TA = coe(6);
a = coe(7);

%...Equation 2.40
rp = h^2/mu/(1 + e);

%...Echo the input data and output the solution to
% the command window:
fprintf('-----')
fprintf('\n Example 5.10')
fprintf('\n\n Input data:\n');
fprintf('\n Slant range (km) = %g', rho);
fprintf('\n Slant range rate (km/s) = %g', rhodot);
fprintf('\n Azimuth (deg) = %g', A);
fprintf('\n Azimuth rate (deg/s) = %g', Adot);
fprintf('\n Elevation (deg) = %g', a);
fprintf('\n Elevation rate (deg/s) = %g', adot);
fprintf('\n Local sidereal time (deg) = %g', theta);
fprintf('\n Latitude (deg) = %g', phi);
fprintf('\n Altitude above sea level (km) = %g', H);
fprintf('\n\n');

fprintf(' Solution:')

fprintf('\n\n State vector:\n');
fprintf('\n r (km) = [%g, %g, %g]', ...
        r(1), r(2), r(3));
fprintf('\n v (km/s) = [%g, %g, %g]', ...
        v(1), v(2), v(3));

fprintf('\n\n Orbital elements:\n')
fprintf('\n Angular momentum (km^2/s) = %g', h)
fprintf('\n Eccentricity = %g', e)
fprintf('\n Inclination (deg) = %g', incl/deg)

```

```

fprintf('\n   RA of ascending node (deg)   = %g', RA/deg)
fprintf('\n   Argument of perigee (deg)    = %g', w/deg)
fprintf('\n   True anomaly (deg)              = %g\n', TA/deg)
fprintf('\n   Semimajor axis (km)                = %g', a)
fprintf('\n   Perigee radius (km)               = %g', rp)
%...If the orbit is an ellipse, output its period:
if e < 1
    T = 2*pi/sqrt(mu)*a^1.5;
    fprintf('\n   Period:')
    fprintf('\n           Seconds                = %g', T)
    fprintf('\n           Minutes                  = %g', T/60)
    fprintf('\n           Hours                    = %g', T/3600)
    fprintf('\n           Days                     = %g', T/24/3600)
end
fprintf('\n-----\n')
% ~~~~~~

```

Output from Example_5_10

Example 5.10

Input data:

Slant range (km)	= 2551
Slant range rate (km/s)	= 0
Azimuth (deg)	= 90
Azimuth rate (deg/s)	= 0.113
Elevation (deg)	= 5168.62
Elevation rate (deg/s)	= 0.05651
Local sidereal time (deg)	= 300
Latitude (deg)	= 60
Altitude above sea level (km)	= 0

Solution:

State vector:

r (km)	= [3830.68, -2216.47, 6605.09]
v (km/s)	= [1.50357, -4.56099, -0.291536]

Orbital elements:

Angular momentum (km ² /s)	= 35621.4
Eccentricity	= 0.619758
Inclination (deg)	= 113.386
RA of ascending node (deg)	= 109.75
Argument of perigee (deg)	= 309.81

```

True anomaly (deg)          = 165.352

Semimajor axis (km)         = 5168.62
Perigee radius (km)         = 1965.32
Period:
  Seconds                    = 3698.05
  Minutes                    = 61.6342
  Hours                      = 1.02724
  Days                      = 0.0428015

```

D.29 Algorithms 5.5 and 5.6: Gauss method of preliminary orbit determination with iterative improvement

Function file: gauss.m

```

% ~~~~~
function [r, v, r_old, v_old] = ...
    gauss(Rho1, Rho2, Rho3, R1, R2, R3, t1, t2, t3)
% ~~~~~
%{
    This function uses the Gauss method with iterative improvement
    (Algorithms 5.5 and 5.6) to calculate the state vector of an
    orbiting body from angles-only observations at three
    closely-spaced times.

    mu                - the gravitational parameter (km^3/s^2)
    t1, t2, t3        - the times of the observations (s)
    tau, tau1, tau3   - time intervals between observations (s)
    R1, R2, R3        - the observation site position vectors
                      at t1, t2, t3 (km)
    Rho1, Rho2, Rho3 - the direction cosine vectors of the
                      satellite at t1, t2, t3
    p1, p2, p3        - cross products among the three direction
                      cosine vectors
    Do                - scalar triple product of Rho1, Rho2 and Rho3
    D                 - Matrix of the nine scalar triple products
                      of R1, R2 and R3 with p1, p2 and p3
    E                 - dot product of R2 and Rho2
    A, B              - constants in the expression relating slant range
                      to geocentric radius
    a,b,c             - coefficients of the 8th order polynomial
                      in the estimated geocentric radius x
    x                 - positive root of the 8th order polynomial
    rho1, rho2, rho3 - the slant ranges at t1, t2, t3
    r1, r2, r3        - the position vectors at t1, t2, t3 (km)

```

```

r_old, v_old      - the estimated state vector at the end of
                    Algorithm 5.5 (km, km/s)

rho1_old,
rho2_old, and
rho3_old          - the values of the slant ranges at t1, t2, t3
                    at the beginning of iterative improvement
                    (Algorithm 5.6) (km)

diff1, diff2,
and diff3         - the magnitudes of the differences between the
                    old and new slant ranges at the end of
                    each iteration

tol              - the error tolerance determining
                    convergence

n                - number of passes through the
                    iterative improvement loop

nmax             - limit on the number of iterations

ro, vo           - magnitude of the position and
                    velocity vectors (km, km/s)

vro             - radial velocity component (km)

a               - reciprocal of the semimajor axis (1/km)

v2              - computed velocity at time t2 (km/s)

r, v            - the state vector at the end of Algorithm 5.6
                    (km, km/s)

```

User m-functions required: kepler_U, f_and_g

User subfunctions required: posroot

```
%}
```

```
% -----
```

```
global mu
```

```
%...Equations 5.98:
```

```
tau1 = t1 - t2;
```

```
tau3 = t3 - t2;
```

```
%...Equation 5.101:
```

```
tau = tau3 - tau1;
```

```
%...Independent cross products among the direction cosine vectors:
```

```
p1 = cross(Rho2,Rho3);
```

```
p2 = cross(Rho1,Rho3);
```

```
p3 = cross(Rho1,Rho2);
```

```
%...Equation 5.108:
```

```
Do = dot(Rho1,p1);
```

```
%...Equations 5.109b, 5.110b and 5.111b:
```

```
D = [[dot(R1,p1) dot(R1,p2) dot(R1,p3)]]
```

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```
[dot(R2,p1) dot(R2,p2) dot(R2,p3)]
[dot(R3,p1) dot(R3,p2) dot(R3,p3)]];

%...Equation 5.115b:
E = dot(R2,Rho2);

%...Equations 5.112b and 5.112c:
A = 1/Do*(-D(1,2)*tau3/tau + D(2,2) + D(3,2)*tau1/tau);
B = 1/6/Do*(D(1,2)*(tau3^2 - tau^2)*tau3/tau ...
    + D(3,2)*(tau^2 - tau1^2)*tau1/tau);

%...Equations 5.117:
a = -(A^2 + 2*A*E + norm(R2)^2);
b = -2*mu*B*(A + E);
c = -(mu*B)^2;

%...Calculate the roots of Equation 5.116 using MATLAB's
% polynomial 'roots' solver:
Roots = roots([1 0 a 0 0 b 0 0 c]);

%...Find the positive real root:
x = posroot(Roots);

%...Equations 5.99a and 5.99b:
f1 = 1 - 1/2*mu*tau1^2/x^3;
f3 = 1 - 1/2*mu*tau3^2/x^3;

%...Equations 5.100a and 5.100b:
g1 = tau1 - 1/6*mu*(tau1/x)^3;
g3 = tau3 - 1/6*mu*(tau3/x)^3;

%...Equation 5.112a:
rho2 = A + mu*B/x^3;

%...Equation 5.113:
rho1 = 1/Do*((6*(D(3,1)*tau1/tau3 + D(2,1)*tau/tau3)*x^3 ...
    + mu*D(3,1)*(tau^2 - tau1^2)*tau1/tau3) ...
    /(6*x^3 + mu*(tau^2 - tau3^2)) - D(1,1));

%...Equation 5.114:
rho3 = 1/Do*((6*(D(1,3)*tau3/tau1 - D(2,3)*tau/tau1)*x^3 ...
    + mu*D(1,3)*(tau^2 - tau3^2)*tau3/tau1) ...
    /(6*x^3 + mu*(tau^2 - tau1^2)) - D(3,3));

%...Equations 5.86:
r1 = R1 + rho1*Rho1;
r2 = R2 + rho2*Rho2;
r3 = R3 + rho3*Rho3;
```

```

%...Equation 5.118:
v2 = (-f3*r1 + f1*r3)/(f1*g3 - f3*g1);

%...Save the initial estimates of r2 and v2:
r_old = r2;
v_old = v2;

%...End of Algorithm 5.5

%...Use Algorithm 5.6 to improve the accuracy of the initial estimates.

%...Initialize the iterative improvement loop and set error tolerance:
rho1_old = rho1; rho2_old = rho2; rho3_old = rho3;
diff1 = 1; diff2 = 1; diff3 = 1;
n = 0;
nmax = 1000;
tol = 1.e-8;

%...Iterative improvement loop:
while ((diff1 > tol) & (diff2 > tol) & (diff3 > tol)) & (n < nmax)
    n = n+1;

%...Compute quantities required by universal Kepler's equation:
    ro = norm(r2);
    vo = norm(v2);
    vro = dot(v2,r2)/ro;
    a = 2/ro - vo^2/mu;

%...Solve universal Kepler's equation at times tau1 and tau3 for
% universal anomalies x1 and x3:
    x1 = kepler_U(tau1, ro, vro, a);
    x3 = kepler_U(tau3, ro, vro, a);

%...Calculate the Lagrange f and g coefficients at times tau1
% and tau3:
    [ff1, gg1] = f_and_g(x1, tau1, ro, a);
    [ff3, gg3] = f_and_g(x3, tau3, ro, a);

%...Update the f and g functions at times tau1 and tau3 by
% averaging old and new:
    f1 = (f1 + ff1)/2;
    f3 = (f3 + ff3)/2;
    g1 = (g1 + gg1)/2;
    g3 = (g3 + gg3)/2;

```

```

%...Equations 5.96 and 5.97:
c1    = g3/(f1*g3 - f3*g1);
c3    = -g1/(f1*g3 - f3*g1);

%...Equations 5.109a, 5.110a and 5.111a:
rho1  = 1/Do*(      -D(1,1) + 1/c1*D(2,1) - c3/c1*D(3,1));
rho2  = 1/Do*(      -c1*D(1,2) +      D(2,2) -      c3*D(3,2));
rho3  = 1/Do*(-c1/c3*D(1,3) + 1/c3*D(2,3) -      D(3,3));

%...Equations 5.86:
r1    = R1 + rho1*Rho1;
r2    = R2 + rho2*Rho2;
r3    = R3 + rho3*Rho3;

%...Equation 5.118:
v2    = (-f3*r1 + f1*r3)/(f1*g3 - f3*g1);

%...Calculate differences upon which to base convergence:
diff1 = abs(rho1 - rho1_old);
diff2 = abs(rho2 - rho2_old);
diff3 = abs(rho3 - rho3_old);

%...Update the slant ranges:
rho1_old = rho1; rho2_old = rho2; rho3_old = rho3;
end
%...End iterative improvement loop

fprintf('\n( **Number of Gauss improvement iterations = %g)\n\n',n)

if n >= nmax
    fprintf('\n\n **Number of iterations exceeds %g \n\n ',nmax);
end

%...Return the state vector for the central observation:
r = r2;
v = v2;

return

% ~~~~~
function x = posroot(Roots)
% ~~~~~
%{
    This subfunction extracts the positive real roots from
    those obtained in the call to MATLAB's 'roots' function.
    If there is more than one positive root, the user is
    prompted to select the one to use.

```

```

x          - the determined or selected positive root
Roots      - the vector of roots of a polynomial
posroots   - vector of positive roots

User M-functions required: none
%}
% ~~~~~

%...Construct the vector of positive real roots:
posroots = Roots(find(Roots>0 & ~imag(Roots)));
npositive = length(posroots);

%...Exit if no positive roots exist:
if npositive == 0
    fprintf('\n\n ** There are no positive roots. \n\n')
    return
end

%...If there is more than one positive root, output the
% roots to the command window and prompt the user to
% select which one to use:
if npositive == 1
    x = posroots;
else
    fprintf('\n\n ** There are two or more positive roots.\n\n')
    for i = 1:npositive
        fprintf('\n root #%g = %g',i,posroots(i))
    end
    fprintf('\n\n Make a choice:\n\n')
    nchoice = 0;
    while nchoice < 1 | nchoice > npositive
        nchoice = input(' Use root #? ');
    end
    x = posroots(nchoice);
    fprintf('\n We will use %g .\n', x)
end

end %posroot

end %gauss
% ~~~~~

```

Script file: Example_5_11.m

```

% ~~~~~
% Example_5_11
% ~~~~~
%{

```

This program uses Algorithms 5.5 and 5.6 (Gauss's method) to compute the state vector from the data provided in Example 5.11.

```
deg      - factor for converting between degrees and radians
pi       - 3.1415926...
mu       - gravitational parameter (km^3/s^2)
Re       - earth's radius (km)
f        - earth's flattening factor
H        - elevation of observation site (km)
phi      - latitude of site (deg)
t        - vector of observation times t1, t2, t3 (s)
ra       - vector of topocentric equatorial right ascensions
          at t1, t2, t3 (deg)
dec      - vector of topocentric equatorial right declinations
          at t1, t2, t3 (deg)
theta    - vector of local sidereal times for t1, t2, t3 (deg)
R        - matrix of site position vectors at t1, t2, t3 (km)
rho      - matrix of direction cosine vectors at t1, t2, t3
fac1, fac2 - common factors
r_old, v_old - the state vector without iterative improvement (km, km/s)
r, v     - the state vector with iterative improvement (km, km/s)
coe      - vector of orbital elements for r, v:
          [h, e, RA, incl, w, TA, a]
          where h = angular momentum (km^2/s)
                 e = eccentricity
                 incl = inclination (rad)
                 w = argument of perigee (rad)
                 TA = true anomaly (rad)
                 a = semimajor axis (km)
coe_old   - vector of orbital elements for r_old, v_old
```

User M-functions required: gauss, coe_from_sv

```
%}
```

```
% -----
```

```
clear all; clc
```

```
global mu
```

```
deg = pi/180;
mu = 398600;
Re = 6378;
f = 1/298.26;
```

```
%...Data declaration for Example 5.11:
```

```
H = 1;
phi = 40*deg;
t = [ 0 118.104 237.577];
```

```

ra    = [ 43.5365   54.4196   64.3178]*deg;
dec   = [-8.78334  -12.0739  -15.1054]*deg;
theta = [ 44.5065   45.000   45.4992]*deg;
%...

%...Equations 5.64, 5.76 and 5.79:
fac1 = Re/sqrt(1-(2*f - f*f)*sin(phi)^2);
fac2 = (Re*(1-f)^2/sqrt(1-(2*f - f*f)*sin(phi)^2) + H)*sin(phi);
for i = 1:3
    R(i,1) = (fac1 + H)*cos(phi)*cos(theta(i));
    R(i,2) = (fac1 + H)*cos(phi)*sin(theta(i));
    R(i,3) = fac2;
    rho(i,1) = cos(dec(i))*cos(ra(i));
    rho(i,2) = cos(dec(i))*sin(ra(i));
    rho(i,3) = sin(dec(i));
end

%...Algorithms 5.5 and 5.6:
[r, v, r_old, v_old] = gauss(rho(1,:), rho(2,:), rho(3,:), ...
                             R(1,:),   R(2,:),   R(3,:), ...
                             t(1),     t(2),     t(3));

%...Algorithm 4.2 for the initial estimate of the state vector
% and for the iteratively improved one:
coe_old = coe_from_sv(r_old,v_old,mu);
coe      = coe_from_sv(r,v,mu);

%...Echo the input data and output the solution to
% the command window:
fprintf('-----')
fprintf('\n Example 5.11: Orbit determination by the Gauss method\n')
fprintf('\n Radius of earth (km)           = %g', Re)
fprintf('\n Flattening factor                   = %g', f)
fprintf('\n Gravitational parameter (km^3/s^2) = %g', mu)
fprintf('\n\n Input data:\n');
fprintf('\n Latitude (deg)                     = %g', phi/deg);
fprintf('\n Altitude above sea level (km) = %g', H);
fprintf('\n\n Observations:')
fprintf('\n                Right')
fprintf('\n                Local')
fprintf('\n Time (s) Ascension (deg) Declination (deg)')
fprintf('\n Sidereal time (deg)')
for i = 1:3
    fprintf('\n %9.4g %11.4f %19.4f %20.4f', ...
            t(i), ra(i)/deg, dec(i)/deg, theta(i)/deg)
end

fprintf('\n\n Solution:\n')

```

```

fprintf('\n Without iterative improvement...\n')
fprintf('\n');
fprintf('\n r (km)                = [%g, %g, %g]', ...
        r_old(1), r_old(2), r_old(3))
fprintf('\n v (km/s)              = [%g, %g, %g]', ...
        v_old(1), v_old(2), v_old(3))
fprintf('\n');

fprintf('\n Angular momentum (km^2/s) = %g', coe_old(1))
fprintf('\n Eccentricity                = %g', coe_old(2))
fprintf('\n RA of ascending node (deg)      = %g', coe_old(3)/deg)
fprintf('\n Inclination (deg)                = %g', coe_old(4)/deg)
fprintf('\n Argument of perigee (deg)         = %g', coe_old(5)/deg)
fprintf('\n True anomaly (deg)               = %g', coe_old(6)/deg)
fprintf('\n Semimajor axis (km)              = %g', coe_old(7))
fprintf('\n Periapse radius (km)             = %g', coe_old(1)^2 ...
        /mu/(1 + coe_old(2)))

%...If the orbit is an ellipse, output the period:
if coe_old(2)<1
    T = 2*pi/sqrt(mu)*coe_old(7)^1.5;
    fprintf('\n Period:')
    fprintf('\n Seconds                = %g', T)
    fprintf('\n Minutes                  = %g', T/60)
    fprintf('\n Hours                    = %g', T/3600)
    fprintf('\n Days                     = %g', T/24/3600)
end

fprintf('\n\n With iterative improvement...\n')
fprintf('\n');
fprintf('\n r (km)                = [%g, %g, %g]', ...
        r(1), r(2), r(3))
fprintf('\n v (km/s)              = [%g, %g, %g]', ...
        v(1), v(2), v(3))
fprintf('\n');
fprintf('\n Angular momentum (km^2/s) = %g', coe(1))
fprintf('\n Eccentricity                = %g', coe(2))
fprintf('\n RA of ascending node (deg) = %g', coe(3)/deg)
fprintf('\n Inclination (deg)          = %g', coe(4)/deg)
fprintf('\n Argument of perigee (deg)  = %g', coe(5)/deg)
fprintf('\n True anomaly (deg)         = %g', coe(6)/deg)
fprintf('\n Semimajor axis (km)        = %g', coe(7))
fprintf('\n Periapse radius (km)       = %g', coe(1)^2 ...
        /mu/(1 + coe(2)))

%...If the orbit is an ellipse, output the period:
if coe(2)<1
    T = 2*pi/sqrt(mu)*coe(7)^1.5;
    fprintf('\n Period:')
    fprintf('\n Seconds                = %g', T)

```

```

        fprintf('\n      Minutes                = %g', T/60)
        fprintf('\n      Hours                = %g', T/3600)
        fprintf('\n      Days                = %g', T/24/3600)
    end
    fprintf('\n-----\n')

% ~ ~ ~ ~ ~

```

Output from Example_5_11

```
( **Number of Gauss improvement iterations = 14)
```

 Example 5.11: Orbit determination by the Gauss method

```

Radius of earth (km)           = 6378
Flattening factor              = 0.00335278
Gravitational parameter (km^3/s^2) = 398600

```

Input data:

```

Latitude (deg)                 = 40
Altitude above sea level (km) = 1

```

Observations:

	Right		Local
Time (s)	Ascension (deg)	Declination (deg)	Sidereal time (deg)
0	43.5365	-8.7833	44.5065
118.1	54.4196	-12.0739	45.0000
237.6	64.3178	-15.1054	45.4992

Solution:

Without iterative improvement...

```

r (km)                        = [5659.03, 6533.74, 3270.15]
v (km/s)                      = [-3.8797, 5.11565, -2.2397]

```

```

Angular momentum (km^2/s)    = 62705.3
Eccentricity                  = 0.097562
RA of ascending node (deg)    = 270.023
Inclination (deg)             = 30.0105
Argument of perigee (deg)     = 88.654
True anomaly (deg)            = 46.3163
Semimajor axis (km)           = 9959.2
Periapse radius (km)          = 8987.56

```

```

Period:
  Seconds           = 9891.17
  Minutes           = 164.853
  Hours             = 2.74755
  Days              = 0.114481

```

With iterative improvement...

```

r (km)              = [5662.04, 6537.95, 3269.05]
v (km/s)            = [-3.88542, 5.12141, -2.2434]

```

```

Angular momentum (km^2/s) = 62816.7
Eccentricity              = 0.0999909
RA of ascending node (deg) = 269.999
Inclination (deg)         = 30.001
Argument of perigee (deg) = 89.9723
True anomaly (deg)        = 45.0284
Semimajor axis (km)       = 9999.48
Periapse radius (km)      = 8999.62
Period:
  Seconds           = 9951.24
  Minutes           = 165.854
  Hours             = 2.76423
  Days              = 0.115176

```

Chapter 6

D.30 Calculate the state vector after a finite-time, constant thrust delta-v maneuver

Function file: `integrate_thrust.m`

```

% ~~~~~
function integrate_thrust
% ~~~~~
%{
  This function uses rkf45 to numerically integrate Equation 6.26 during
  the delta-v burn and then find the apogee of the post-burn orbit.

  The input data are for the first part of Example 6.15.

  mu          - gravitational parameter (km^3/s^2)
  RE          - earth radius (km)
  g0          - sea-level acceleration of gravity (m/s^2)

```

```

T          - rated thrust of rocket engine (kN)
Isp        - specific impulse of rocket engine (s)
m0         - initial spacecraft mass (kg)
r0         - initial position vector (km)
v0         - initial velocity vector (km/s)
t0         - initial time (s)
t_burn     - rocket motor burn time (s)
y0         - column vector containing r0, v0 and m0
t          - column vector of the times at which the solution is found (s)
y          - a matrix whose elements are:
              columns 1, 2 and 3:
                  The solution for the x, y and z components of the
                  position vector r at the times t
              columns 4, 5 and 6:
                  The solution for the x, y and z components of the
                  velocity vector v at the times t
              column 7:
                  The spacecraft mass m at the times t
r1         - position vector after the burn (km)
v1         - velocity vector after the burn (km/s)
m1         - mass after the burn (kg)
coe        - orbital elements of the post-burn trajectory
              (h e RA incl w TA a)
ra         - position vector vector at apogee (km)
va         - velocity vector at apogee (km)
rmax       - apogee radius (km)

User M-functions required: rkf45, coe_from_sv, rv_from_r0v0_ta
User subfunctions required: rates, output
%}
% -----

%...Preliminaries:
clear all; close all; clc
global mu
deg      = pi/180;
mu       = 398600;
RE       = 6378;
g0       = 9.807;

%...Input data:
r0       = [RE+480    0    0];
v0       = [ 0    7.7102 0];
t0       = 0;
t_burn   = 261.1127;

m0       = 2000;
T        = 10;

```

```

Isp      = 300;
%...end Input data

%...Integrate the equations of motion over the burn time:
y0      = [r0 v0 m0]';
[t,y] = rkf45(@rates, [t0 t_burn], y0, 1.e-16);

%...Compute the state vector and mass after the burn:
r1      = [y(end,1) y(end,2) y(end,3)];
v1      = [y(end,4) y(end,5) y(end,6)];
m1      = y(end,7);
coe     = coe_from_sv(r1,v1,mu);
e       = coe(2); %eccentricity
TA      = coe(6); %true anomaly (radians)
a       = coe(7); %semimajor axis

%...Find the state vector at apogee of the post-burn trajectory:
if TA <= pi
    dtheta = pi - TA;
else
    dtheta = 3*pi - TA;
end
[ra,va] = rv_from_r0v0_ta(r1, v1, dtheta/deg, mu);
rmax    = norm(ra);

output

%...Subfunctions:

%~~~~~
function dfdt = rates(t,f)
%~~~~~
%{
    This function calculates the acceleration vector using Equation 6.26.

    t          - time (s)
    f          - column vector containing the position vector, velocity
                  vector and the mass at time t
    x, y, z    - components of the position vector (km)
    vx, vy, vz - components of the velocity vector (km/s)
    m          - mass (kg)
    r          - magnitude of the position vector (km)
    v          - magnitude of the velocity vector (km/s)
    ax, ay, az - components of the acceleration vector (km/s^2)
    mdot       - rate of change of mass (kg/s)
    dfdt       - column vector containing the velocity and acceleration
                  components and the mass rate
%}

```

```

% -----
x = f(1); y = f(2); z = f(3);
vx = f(4); vy = f(5); vz = f(6);
m = f(7);

r = norm([x y z]);
v = norm([vx vy vz]);
ax = -mu*x/r^3 + T/m*vx/v;
ay = -mu*y/r^3 + T/m*vy/v;
az = -mu*z/r^3 + T/m*vz/v;
mdot = -T*1000/g0/Isp;

dfdt = [vx vy vz ax ay az mdot]';

end %rates

%~~~~~
function output
%~~~~~
fprintf('\n\n-----\n')
fprintf('\nBefore ignition:')
fprintf('\n Mass = %g kg', m0)
fprintf('\n State vector:')
fprintf('\n r = [%10g, %10g, %10g] (km)', r0(1), r0(2), r0(3))
fprintf('\n Radius = %g', norm(r0))
fprintf('\n v = [%10g, %10g, %10g] (km/s)', v0(1), v0(2), v0(3))
fprintf('\n Speed = %g\n', norm(v0))
fprintf('\nThrust = %12g kN', T)
fprintf('\nBurn time = %12.6f s', t_burn)
fprintf('\nMass after burn = %12.6E kg\n', m1)
fprintf('\nEnd-of-burn-state vector:')
fprintf('\n r = [%10g, %10g, %10g] (km)', r1(1), r1(2), r1(3))
fprintf('\n Radius = %g', norm(r1))
fprintf('\n v = [%10g, %10g, %10g] (km/s)', v1(1), v1(2), v1(3))
fprintf('\n Speed = %g\n', norm(v1))
fprintf('\nPost-burn trajectory:')
fprintf('\n Eccentricity = %g', e)
fprintf('\n Semimajor axis = %g km', a)
fprintf('\n Apogee state vector:')
fprintf('\n r = [%17.10E, %17.10E, %17.10E] (km)', ra(1), ra(2), ra(3))
fprintf('\n Radius = %g', norm(ra))
fprintf('\n v = [%17.10E, %17.10E, %17.10E] (km/s)', va(1), va(2), va(3))
fprintf('\n Speed = %g', norm(va))
fprintf('\n\n-----\n\n')

end %output

end %integrate_thrust

```

Chapter 7

D.31 Algorithm 7.1: Find the position, velocity, and acceleration of *B* relative to *A*'s LVLH frame

Function file: rva_relative.m

```
% ~~~~~
function [r_rel_x, v_rel_x, a_rel_x] = rva_relative(rA,vA,rB,vB)
% ~~~~~
%{
    This function uses the state vectors of spacecraft A and B
    to find the position, velocity and acceleration of B relative
    to A in the LVLH frame attached to A (see Figure 7.1).

    rA,vA      - state vector of A (km, km/s)
    rB,vB      - state vector of B (km, km/s)
    mu         - gravitational parameter (km^3/s^2)
    hA         - angular momentum vector of A (km^2/s)
    i, j, k    - unit vectors along the x, y and z axes of A's
                  LVLH frame
    QXx        - DCM of the LVLH frame relative to the geocentric
                  equatorial frame (GEF)
    Omega      - angular velocity of the LVLH frame (rad/s)
    Omega_dot  - angular acceleration of the LVLH frame (rad/s^2)
    aA, aB     - absolute accelerations of A and B (km/s^2)
    r_rel      - position of B relative to A in GEF (km)
    v_rel      - velocity of B relative to A in GEF (km/s)
    a_rel      - acceleration of B relative to A in GEF (km/s^2)
    r_rel_x    - position of B relative to A in the LVLH frame
    v_rel_x    - velocity of B relative to A in the LVLH frame
    a_rel_x    - acceleration of B relative to A in the LVLH frame

    User M-functions required: None
%}
% -----

global mu

%...Calculate the vector hA:
hA = cross(rA, vA);

%...Calculate the unit vectors i, j and k:
i = rA/norm(rA);
```

```

k = hA/norm(hA);
j = cross(k,i);

%...Calculate the transformation matrix Qxx:
QXx = [i; j; k];

%...Calculate Omega and Omega_dot:
Omega      = hA/norm(rA)^2;           % Equation 7.5
Omega_dot = -2*dot(rA,vA)/norm(rA)^2*Omega;% Equation 7.6

%...Calculate the accelerations aA and aB:
aA = -mu*rA/norm(rA)^3;
aB = -mu*rB/norm(rB)^3;

%...Calculate r_rel:
r_rel = rB - rA;

%...Calculate v_rel:
v_rel = vB - vA - cross(Omega,r_rel);

%...Calculate a_rel:
a_rel = aB - aA - cross(Omega_dot,r_rel)...
        - cross(Omega,cross(Omega,r_rel))...
        - 2*cross(Omega,v_rel);

%...Calculate r_rel_x, v_rel_x and a_rel_x:
r_rel_x = QXx*r_rel';
v_rel_x = QXx*v_rel';
a_rel_x = QXx*a_rel';

end %rva_relative

% ~~~~~

```

Script file: Example_7_01.m

```

% ~~~~~
% Example_7_01
% ~~~~~
%{
    This program uses the data of Example 7.1 to calculate the position,
    velocity and acceleration of an orbiting chaser B relative to an
    orbiting target A.

    mu           - gravitational parameter (km^3/s^2)
    deg           - conversion factor from degrees to radians

```

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```

                                Spacecraft A & B:
h_A, h_B      -   angular momentum (km^2/s)
e_A, E_B      -   eccentricity
i_A, i_B      -   inclination (radians)
RAAN_A, RAAN_B -   right ascension of the ascending node (radians)
omega_A, omega_B - argument of perigee (radians)
theta_A, theta_A - true anomaly (radians)

rA, vA        -   inertial position (km) and velocity (km/s) of A
rB, vB        -   inertial position (km) and velocity (km/s) of B
r             -   position (km) of B relative to A in A's
                  co-moving frame
v             -   velocity (km/s) of B relative to A in A's
                  co-moving frame
a             -   acceleration (km/s^2) of B relative to A in A's
                  co-moving frame

User M-function required:  sv_from_coe, rva_relative
User subfunctions required: none
%}
% -----

clear all; clc
global mu
mu = 398600;
deg = pi/180;

%...Input data:

%   Spacecraft A:
h_A    = 52059;
e_A    = 0.025724;
i_A    = 60*deg;
RAAN_A = 40*deg;
omega_A = 30*deg;
theta_A = 40*deg;

%   Spacecraft B:
h_B    = 52362;
e_B    = 0.0072696;
i_B    = 50*deg;
RAAN_B = 40*deg;
omega_B = 120*deg;
theta_B = 40*deg;

%...End input data
```

```

%...Compute the initial state vectors of A and B using Algorithm 4.5:
[rA,vA] = sv_from_coe([h_A e_A RAAN_A i_A omega_A theta_A],mu);
[rB,vB] = sv_from_coe([h_B e_B RAAN_B i_B omega_B theta_B],mu);

%...Compute relative position, velocity and acceleration using
%   Algorithm 7.1:
[r,v,a] = rva_relative(rA,vA,rB,vB);

%...Output
fprintf('\n\n-----\n\n')
fprintf('\nOrbital parameters of spacecraft A:')
fprintf('\n   angular momentum   = %g (km^2/s)', h_A)
fprintf('\n   eccentricity         = %g', e_A)
fprintf('\n   inclination          = %g (deg)', i_A/deg)
fprintf('\n   RAAN                 = %g (deg)', RAAN_A/deg)
fprintf('\n   argument of perigee = %g (deg)', omega_A/deg)
fprintf('\n   true anomaly         = %g (deg)\n', theta_A/deg)

fprintf('\nState vector of spacecraft A:')
fprintf('\n   r = [%g, %g, %g]', rA(1), rA(2), rA(3))
fprintf('\n           (magnitude = %g)', norm(rA))
fprintf('\n   v = [%g, %g, %g]', vA(1), vA(2), vA(3))
fprintf('\n           (magnitude = %g)\n', norm(vA))

fprintf('\nOrbital parameters of spacecraft B:')
fprintf('\n   angular momentum   = %g (km^2/s)', h_B)
fprintf('\n   eccentricity         = %g', e_B)
fprintf('\n   inclination          = %g (deg)', i_B/deg)
fprintf('\n   RAAN                 = %g (deg)', RAAN_B/deg)
fprintf('\n   argument of perigee = %g (deg)', omega_B/deg)
fprintf('\n   true anomaly         = %g (deg)\n', theta_B/deg)

fprintf('\nState vector of spacecraft B:')
fprintf('\n   r = [%g, %g, %g]', rB(1), rB(2), rB(3))
fprintf('\n           (magnitude = %g)', norm(rB))
fprintf('\n   v = [%g, %g, %g]', vB(1), vB(2), vB(3))
fprintf('\n           (magnitude = %g)\n', norm(vB))

fprintf('\nIn the co-moving frame attached to A:')
fprintf('\n   Position of B relative to A = [%g, %g, %g]', ...
        r(1), r(2), r(3))
fprintf('\n           (magnitude = %g)\n', norm(r))
fprintf('\n   Velocity of B relative to A = [%g, %g, %g]', ...
        v(1), v(2), v(3))
fprintf('\n           (magnitude = %g)\n', norm(v))
fprintf('\n   Acceleration of B relative to A = [%g, %g, %g]', ...
        a(1), a(2), a(3))
fprintf('\n           (magnitude = %g)\n', norm(a))

```

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```
fprintf('\n\n-----\n\n')
```

```
% ~~~~~
```

Output from Example_7_01.m

Orbital parameters of spacecraft A:

```
angular momentum = 52059 (km^2/s)
eccentricity      = 0.025724
inclination      = 60 (deg)
RAAN             = 40 (deg)
argument of perigee = 30 (deg)
true anomaly     = 40 (deg)
```

State vector of spacecraft A:

```
r = [-266.768, 3865.76, 5426.2]
    (magnitude = 6667.75)
v = [-6.48356, -3.61975, 2.41562]
    (magnitude = 7.8086)
```

Orbital parameters of spacecraft B:

```
angular momentum = 52362 (km^2/s)
eccentricity      = 0.0072696
inclination      = 50 (deg)
RAAN             = 40 (deg)
argument of perigee = 120 (deg)
true anomaly     = 40 (deg)
```

State vector of spacecraft B:

```
r = [-5890.71, -2979.76, 1792.21]
    (magnitude = 6840.43)
v = [0.935828, -5.2403, -5.50095]
    (magnitude = 7.65487)
```

In the co-moving frame attached to A:

```
Position of B relative to A = [-6701.15, 6828.27, -406.261]
    (magnitude = 9575.79)
```

```
Velocity of B relative to A = [0.316667, 0.111993, 1.24696]
    (magnitude = 1.29141)
```

```
Acceleration of B relative to A = [-0.000222229, -0.000180743, 0.000505932]
    (magnitude = 0.000581396)
```

D.32 Plot the position of one spacecraft relative to another

Script file: Example_7_02.m

```
% ~~~~~
% Example_7_02
% ~~~~~
%{
    This program produces a 3D plot of the motion of spacecraft B
    relative to A in Example 7.1. See Figure 7.4.

    User M-functions required: rv_from_r0v0 (Algorithm 3.4)
                             sv_from_coe (Algorithm 4.5)
                             rva_relative (Algorithm 7.1)
%}
% -----

clear all; close all; clc

global mu

%...Gravitational parameter and earth radius:
mu = 398600;
RE = 6378;

%...Conversion factor from degrees to radians:
deg = pi/180;

%...Input data:
% Initial orbital parameters (angular momentum, eccentricity,
% inclination, RAAN, argument of perigee and true anomaly).
% Spacecraft A:
h_A = 52059;
e_A = 0.025724;
i_A = 60*deg;
RAAN_A = 40*deg;
omega_A = 30*deg;
theta_A = 40*deg;

% Spacecraft B:
h_B = 52362;
e_B = 0.0072696;
i_B = 50*deg;
RAAN_B = 40*deg;
omega_B = 120*deg;
theta_B = 40*deg;
```

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```
vdir = [1 1 1];

%...End input data

%...Compute the initial state vectors of A and B using Algorithm 4.5:
[rA0,vA0] = sv_from_coe([h_A e_A RAAN_A i_A omega_A theta_A],mu);
[rB0,vB0] = sv_from_coe([h_B e_B RAAN_B i_B omega_B theta_B],mu);

h0 = cross(rA0,vA0);

%...Period of A:
TA = 2*pi/mu^2*(h_A/sqrt(1 - e_A^2))^3;

%...Number of time steps per period of A's orbit:
n = 100;

%...Time step as a fraction of A's period:
dt = TA/n;

%...Number of periods of A's orbit for which the trajectory
% will be plotted:
n_Periods = 60;

%...Initialize the time:
t = - dt;

%...Generate the trajectory of B relative to A:
for count = 1:n_Periods*n

%...Update the time:
    t = t + dt;

%...Update the state vector of both orbits using Algorithm 3.4:
    [rA,vA] = rv_from_r0v0(rA0, vA0, t);
    [rB,vB] = rv_from_r0v0(rB0, vB0, t);

%...Compute r_rel using Algorithm 7.1:
    [r_rel, v_rel, a_rel] = rva_relative(rA,vA,rB,vB);

%...Store the components of the relative position vector
% at this time step in the vectors x, y and z, respectively:
    x(count) = r_rel(1);
    y(count) = r_rel(2);
    z(count) = r_rel(3);
    r(count) = norm(r_rel);
    T(count) = t;
end
```

```

%...Plot the trajectory of B relative to A:
figure(1)
plot3(x, y, z)
hold on
axis equal
axis on
grid on
box off
view(vdir)
% Draw the co-moving x, y and z axes:
line([0 4000], [0 0], [0 0]); text(4000, 0, 0, 'x')
line([0 0], [0 7000], [0 0]); text(0, 7000, 0, 'y')
line([0 0], [0 0], [0 4000]); text(0, 0, 4000, 'z')

% Label the origin of the moving frame attached to A:
text(0, 0, 0, 'A')

% Label the start of B's relative trajectory:
text(x(1), y(1), z(1), 'B')

% Draw the initial position vector of B:
line([0 x(1)], [0 y(1)], [0 z(1)])
% ~~~~~

```

D.33 Solution of the linearized equations of relative motion with an elliptical reference orbit

Function file: Example_7_03.m

```

% ~~~~~
function Example_7_03
% ~~~~~
%{
    This function plots the motion of chaser B relative to target A
    for the data in Example 7.3. See Figures 7.6 and 7.7.

    mu      - gravitational parameter (km^3/s^2)
    RE      - radius of the earth (km)

    Target orbit at time t = 0:
    rp      - perigee radius (km)
    e       - eccentricity
    i       - inclination (rad)
    RA      - right ascension of the ascending node (rad)
    omega   - argument of perigee (rad)
    theta   - true anomaly (rad)
%}

```

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```
ra      - apogee radius (km)
h      - angular momentum (km^2/s)
a      - semimajor axis (km)
T      - period (s)
n      - mean motion (rad/s)

dr0, dv0 - initial relative position (km) and relative velocity (km/s)
           of B in the co-moving frame
t0, tf  - initial and final times (s) for the numerical integration
R0, V0  - initial position (km) and velocity (km/s) of A in the
           geocentric equatorial frame
y0      - column vector containing r0, v0
%}
% User M-functions required: sv_from_coe, rkf45
% User subfunctions required: rates
% -----

clear all; close all; clc

global mu

mu = 398600;
RE = 6378;

%...Input data:
% Prescribed initial orbital parameters of target A:
rp = RE + 300;
e = 0.1;
i = 0;
RA = 0;
omega = 0;
theta = 0;

% Additional computed parameters:
ra = rp*(1 + e)/(1 - e);
h = sqrt(2*mu*rp*ra/(ra + rp));
a = (rp + ra)/2;
T = 2*pi/sqrt(mu)*a^1.5;
n = 2*pi/T;

% Prescribed initial state vector of chaser B in the co-moving frame:
dr0 = [-1 0 0];
dv0 = [0 0 -2*n*dr0(1) 0];
t0 = 0;
tf = 5*T;
%...End input data
```

```

%...Calculate the target's initial state vector using Algorithm 4.5:
[R0,V0] = sv_from_coe([h e RA i omega theta],mu);

%...Initial state vector of B's orbit relative to A
y0 = [dr0 dv0]';

%...Integrate Equations 7.34 using Algorithm 1.3:
[t,y] = rkf45(@rates, [t0 tf], y0);

plotit

return

% ~~~~~~
function dydt = rates(t,f)
% ~~~~~~
%{
    This function computes the components of f(t,y) in Equation 7.36.

    t          - time
    f          - column vector containing the relative position and
                  velocity vectors of B at time t

    R, V       - updated state vector of A at time t
    X, Y, Z    - components of R
    VX, VY, VZ - components of V
    R_         - magnitude of R
    RdotV      - dot product of R and V
    h          - magnitude of the specific angular momentum of A

    dx , dy , dz - components of the relative position vector of B
    dvx, dvy, dvz - components of the relative velocity vector of B
    dax, day, daz - components of the relative acceleration vector of B
    dydt       - column vector containing the relative velocity
                  and acceleration components of B at time t

    User M-function required: rv_from_r0v0
%}
% -----
%...Update the state vector of the target orbit using Algorithm 3.4:
[R,V] = rv_from_r0v0(R0, V0, t);

X  = R(1); Y  = R(2); Z  = R(3);
VX = V(1); VY = V(2); VZ = V(3);

R_  = norm([X Y Z]);
RdotV = dot([X Y Z], [VX VY VZ]);
h    = norm(cross([X Y Z], [VX VY VZ]));

```

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```
dx    = f(1); dy  = f(2); dz  = f(3);
dvx   = f(4); dvy = f(5); dvz = f(6);

dax   = (2*mu/R_^3 + h^2/R_^4)*dx - 2*RdotV/R_^4*h*dy + 2*h/R_^2*dvy;
day   = -(mu/R_^3 - h^2/R_^4)*dy + 2*RdotV/R_^4*h*dx - 2*h/R_^2*dvx;
daz   = -mu/R_^3*dz;

dydt  = [dvx dvy dvz dax day daz]';
end %rates

% ~~~~~~
function plotit
% ~~~~~~
%...Plot the trajectory of B relative to A:
% -----
hold on
plot(y(:,2), y(:,1))
axis on
axis equal
axis ([0 40 -5 5])
xlabel('y (km)')
ylabel('x (km)')
grid on
box on
%...Label the start of B's trajectory relative to A:
text(y(1,2), y(1,1), 'o')
end %plotit

end %Example_7_03
% ~~~~~~
```

Chapter 8

D.34 Convert the numerical designation of a month or a planet into its name

The following trivial script can be used in programs that input of the numerical values for a month and/or a planet.

Function file: month_planet_names.m

```
% ~~~~~~
function [month, planet] = month_planet_names(month_id, planet_id)
% ~~~~~~
```

```

%{
    This function returns the name of the month and the planet
    corresponding, respectively, to the numbers "month_id" and
    "planet_id".

    months    - a vector containing the names of the 12 months
    planets    - a vector containing the names of the 9 planets
    month_id   - the month number (1 - 12)
    planet_id  - the planet number (1 - 9)

    User M-functions required: none
%}
% -----

months = ['January '
          'February '
          'March '
          'April '
          'May '
          'June '
          'July '
          'August '
          'September'
          'October '
          'November '
          'December '];

planets = ['Mercury'
           'Venus '
           'Earth '
           'Mars '
           'Jupiter'
           'Saturn '
           'Uranus '
           'Neptune'
           'Pluto '];

month = months(month_id, 1:9);
planet = planets(planet_id, 1:7);
% ~~~~~~
end %month_planet_names

```

D.35 Algorithm 8.1: Calculation of the heliocentric state vector of a planet at a given epoch

Function file: planet_elements_and_sv.m

```
% ~~~~~
function [coe, r, v, jd] = planet_elements_and_sv ...
    (planet_id, year, month, day, hour, minute, second)
% ~~~~~
%{
    This function calculates the orbital elements and the state
    vector of a planet from the date (year, month, day)
    and universal time (hour, minute, second).

    mu      - gravitational parameter of the sun (km^3/s^2)
    deg      - conversion factor between degrees and radians
    pi       - 3.1415926...

    coe      - vector of heliocentric orbital elements
               [h e RA incl w TA a w_hat L M E],
               where
                   h      = angular momentum          (km^2/s)
                   e      = eccentricity
                   RA     = right ascension           (deg)
                   incl   = inclination               (deg)
                   w      = argument of perihelion     (deg)
                   TA     = true anomaly              (deg)
                   a      = semimajor axis            (km)
                   w_hat  = longitude of perihelion ( = RA + w) (deg)
                   L      = mean longitude ( = w_hat + M) (deg)
                   M      = mean anomaly              (deg)
                   E      = eccentric anomaly         (deg)

    planet_id - planet identifier:
                1 = Mercury
                2 = Venus
                3 = Earth
                4 = Mars
                5 = Jupiter
                7 = Uranus
                8 = Neptune
                9 = Pluto

    year      - range: 1901 - 2099
    month     - range: 1 - 12
    day       - range: 1 - 31
    hour      - range: 0 - 23
```

```

minute    - range: 0 - 60
second    - range: 0 - 60

j0         - Julian day number of the date at 0 hr UT
ut         - universal time in fractions of a day
jd         - Julian day number of the date and time

J2000_coe  - row vector of J2000 orbital elements from Table 9.1
rates      - row vector of Julian centennial rates from Table 9.1
t0         - Julian centuries between J2000 and jd
elements   - orbital elements at jd

r          - heliocentric position vector
v          - heliocentric velocity vector

User M-functions required: J0, kepler_E, sv_from_coe
User subfunctions required: planetary_elements, zero_to_360
%}
% -----

global mu
deg       = pi/180;

%...Equation 5.48:
j0        = J0(year, month, day);

ut        = (hour + minute/60 + second/3600)/24;

%...Equation 5.47
jd        = j0 + ut;

%...Obtain the data for the selected planet from Table 8.1:
[J2000_coe, rates] = planetary_elements(planet_id);

%...Equation 8.93a:
t0        = (jd - 2451545)/36525;

%...Equation 8.93b:
elements  = J2000_coe + rates*t0;

a         = elements(1);
e         = elements(2);

%...Equation 2.71:
h         = sqrt(mu*a*(1 - e^2));

%...Reduce the angular elements to within the range 0 - 360 degrees:
incl      = elements(3);

```

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```
RA      = zero_to_360(elements(4));
w_hat   = zero_to_360(elements(5));
L       = zero_to_360(elements(6));
w       = zero_to_360(w_hat - RA);
M       = zero_to_360((L - w_hat));

%...Algorithm 3.1 (for which M must be in radians)
E       = kepler_E(e, M*deg);

%...Equation 3.13 (converting the result to degrees):
TA      = zero_to_360...
        (2*atan(sqrt((1 + e)/(1 - e))*tan(E/2))/deg);

coe     = [h e RA incl w TA a w_hat L M E/deg];

%...Algorithm 4.5 (for which all angles must be in radians):
[r, v] = sv_from_coe([h e RA*deg incl*deg w*deg TA*deg],mu);

return

% ~~~~~
function [J2000_coe, rates] = planetary_elements(planet_id)
% ~~~~~
%{
This function extracts a planet's J2000 orbital elements and
centennial rates from Table 8.1.

planet_id      - 1 through 9, for Mercury through Pluto

J2000_elements - 9 by 6 matrix of J2000 orbital elements for the nine
planets Mercury through Pluto. The columns of each
row are:
    a      = semimajor axis (AU)
    e      = eccentricity
    i      = inclination (degrees)
    RA     = right ascension of the ascending
             node (degrees)
    w_hat  = longitude of perihelion (degrees)
    L      = mean longitude (degrees)

cent_rates     - 9 by 6 matrix of the rates of change of the
J2000_elements per Julian century (Cy). Using "dot"
for time derivative, the columns of each row are:
    a_dot    (AU/Cy)
    e_dot    (1/Cy)
    i_dot    (arcseconds/Cy)
    RA_dot   (arcseconds/Cy)
    w_hat_dot (arcseconds/Cy)
```

```

                                Ldot      (arcseconds/Cy)

J2000_coe    - row vector of J2000_elements corresponding
               to "planet_id", with au converted to km
rates        - row vector of cent_rates corresponding to
               "planet_id", with au converted to km and
               arcseconds converted to degrees

au           - astronomical unit (km)
%}
% -----

J2000_elements = ...
[ 0.38709893  0.20563069  7.00487   48.33167   77.45645  252.25084
  0.72333199  0.00677323  3.39471   76.68069  131.53298  181.97973
  1.00000011  0.01671022  0.00005  -11.26064  102.94719  100.46435
  1.52366231  0.09341233  1.85061   49.57854  336.04084  355.45332
  5.20336301  0.04839266  1.30530  100.55615   14.75385   34.40438
  9.53707032  0.05415060  2.48446  113.71504   92.43194   49.94432
 19.19126393  0.04716771  0.76986   74.22988  170.96424  313.23218
 30.06896348  0.00858587  1.76917  131.72169   44.97135   304.88003
 39.48168677  0.24880766  17.14175  110.30347  224.06676  238.92881];

cent_rates = ...
[ 0.00000066  0.00002527 -23.51   -446.30   573.57  538101628.29
  0.00000092 -0.00004938  -2.86    -996.89  -108.80  210664136.06
 -0.00000005 -0.00003804 -46.94  -18228.25  1198.28  129597740.63
 -0.00007221  0.00011902 -25.47  -1020.19  1560.78  68905103.78
  0.00060737 -0.00012880  -4.15   1217.17   839.93  10925078.35
 -0.00301530 -0.00036762   6.11  -1591.05 -1948.89  4401052.95
  0.00152025 -0.00019150  -2.09  -1681.4   1312.56  1542547.79
 -0.00125196  0.00002514  -3.64  -151.25  -844.43   786449.21
 -0.00076912  0.00006465  11.07   -37.33  -132.25  522747.90];

J2000_coe    = J2000_elements(planet_id,:);
rates        = cent_rates(planet_id,:);

%...Convert from AU to km:
au           = 149597871;
J2000_coe(1) = J2000_coe(1)*au;
rates(1)     = rates(1)*au;

%...Convert from arcseconds to fractions of a degree:
rates(3:6)   = rates(3:6)/3600;

end %planetary_elements

```

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```
% ~~~~~~
function y = zero_to_360(x)
% ~~~~~~
%{
    This function reduces an angle to lie in the range 0 - 360 degrees.

    x - the original angle in degrees
    y - the angle reduced to the range 0 - 360 degrees

%}
% -----

if x >= 360
    x = x - fix(x/360)*360;
elseif x < 0
    x = x - (fix(x/360) - 1)*360;
end

y = x;

end %zero_to_360

end %planet_elements_and_sv
% ~~~~~~
```

Script file: Example_8_07.m

```
% ~~~~~~
% Example_8_07
% ~~~~~~
%
% This program uses Algorithm 8.1 to compute the orbital elements
% and state vector of the earth at the date and time specified
% in Example 8.10. To obtain the same results for Mars, set
% planet_id = 4.
%
% mu      - gravitational parameter of the sun (km^3/s^2)
% deg     - conversion factor between degrees and radians
% pi      - 3.1415926...
%
% coe     - vector of heliocentric orbital elements
%           [h e RA incl w TA a w_hat L M E],
%           where
%           h      = angular momentum                (km^2/s)
%           e      = eccentricity
%           RA     = right ascension                 (deg)
%           incl   = inclination                     (deg)
```

```

%          w      = argument of perihelion          (deg)
%          TA     = true anomaly                   (deg)
%          a      = semimajor axis                 (km)
%          w_hat  = longitude of perihelion ( = RA + w) (deg)
%          L      = mean longitude ( = w_hat + M)   (deg)
%          M      = mean anomaly                   (deg)
%          E      = eccentric anomaly              (deg)
%
% r          - heliocentric position vector (km)
% v          - heliocentric velocity vector (km/s)
%
% planet_id - planet identifier:
%           1 = Mercury
%           2 = Venus
%           3 = Earth
%           4 = Mars
%           5 = Jupiter
%           6 = Saturn
%           7 = Uranus
%           8 = Neptune
%           9 = Pluto
%
% year       - range: 1901 - 2099
% month      - range: 1 - 12
% day        - range: 1 - 31
% hour       - range: 0 - 23
% minute     - range: 0 - 60
% second     - range: 0 - 60
%
% User M-functions required: planet_elements_and_sv,
%                           month_planet_names
% -----

global mu
mu = 1.327124e11;
deg = pi/180;

%...Input data
planet_id = 3;
year      = 2003;
month     = 8;
day       = 27;
hour      = 12;
minute    = 0;
second    = 0;
%...

```

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```
%...Algorithm 8.1:
[coe, r, v, jd] = planet_elements_and_sv ...
    (planet_id, year, month, day, hour, minute, second);

%...Convert the planet_id and month numbers into names for output:
[month_name, planet_name] = month_planet_names(month, planet_id);

%...Echo the input data and output the solution to
%   the command window:
fprintf('-----')
fprintf('\n Example 8.7')
fprintf('\n\n Input data:\n');
fprintf('\n   Planet: %s', planet_name)
fprintf('\n   Year  : %g', year)
fprintf('\n   Month : %s', month_name)
fprintf('\n   Day   : %g', day)
fprintf('\n   Hour  : %g', hour)
fprintf('\n   Minute: %g', minute)
fprintf('\n   Second: %g', second)
fprintf('\n\n   Julian day: %11.3f', jd)

fprintf('\n\n');
fprintf(' Orbital elements:')
fprintf('\n');

fprintf('\n   Angular momentum (km^2/s)           = %g', coe(1));
fprintf('\n   Eccentricity                        = %g', coe(2));
fprintf('\n   Right ascension of the ascending node (deg) = %g', coe(3));
fprintf('\n   Inclination to the ecliptic (deg)         = %g', coe(4));
fprintf('\n   Argument of perihelion (deg)             = %g', coe(5));
fprintf('\n   True anomaly (deg)                      = %g', coe(6));
fprintf('\n   Semimajor axis (km)                    = %g', coe(7));

fprintf('\n');

fprintf('\n   Longitude of perihelion (deg)           = %g', coe(8));
fprintf('\n   Mean longitude (deg)                   = %g', coe(9));
fprintf('\n   Mean anomaly (deg)                     = %g', coe(10));
fprintf('\n   Eccentric anomaly (deg)                 = %g', coe(11));

fprintf('\n\n');
fprintf(' State vector:')
fprintf('\n');

fprintf('\n   Position vector (km) = [%g %g %g]', r(1), r(2), r(3))
fprintf('\n   Magnitude            = %g\n', norm(r))
fprintf('\n   Velocity (km/s)      = [%g %g %g]', v(1), v(2), v(3))
fprintf('\n   Magnitude            = %g', norm(v))
```

```
fprintf('\n-----\n')
% ~~~~~
```

Output from Example_8_07

Example 8.7

Input data:

```
Planet: Earth
Year  : 2003
Month : August
Day   : 27
Hour  : 12
Minute: 0
Second: 0
```

Julian day: 2452879.000

Orbital elements:

```
Angular momentum (km^2/s)      = 4.4551e+09
Eccentricity                    = 0.0167088
Right ascension of the ascending node (deg) = 348.554
Inclination to the ecliptic (deg) = -0.000426218
Argument of perihelion (deg)    = 114.405
True anomaly (deg)              = 230.812
Semimajor axis (km)             = 1.49598e+08

Longitude of perihelion (deg)    = 102.959
Mean longitude (deg)            = 335.267
Mean anomaly (deg)              = 232.308
Eccentric anomaly (deg)         = 231.558
```

State vector:

```
Position vector (km) = [1.35589e+08 -6.68029e+07 286.909]
Magnitude             = 1.51152e+08

Velocity (km/s)      = [12.6804 26.61 -0.000212731]
Magnitude             = 29.4769
```

D.36 Algorithm 8.2: Calculation of the spacecraft trajectory from planet 1 to planet 2

Function file: `interplanetary.m`

```
% ~~~~~~
function ...
[planet1, planet2, trajectory] = interplanetary(depart, arrive)
% ~~~~~~
%{
    This function determines the spacecraft trajectory from the sphere
    of influence of planet 1 to that of planet 2 using Algorithm 8.2.

    mu          - gravitational parameter of the sun (km^3/s^2)
    dum          - a dummy vector not required in this procedure

    planet_id   - planet identifier:
                  1 = Mercury
                  2 = Venus
                  3 = Earth
                  4 = Mars
                  5 = Jupiter
                  6 = Saturn
                  7 = Uranus
                  8 = Neptune
                  9 = Pluto

    year        - range: 1901 - 2099
    month       - range: 1 - 12
    day         - range: 1 - 31
    hour        - range: 0 - 23
    minute      - range: 0 - 60
    second      - range: 0 - 60

    jd1, jd2    - Julian day numbers at departure and arrival
    tof         - time of flight from planet 1 to planet 2 (s)

    Rp1, Vp1    - state vector of planet 1 at departure (km, km/s)
    Rp2, Vp2    - state vector of planet 2 at arrival (km, km/s)
    R1, V1      - heliocentric state vector of spacecraft at
                  departure (km, km/s)
    R2, V2      - heliocentric state vector of spacecraft at
                  arrival (km, km/s)

    depart      - [planet_id, year, month, day, hour, minute, second]
                  at departure
    arrive      - [planet_id, year, month, day, hour, minute, second]
```

```

                                at arrival

planet1    - [Rp1, Vp1, jd1]
planet2    - [Rp2, Vp2, jd2]
trajectory - [V1, V2]

User M-functions required: planet_elements_and_sv, lambert
%}
% -----

global mu

planet_id = depart(1);
year      = depart(2);
month     = depart(3);
day       = depart(4);
hour      = depart(5);
minute    = depart(6);
second    = depart(7);

%...Use Algorithm 8.1 to obtain planet 1's state vector (don't
%...need its orbital elements ["dum"]):
[dum, Rp1, Vp1, jd1] = planet_elements_and_sv ...
    (planet_id, year, month, day, hour, minute, second);

planet_id = arrive(1);
year      = arrive(2);
month     = arrive(3);
day       = arrive(4);
hour      = arrive(5);
minute    = arrive(6);
second    = arrive(7);

%...Likewise use Algorithm 8.1 to obtain planet 2's state vector:
[dum, Rp2, Vp2, jd2] = planet_elements_and_sv ...
    (planet_id, year, month, day, hour, minute, second);

tof = (jd2 - jd1)*24*3600;

%...Patched conic assumption:
R1 = Rp1;
R2 = Rp2;

%...Use Algorithm 5.2 to find the spacecraft's velocity at
% departure and arrival, assuming a prograde trajectory:
[V1, V2] = lambert(R1, R2, tof, 'pro');

planet1    = [Rp1, Vp1, jd1];

```

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```
planet2    = [Rp2, Vp2, jd2];
trajectory = [V1, V2];

end %interplanetary
% ~~~~~
```

Script file: Example_8_08.m

```
% ~~~~~
% Example_8_08
% ~~~~~
%{
    This program uses Algorithm 8.2 to solve Example 8.8.

    mu          - gravitational parameter of the sun (km^3/s^2)
    deg          - conversion factor between degrees and radians
    pi           - 3.1415926...

    planet_id    - planet identifier:
                   1 = Mercury
                   2 = Venus
                   3 = Earth
                   4 = Mars
                   5 = Jupiter
                   6 = Saturn
                   7 = Uranus
                   8 = Neptune
                   9 = Pluto

    year         - range: 1901 - 2099
    month        - range: 1 - 12
    day          - range: 1 - 31
    hour         - range: 0 - 23
    minute       - range: 0 - 60
    second       - range: 0 - 60

    depart       - [planet_id, year, month, day, hour, minute, second]
                  at departure
    arrive       - [planet_id, year, month, day, hour, minute, second]
                  at arrival

    planet1      - [Rp1, Vp1, jd1]
    planet2      - [Rp2, Vp2, jd2]
    trajectory    - [V1, V2]

    coe          - orbital elements [h e RA incl w TA]
                  where
```

```

        h    = angular momentum (km^2/s)
        e    = eccentricity
        RA   = right ascension of the ascending
              node (rad)
        incl = inclination of the orbit (rad)
        w    = argument of perigee (rad)
        TA   = true anomaly (rad)
        a    = semimajor axis (km)

jd1, jd2 - Julian day numbers at departure and arrival
tof      - time of flight from planet 1 to planet 2 (days)

Rp1, Vp1 - state vector of planet 1 at departure (km, km/s)
Rp2, Vp2 - state vector of planet 2 at arrival (km, km/s)
R1, V1   - heliocentric state vector of spacecraft at
           departure (km, km/s)
R2, V2   - heliocentric state vector of spacecraft at
           arrival (km, km/s)

vinf1, vinf2 - hyperbolic excess velocities at departure
              and arrival (km/s)

User M-functions required: interplanetary, coe_from_sv,
                          month_planet_names
%}
% -----

clear all; clc
global mu
mu = 1.327124e11;
deg = pi/180;

%...Data declaration for Example 8.8:

%...Departure
planet_id = 3;
year      = 1996;
month     = 11;
day       = 7;
hour      = 0;
minute    = 0;
second    = 0;
depart = [planet_id year month day hour minute second];

%...Arrival
planet_id = 4;
year      = 1997;
month     = 9;

```

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```
day      = 12;
hour     = 0;
minute   = 0;
second   = 0;
arrive = [planet_id year month day hour minute second];

%...

%...Algorithm 8.2:
[planet1, planet2, trajectory] = interplanetary(depart, arrive);

R1 = planet1(1,1:3);
Vp1 = planet1(1,4:6);
jd1 = planet1(1,7);

R2 = planet2(1,1:3);
Vp2 = planet2(1,4:6);
jd2 = planet2(1,7);

V1 = trajectory(1,1:3);
V2 = trajectory(1,4:6);

tof = jd2 - jd1;

%...Use Algorithm 4.2 to find the orbital elements of the
% spacecraft trajectory based on [Rp1, V1]...
coe = coe_from_sv(R1, V1, mu);
% ... and [R2, V2]
coe2 = coe_from_sv(R2, V2, mu);

%...Equations 8.94 and 8.95:
vinf1 = V1 - Vp1;
vinf2 = V2 - Vp2;

%...Echo the input data and output the solution to
% the command window:
fprintf('-----')
fprintf('\n Example 8.8')
fprintf('\n\n Departure:\n');
fprintf('\n Planet: %s', planet_name(depart(1)))
fprintf('\n Year : %g', depart(2))
fprintf('\n Month : %s', month_name(depart(3)))
fprintf('\n Day : %g', depart(4))
fprintf('\n Hour : %g', depart(5))
fprintf('\n Minute: %g', depart(6))
fprintf('\n Second: %g', depart(7))
fprintf('\n\n Julian day: %11.3f\n', jd1)
fprintf('\n Planet position vector (km) = [%g %g %g]', ...
```

```

R1(1),R1(2), R1(3))

fprintf('\n    Magnitude                                = %g\n', norm(R1))

fprintf('\n    Planet velocity (km/s)                        = [%g  %g  %g]', ...
        Vp1(1), Vp1(2), Vp1(3))

fprintf('\n    Magnitude                                = %g\n', norm(Vp1))

fprintf('\n    Spacecraft velocity (km/s)                    = [%g  %g  %g]', ...
        V1(1), V1(2), V1(3))

fprintf('\n    Magnitude                                = %g\n', norm(V1))

fprintf('\n    v-infinity at departure (km/s) = [%g  %g  %g]', ...
        vinf1(1), vinf1(2), vinf1(3))

fprintf('\n    Magnitude                                = %g\n', norm(vinf1))

fprintf('\n\n Time of flight = %g days\n', tof)

fprintf('\n\n Arrival:\n');
fprintf('\n    Planet: %s', planet_name(arrive(1)))
fprintf('\n    Year  : %g', arrive(2))
fprintf('\n    Month : %s', month_name(arrive(3)))
fprintf('\n    Day   : %g', arrive(4))
fprintf('\n    Hour  : %g', arrive(5))
fprintf('\n    Minute: %g', arrive(6))
fprintf('\n    Second: %g', arrive(7))
fprintf('\n\n    Julian day: %11.3f\n', jd2)
fprintf('\n    Planet position vector (km) = [%g  %g  %g]', ...
        R2(1), R2(2), R2(3))

fprintf('\n    Magnitude                                = %g\n', norm(R1))

fprintf('\n    Planet velocity (km/s)                        = [%g  %g  %g]', ...
        Vp2(1), Vp2(2), Vp2(3))

fprintf('\n    Magnitude                                = %g\n', norm(Vp2))

fprintf('\n    Spacecraft Velocity (km/s)                    = [%g  %g  %g]', ...
        V2(1), V2(2), V2(3))

fprintf('\n    Magnitude                                = %g\n', norm(V2))

fprintf('\n    v-infinity at arrival (km/s) = [%g  %g  %g]', ...
        vinf2(1), vinf2(2), vinf2(3))

```

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```
fprintf('\n    Magnitude                = %g', norm(vinf2))

fprintf('\n\n\n Orbital elements of flight trajectory:\n')

fprintf('\n    Angular momentum (km^2/s)          = %g',...
        coe(1))
fprintf('\n    Eccentricity                        = %g',...
        coe(2))
fprintf('\n    Right ascension of the ascending node (deg) = %g',...
        coe(3)/deg)
fprintf('\n    Inclination to the ecliptic (deg)         = %g',...
        coe(4)/deg)
fprintf('\n    Argument of perihelion (deg)             = %g',...
        coe(5)/deg)
fprintf('\n    True anomaly at departure (deg)          = %g',...
        coe(6)/deg)
fprintf('\n    True anomaly at arrival (deg)            = %g\n', ...
        coe2(6)/deg)
fprintf('\n    Semimajor axis (km)                     = %g',...
        coe(7))

% If the orbit is an ellipse, output the period:
if coe(2) < 1
    fprintf('\n    Period (days)                        = %g', ...
            2*pi/sqrt(mu)*coe(7)^1.5/24/3600)
end
fprintf('\n-----\n')
% ~~~~~~
```

Output from Example_8_08

Example 8.8

Departure:

Planet: Earth
Year : 1996
Month : November
Day : 7
Hour : 0
Minute : 0
Second : 0

Julian day: 2450394.500

Planet position vector (km) = [1.04994e+08 1.04655e+08 988.331]
Magnitude = 1.48244e+08

```
Planet velocity (km/s)      = [-21.515  20.9865  0.000132284]
Magnitude                   = 30.0554
```

```
Spacecraft velocity (km/s)  = [-24.4282  21.7819  0.948049]
Magnitude                   = 32.7427
```

```
v-infinity at departure (km/s) = [-2.91321  0.79542  0.947917]
Magnitude                   = 3.16513
```

Time of flight = 309 days

Arrival:

```
Planet: Mars
Year   : 1997
Month  : September
Day    : 12
Hour   : 0
Minute : 0
Second : 0
```

Julian day: 2450703.500

```
Planet position vector (km) = [-2.08329e+07  -2.18404e+08  -4.06287e+06]
Magnitude                   = 1.48244e+08
```

```
Planet velocity (km/s)      = [25.0386  -0.220288  -0.620623]
Magnitude                   = 25.0472
```

```
Spacecraft Velocity (km/s)  = [22.1581  -0.19666  -0.457847]
Magnitude                   = 22.1637
```

```
v-infinity at arrival (km/s) = [-2.88049  0.023628  0.162776]
Magnitude                   = 2.88518
```

Orbital elements of flight trajectory:

```
Angular momentum (km^2/s)   = 4.84554e+09
Eccentricity                 = 0.205785
Right ascension of the ascending node (deg) = 44.8942
Inclination to the ecliptic (deg) = 1.6621
Argument of perihelion (deg) = 19.9738
True anomaly at departure (deg) = 340.039
True anomaly at arrival (deg) = 199.695
```

Semimajor axis (km)	= 1.84742e+08
Period (days)	= 501.254

Chapter 9

D.37 Algorithm 9.1: Calculate the direction cosine matrix from the quaternion

Function file: dcm_from_q.m

```
% ~~~~~
function Q = dcm_from_q(q)
% ~~~~~
%{
    This function calculates the direction cosine matrix
    from the quaternion.

    q - quaternion (where q(4) is the scalar part)
    Q - direction cosine matrix
%}
% -----

q1 = q(1); q2 = q(2); q3 = q(3); q4 = q(4);

Q = [q1^2-q2^2-q3^2+q4^2,      2*(q1*q2+q3*q4),      2*(q1*q3-q2*q4);
     2*(q1*q2-q3*q4), -q1^2+q2^2-q3^2+q4^2,      2*(q2*q3+q1*q4);
     2*(q1*q3+q2*q4),      2*(q2*q3-q1*q4), -q1^2-q2^2+q3^2+q4^2 ];

end %dcm_from_q
% ~~~~~
```

D.38 Algorithm 9.2: Calculate the quaternion from the direction cosine matrix

Function file: q_from_dcm.m

```
% ~~~~~
function q = q_from_dcm(Q)
% ~~~~~
%{
    This function calculates the quaternion from the direction
    cosine matrix.
%}
```

```

Q - direction cosine matrix
q - quaternion (where q(4) is the scalar part)

%}
% -----

K3 = ...
[Q(1,1)-Q(2,2)-Q(3,3), Q(2,1)+Q(1,2), Q(3,1)+Q(1,3), Q(2,3)-Q(3,2);
 Q(2,1)+Q(1,2), Q(2,2)-Q(1,1)-Q(3,3), Q(3,2)+Q(2,3), Q(3,1)-Q(1,3);
 Q(3,1)+Q(1,3), Q(3,2)+Q(2,3), Q(3,3)-Q(1,1)-Q(2,2), Q(1,2)-Q(2,1);
 Q(2,3)-Q(3,2), Q(3,1)-Q(1,3), Q(1,2)-Q(2,1), Q(1,1)+Q(2,2)+Q(3,3)]/3;

[eigvec, eigval] = eig(K3);

[x,i] = max(diag(eigval));

q = eigvec(:,i);
end %q_from_dcm
% ~~~~~

```

D.39 Example 9.23: Solution of the spinning top problem

Function file: Example_9_23.m

```

%~~~~~
function Example_9_23
% ~~~~~
%{
    This program numerically integrates Euler's equations of motion
    for the spinning top (Example 9.23, Equations (a)). The
    quaternion is used to obtain the time history of the top's
    orientation. See Figure 9.26.

    User M-functions required: rkf45, q_from_dcm, dcm_from_q, dcm_to_euler
    User subfunction required: rates
%}
%-----

clear all; close all; clc

%...Data from Example 9.15:
g      = 9.807;           % Acceleration of gravity (m/s^2)
m      = 0.5;             % Mass in kg
d      = 0.05;            % Distance of center of mass from pivot point (m)
A      = 12.e-4;          % Moment of inertia about body x (kg-m^2)
B      = 12.e-4;          % Moment of inertia about body y (kg-m^2)
C      = 4.5e-4;          % Moment of inertia about body z (kg-m^2)

```

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```
wspin = 1000*2*pi/60; % Spin rate (rad/s)
theta = 60;           % Initial nutation angle (deg)

z = [sind(theta) 0 cosd(theta)]; % Initial z-axis direction:
p = [0 1 0];                % Initial x-axis direction
                                % (or a line defining x-z plane)

y = cross(z,p); % y-axis direction (normal to x-z plane)
x = cross(y,z); % x-axis direction (normal to y-z plane)
i = x/norm(x);  % Unit vector along x axis
j = y/norm(y);  % Unit vector along y axis
k = z/norm(z);  % Unit vector along z axis
QXx = [i; j; k]; % Initial direction cosine matrix
q0 = q_from_dcm(QXx); % Initial quaternion
w0 = [0 0 wspin]'; % Initial body-frame angular velocities (rad/s)
t0 = 0; % Initial time (s)
tf = 2; % Final time (s)
f0 = [q0; w0]; % Initial conditions vector
            % (quaternion & angular velocities):

[t,f] = rkf45(@rates, [t0,tf], f0); % RK4(5) numerical ODE solver.
            % Time derivatives computed in
            % function 'rates' below.

q = f(:,1:4); % Solution for quaternion at 'nsteps' times from t0 to tf
wx = f(:,5); % Solution for angular velocities
wy = f(:,6); % at 'nsteps' times
wz = f(:,7); % from t0 to tf

for m = 1:length(t)
    QXx = dcm_from_q(q(m,:)); % DCM from quaternion
    [prec(m) nut(m) spin(m)] = dcm_to_euler(QXx); % Euler angles from DCM
end

plotit

%~~~~~
function dfdt = rates(t,f)
%~~~~~
q = f(1:4); % components of quaternion
wx = f(5); % angular velocity along x
wy = f(6); % angular velocity along y
wz = f(7); % angular velocity along z

q = q/norm(q); % normalize the quaternion

Q = dcm_from_q(q); % DCM from quaternion
```

```

%...Body frame components of the moment of the weight vector
%   about the pivot point:
M = Q*[-m*g*d*Q(3,2)
        m*g*d*Q(3,1)
        0];
%...Skew-symmetric matrix of angular velocities:
Omega = [ 0   wz  -wy   wx
          -wz   0   wx   wy
           wy  -wx   0   wz
          -wx  -wy  -wz   0];
q_dot  = Omega*q/2; % time derivative of quaternion

%...Euler's equations:
wx_dot = M(1)/A - (C - B)*wy*wz/A; % time derivative of wx
wy_dot = M(2)/B - (A - C)*wz*wx/B; % time derivative of wy
wz_dot = M(3)/C - (B - A)*wx*wy/C; % time derivative of wz

%...Return the rates in a column vector:
dfdt = [q_dot; wx_dot; wy_dot; wz_dot];
end %rates

%~~~~~
function plotit
%~~~~~

figure(1) % Euler angles

subplot(311)
plot(t, prec)
xlabel('time (s)')
ylabel('Precession angle (deg)')
axis([-inf, inf, -inf, inf])
grid

subplot(312)
plot(t, nut)
xlabel('time (s)')
ylabel('Nutation angle (deg)')
axis([-inf, inf, -inf, inf])
grid

subplot(313)
plot(t, spin)
xlabel('time (s)')
ylabel('Spin angle (deg)')
axis([-inf, inf, -inf, inf])
grid

```

```

end %plotit

end %Example_9_23
%~~~~~

```

Chapter 11

D.40 Calculation of a gravity-turn trajectory

Function file: Example_11_03.m

```

% ~~~~~
function Example_11_03
% ~~~~~
%{
    This program numerically integrates Equations 11.6 through
    11.8 for a gravity turn trajectory.

    User M-functions required: rkf45
    User subfunction required: rates
%}
% -----
clear all;close all;clc

deg    = pi/180;      % ...Convert degrees to radians
g0     = 9.81;        % ...Sea-level acceleration of gravity (m/s)
Re     = 6378e3;      % ...Radius of the earth (m)
hscale = 7.5e3;       % ...Density scale height (m)
rho0   = 1.225;       % ...Sea level density of atmosphere (kg/m^3)

diam   = 196.85/12 ...
        *0.3048;      % ...Vehicle diameter (m)

A      = pi/4*(diam)^2; % ...Frontal area (m^2)
CD     = 0.5;          % ...Drag coefficient (assumed constant)
m0     = 149912*.4536; % ...Lift-off mass (kg)
n      = 15;           % ...Mass ratio
T2W    = 1.4;          % ...Thrust to weight ratio
Isp    = 390;          % ...Specific impulse (s)

mfinal = m0/n;         % ...Burnout mass (kg)
Thrust  = T2W*m0*g0;   % ...Rocket thrust (N)
m_dot   = Thrust/Isp/g0; % ...Propellant mass flow rate (kg/s)
mprop   = m0 - mfinal; % ...Propellant mass (kg)

```

```

tburn = mprop/m_dot; % ...Burn time (s)
hturn = 130; % ...Height at which pitchover begins (m)
t0 = 0; % ...Initial time for the numerical integration
tf = tburn; % ...Final time for the numerical integration
tspan = [t0,tf]; % ...Range of integration

% ...Initial conditions:
v0 = 0; % ...Initial velocity (m/s)
gamma0 = 89.85*deg; % ...Initial flight path angle (rad)
x0 = 0; % ...Initial downrange distance (km)
h0 = 0; % ...Initial altitude (km)
vD0 = 0; % ...Initial value of velocity loss due
% to drag (m/s)
vG0 = 0; % ...Initial value of velocity loss due
% to gravity (m/s)

%...Initial conditions vector:
f0 = [v0; gamma0; x0; h0; vD0; vG0];

%...Call to Runge-Kutta numerical integrator 'rkf45'
% rkf45 solves the system of equations df/dt = f(t):

[t,f] = rkf45(@rates, tspan, f0);

%...t is the vector of times at which the solution is evaluated
%...f is the solution vector f(t)
%...rates is the embedded function containing the df/dt's

% ...Solution f(t) returned on the time interval [t0 tf]:
v = f(:,1)*1.e-3; % ...Velocity (km/s)
gamma = f(:,2)/deg; % ...Flight path angle (degrees)
x = f(:,3)*1.e-3; % ...Downrange distance (km)
h = f(:,4)*1.e-3; % ...Altitude (km)
vD = -f(:,5)*1.e-3; % ...Velocity loss due to drag (km/s)
vG = -f(:,6)*1.e-3; % ...Velocity loss due to gravity (km/s)

for i = 1:length(t)
    Rho = rho0 * exp(-h(i)*1000/hscale); %...Air density
    q(i) = 1/2*Rho*v(i)^2; %...Dynamic pressure
end

output

return

%~~~~~
function dydt = rates(t,y)
%~~~~~

```

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```
% Calculates the time rates df/dt of the variables f(t)
% in the equations of motion of a gravity turn trajectory.
%-----

%...Initialize dfdt as a column vector:
dfdt = zeros(6,1);

v      = y(1); % ...Velocity
gamma  = y(2); % ...Flight path angle
x      = y(3); % ...Downrange distance
h      = y(4); % ...Altitude
vD     = y(5); % ...Velocity loss due to drag
vG     = y(6); % ...Velocity loss due to gravity

%...When time t exceeds the burn time, set the thrust
% and the mass flow rate equal to zero:
if t < tburn
    m = m0 - m_dot*t;      % ...Current vehicle mass
    T = Thrust;            % ...Current thrust
else
    m = m0 - m_dot*tburn; % ...Current vehicle mass
    T = 0;                % ...Current thrust
end

g      = g0/(1 + h/Re)^2;      % ...Gravitational variation
                        % with altitude h
rho    = rho0 * exp(-h/hscale); % ...Exponential density variation
                        % with altitude
D      = 1/2 * rho*v^2 * A * CD; % ...Drag [Equation 11.1]

%...Define the first derivatives of v, gamma, x, h, vD and vG
% ("dot" means time derivative):
%v_dot = T/m - D/m - g*sin(gamma); % ...Equation 11.6

%...Start the gravity turn when h = hturn:
if h <= hturn
    gamma_dot = 0;
    v_dot     = T/m - D/m - g;
    x_dot     = 0;
    h_dot     = v;
    vG_dot    = -g;
else
    v_dot = T/m - D/m - g*sin(gamma);
    gamma_dot = -1/v*(g - v^2/(Re + h))*cos(gamma); % ...Equation 11.7
    x_dot    = Re/(Re + h)*v*cos(gamma);           % ...Equation 11.8(1)
    h_dot    = v*sin(gamma);                       % ...Equation 11.8(2)
    vG_dot   = -g*sin(gamma);                      % ...Gravity loss rate
end
```

```

vD_dot    = -D/m;                                % ...Drag loss rate

%...Load the first derivatives of f(t) into the vector dfdt:
dydt(1)   = v_dot;
dydt(2)   = gamma_dot;
dydt(3)   = x_dot;
dydt(4)   = h_dot;
dydt(5)   = vD_dot;
dydt(6)   = vG_dot;
end

%~~~~~
function output
%~~~~~
fprintf('\n\n -----\n')
fprintf('\n Initial flight path angle = %10g deg ',gamma0/deg)
fprintf('\n Pitchover altitude       = %10g m   ',hturn)
fprintf('\n Burn time                     = %10g s   ',tburn)
fprintf('\n Final speed                     = %10g km/s',v(end))
fprintf('\n Final flight path angle       = %10g deg ',gamma(end))
fprintf('\n Altitude                       = %10g km   ',h(end))
fprintf('\n Downrange distance             = %10g km   ',x(end))
fprintf('\n Drag loss                      = %10g km/s',vD(end))
fprintf('\n Gravity loss                   = %10g km/s',vG(end))
fprintf('\n\n -----\n')

figure(1)
plot(x, h)
axis equal
xlabel('Downrange Distance (km)')
ylabel('Altitude (km)')
axis([-inf, inf, 0, inf])
grid

figure(2)
subplot(2,1,1)
plot(h, v)
xlabel('Altitude (km)')
ylabel('Speed (km/s)')
axis([-inf, inf, -inf, inf])
grid

subplot(2,1,2)
plot(t, gamma)
xlabel('Time (s)')
ylabel('Flight path angle (deg)')
axis([-inf, inf, -inf, inf])
grid

```

```

figure(3)
plot(h, q)
xlabel('Altitude (km)')
ylabel('Dynamic pressure (N/m^2)')
axis([-inf, inf, -inf, inf])
grid

end %output

end %Example_11_03
% ~~~~~

```

Chapter 12

D.41 U.S. Standard Atmosphere 1976

Function file: atmosphere.m

```

%~~~~~
function density = atmosphere(z)
%
% ATMOSPHERE calculates density for altitudes from sea level
% through 1000 km using exponential interpolation.
%~~~~~

%...Geometric altitudes (km):
h = ...
[ 0 25 30 40 50 60 70 ...
 80 90 100 110 120 130 140 ...
150 180 200 250 300 350 400 ...
450 500 600 700 800 900 1000];

%...Corresponding densities (kg/m^3) from USSA76:
r = ...
[1.225      4.008e-2  1.841e-2  3.996e-3  1.027e-3  3.097e-4  8.283e-5 ...
1.846e-5  3.416e-6  5.606e-7  9.708e-8  2.222e-8  8.152e-9  3.831e-9 ...
2.076e-9  5.194e-10 2.541e-10 6.073e-11 1.916e-11 7.014e-12 2.803e-12 ...
1.184e-12 5.215e-13 1.137e-13 3.070e-14 1.136e-14 5.759e-15 3.561e-15];

%...Scale heights (km):
H = ...
[ 7.310  6.427  6.546  7.360  8.342  7.583  6.661 ...

```

```

    5.927  5.533  5.703   6.782   9.973  13.243  16.322 ...
21.652 27.974 34.934  43.342  49.755  54.513  58.019 ...
60.980 65.654 76.377 100.587 147.203 208.020];

```

```

%...Handle altitudes outside of the range:

```

```

if z > 1000
    z = 1000;
elseif z < 0
    z = 0;
end

```

```

%...Determine the interpolation interval:

```

```

for j = 1:27
    if z >= h(j) && z < h(j+1)
        i = j;
    end
end
if z == 1000
    i = 27;
end

```

```

%...Exponential interpolation:

```

```

density = r(i)*exp(-(z - h(i))/H(i));

```

```

end %atmosphere

```

```

%~~~~~

```

Function file: Example_12_01.m

```

% ~~~~~

```

```

function Example_12_01

```

```

% ~~~~~

```

```

%

```

```

% This function solves Example 12.1 by using MATLAB's ode45 to numerically
% integrate Equation 12.2 for atmospheric drag.

```

```

%

```

```

% User M-functions required: sv_from_coe, atmosphere

```

```

% User subfunctions required: rates, terminate

```

```

% -----

```

```

%...Preliminaries:

```

```

close all, clear all, clc

```

```

%...Conversion factors:

```

```

hours    = 3600;           %Hours to seconds
days    = 24*hours;       %Days to seconds
deg       = pi/180;        %Degrees to radians

```

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[illegible]

```

%...Extract the locally extreme altitudes:
altitude = sqrt(sum(y(:,1:3).^2,2)) - RE;    %Altitude at each time
[max_altitude,imax,min_altitude,imin] = extrema(altitude);
maxima    = [t(imax) max_altitude]; %Maximum altitudes and times
minima    = [t(imin) min_altitude]; %Minimum altitudes and times
apogee    = sortrows(maxima,1);      %Maxima sorted with time
perigee    = sortrows(minima,1);     %Minima sorted with time

figure(1)
apogee(1,2) = NaN;
%...Plot perigee and apogee history on the same figure:
plot(apogee(:,1)/days, apogee(:,2),'b','linewidth',2)
hold on
plot(perigee(:,1)/days, perigee(:,2),'r','linewidth',2)
grid on
grid minor
xlabel('Time (days)')
ylabel('Altitude (km)')
ylim([0 1000]);

%...Subfunctions:
% ~~~~~~
function dfdt = rates(t,f)
% ~~~~~~
%
% This function calculates the spacecraft acceleration from its
% position and velocity at time t.
% -----
R    = f(1:3)';           %Position vector (km/s)
r    = norm(R);           %Distance from earth's center (km)
alt  = r - RE;            %Altitude (km)
rho  = atmosphere(alt);   %Air density from US Standard Model (kg/m^3)
V    = f(4:6)';           %Velocity vector (km/s)
Vrel = V - cross(wE,R);   %Velocity relative to the atmosphere (km/s)
vrel = norm(Vrel);        %Speed relative to the atmosphere (km/s)
uv    = Vrel/vrel;        %Relative velocity unit vector
ap    = -CD*A/m*rho*...    %Acceleration due to drag (m/s^2)
        (1000*vrel)^2/2*uv; % (converting units of vrel from km/s to m/s)
a0    = -mu*R/r^3;         %Gravitational acceleration (km/s^2)
a      = a0 + ap/1000;     %Total acceleration (km/s^2)
dfdt  = [V a]';           %Velocity and the acceleration returned to ode45
end %rates
% ~~~~~~

% ~~~~~~
function [lookfor stop direction] = terminate(t,y)
% ~~~~~~
%

```

```
% This function specifies the event at which ode45 terminates.
% -----
lookfor = alt - 100; % = 0 when altitude = 100 km
stop     = 1;        % 1 means terminate at lookfor = 0; Otherwise 0
direction = -1;      % -1 means zero crossing is from above
end %terminate
% ~~~~~

end %Example_12_01
% ~~~~~
```

D.43 J_2 perturbation of an orbit using Encke's method

Function file: Example_12_02.m

```
% ~~~~~
function Example_12_02
% ~~~~~
%
% This function solves Example 12.2 by using Encke's method together
% with MATLAB's ode45 to integrate Equation 12.2 for a  $J_2$  gravitational
% perturbation given by Equation 12.30.
%
% User M-functions required: sv_from_coe, coe_from_sv, rv_from_r0v0
% User subfunction required: rates
% -----
%...Preliminaries:
clc, close all, clear all

%...Conversion factors:
hours = 3600;           %Hours to seconds
days = 24*hours;       %Days to seconds
deg   = pi/180;         %Degrees to radians

%...Constants:
global mu
mu     = 398600;         %Gravitational parameter (km^3/s^2)
RE     = 6378;          %Earth's radius (km)
J2     = 1082.63e-6;

%...Initial orbital parameters (given):
zp0 = 300;              %Perigee altitude (km)
za0 = 3062;             %Apogee altitude (km)
RA0 = 45*deg;           %Right ascension of the node (radians)
i0  = 28*deg;           %Inclination (radians)
w0  = 30*deg;           %Argument of perigee (radians)
```

```

TA0 = 40*deg;                                %True anomaly (radians)

%...Initial orbital parameters (inferred):
rp0 = RE + zp0;                               %Perigee radius (km)
ra0 = RE + za0;                               %Apogee radius (km)
e0  = (ra0 - rp0)/(ra0 + rp0); %Eccentricity
a0  = (ra0 + rp0)/2;                         %Semimajor axis (km)
h0  = sqrt(rp0*mu*(1+e0));                   %Angular momentum (km^2/s)
T0  = 2*pi/sqrt(mu)*a0^1.5;                  %Period (s)

t0  = 0; tf = 2*days;                       %Initial and final time (s)
%...end Input data

%Store the initial orbital elements in the array coe0:
coe0 = [h0 e0 RA0 i0 w0 TA0];

%...Obtain the initial state vector from Algorithm 4.5 (sv_from_coe):
[R0 V0] = sv_from_coe(coe0, mu); %R0 is the initial position vector
                                     %R0 is the initial position vector
r0 = norm(R0); v0 = norm(V0);           %Magnitudes of T0 and V0

del_t = T0/100;                             %Time step for Encke procedure
options = odeset('maxstep', del_t);

%...Begin the Encke integration;
t      = t0;                                %Initialize the time scalar
tsave  = t0;                                %Initialize the vector of solution times
y      = [R0 V0];                           %Initialize the state vector
del_y0 = zeros(6,1);                        %Initialize the state vector perturbation

t = t + del_t;                               %First time step

% Loop over the time interval [t0, tf] with equal increments del_t:
while t <= tf + del_t/2

% Integrate Equation 12.7 over the time increment del_t:
[dum,z] = ode45(@rates, [t0 t], del_y0, options);

% Compute the osculating state vector at time t:
[Rosc,Vosc] = rv_from_r0v0(R0, V0, t-t0);

% Rectify:
R0  = Rosc + z(end,1:3);
V0  = Vosc + z(end,4:6);
t0  = t;

% Prepare for next time step:

```

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```
    tsave = [tsave;t];
    y      = [y; [R0 V0]];
    t      = t + del_t;
    del_y0 = zeros(6,1);
end
% End the loop

t = tsave; %t is the vector of equispaced solution times
%...End the Encke integration;

%...At each solution time extract the orbital elements from the state
% vector using Algorithm 4.2:
n_times = length(t); %n_times is the number of solution times
for j = 1:n_times
    R      = [y(j,1:3)];
    V      = [y(j,4:6)];
    r(j)   = norm(R);
    v(j)   = norm(V);
    coe    = coe_from_sv(R,V, mu);
    h(j)   = coe(1);
    e(j)   = coe(2);
    RA(j)  = coe(3);
    i(j)   = coe(4);
    w(j)   = coe(5);
    TA(j)  = coe(6);
end

%...Plot selected osculating elements:

figure(1)
subplot(2,1,1)
plot(t/3600,(RA - RA0)/deg)
title('Variation of Right Ascension')
xlabel('hours')
ylabel('{\it\Delta\omega} (deg)')
grid on
grid minor
axis tight

subplot(2,1,2)
plot(t/3600,(w - w0)/deg)
title('Variation of Argument of Perigee')
xlabel('hours')
ylabel('{\it\Delta\omega} (deg)')
grid on
grid minor
axis tight
```

```

figure(2)
subplot(3,1,1)
plot(t/3600,h - h0)
title('Variation of Angular Momentum')
xlabel('hours')
ylabel('{\it\Delta h} (km^2/s)')
grid on
grid minor
axis tight

subplot(3,1,2)
plot(t/3600,e - e0)
title('Variation of Eccentricity')
xlabel('hours')
ylabel('{\it\Delta e}')
grid on
grid minor
axis tight

subplot(3,1,3)
plot(t/3600,(i - i0)/deg)
title('Variation of Inclination')
xlabel('hours')
ylabel('{\it\Delta i} (deg)')
grid on
grid minor
axis tight

%...Subfunction:
%~~~~~
function dfdt = rates(t,f)
%~~~~~
%
% This function calculates the time rates of Encke's deviation in position
% del_r and velocity del_v.
% -----

del_r = f(1:3)';    %Position deviation
del_v = f(4:6)';    %Velocity deviation

%...Compute the state vector on the osculating orbit at time t
% (Equation 12.5) using Algorithm 3.4:
[Rosc,Vosc] = rv_from_r0v0(R0, V0, t-t0);

%...Calculate the components of the state vector on the perturbed orbit
% and their magnitudes:
Rpp = Rosc + del_r;
Vpp = Vosc + del_v;

```

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```
rosc = norm(Rosc);
rpp = norm(Rpp);

%...Compute the J2 perturbing acceleration from Equation 12.30:
xx = Rpp(1); yy = Rpp(2); zz = Rpp(3);

fac = 3/2*J2*(mu/rpp^2)*(RE/rpp)^2;
ap = -fac*[(1 - 5*(zz/rpp)^2)*(xx/rpp) ...
            (1 - 5*(zz/rpp)^2)*(yy/rpp) ...
            (3 - 5*(zz/rpp)^2)*(zz/rpp)];

%...Compute the total perturbing acceleration from Equation 12.7:
F = 1 - (rosc/rpp)^3;
del_a = -mu/rosc^3*(del_r - F*Rpp) + ap;

dfdt = [del_v(1) del_v(2) del_v(3) del_a(1) del_a(2) del_a(3)]';
dfdt = [del_v del_a]'; %Return the derivative velocity and acceleration
                        %to ode45.

end %rates
%~~~~~

end %Example_12_02
%~~~~~
```

D.44 Example 12.6: Using Gauss variational equations to assess J_2 effect on orbital elements

Function file: Example_12_06.m

```
% ~~~~~
function Example_12_6
% ~~~~~
%
% This function solves Example 12.6 by using MATLAB's ode45 to numerically
% integrate Equations 12.89 (the Gauss planetary equations) to determine
% the J2 perturbation of the orbital elements.
%
% User M-functions required: None
% User subfunctions required: rates
% -----

%...Preliminaries:
close all; clear all; clc

%...Conversion factors:
```

```

hours    = 3600;                %Hours to seconds
days    = 24*hours;            %Days to seconds
deg       = pi/180;              %Degrees to radians

%...Constants:
mu        = 398600;              %Gravitational parameter (km^3/s^2)
RE        = 6378;                %Earth's radius (km)
J2        = 1082.63e-6;          %Earth's J2

%...Initial orbital parameters (given):
rp0       = RE + 300;            %perigee radius (km)
ra0       = RE + 3062;           %apogee radius (km)
RA0       = 45*deg;              %Right ascension of the node (radians)
i0        = 28*deg;              %Inclination (radians)
w0        = 30*deg;              %Argument of perigee (radians)
TA0       = 40*deg;              %True anomaly (radians)

%...Initial orbital parameters (inferred):
e0        = (ra0 - rp0)/(ra0 + rp0); %eccentricity
h0        = sqrt(rp0*mu*(1 + e0));   %angular momentum (km^2/s)
a0        = (rp0 + ra0)/2;           %Semimajor axis (km)
T0        = 2*pi/sqrt(mu)*a0^1.5;    %Period (s)

%...Store initial orbital elements (from above) in the vector coe0:
coe0      = [h0 e0 RA0 i0 w0 TA0];

%...Use ODE45 to integrate the Gauss variational equations (Equations
% 12.89) from t0 to tf:
t0        = 0;
tf        = 2*days;
nout      = 5000; %Number of solution points to output for plotting purposes
tspan     = linspace(t0, tf, nout);
options   = odeset(...
            'reltol',    1.e-8, ...
            'abstol',    1.e-8, ...
            'initialstep', T0/1000);
y0        = coe0';
[t,y]     = ode45(@rates, tspan, y0, options);

%...Assign the time histories mnemonic variable names:
h = y(:,1);
e = y(:,2);
RA = y(:,3);
i = y(:,4);
w = y(:,5);
TA = y(:,6);

%...Plot the time histories of the osculating elements:

```

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```
figure(1)
subplot(5,1,1)
plot(t/3600,(RA - RA0)/deg)
title('Right Ascension (degrees)')
xlabel('hours')
grid on
grid minor
axis tight

subplot(5,1,2)
plot(t/3600,(w - w0)/deg)
title('Argument of Perigee (degrees)')
xlabel('hours')
grid on
grid minor
axis tight

subplot(5,1,3)
plot(t/3600,h - h0)
title('Angular Momentum (km^2/s)')
xlabel('hours')
grid on
grid minor
axis tight

subplot(5,1,4)
plot(t/3600,e - e0)
title('Eccentricity')
xlabel('hours')
grid on
grid minor
axis tight

subplot(5,1,5)
plot(t/3600,(i - i0)/deg)
title('Inclination (degrees)')
xlabel('hours')
grid on
grid minor
axis tight

%...Subfunction:

%~~~~~
function dfdt = rates(t,f)
%~~~~~
%
% This function calculates the time rates of the orbital elements
```

```

% from Gauss's variational equations (Equations 12.89).
%-----

%...The orbital elements at time t:
h    = f(1);
e    = f(2);
RA   = f(3);
i    = f(4);
w    = f(5);
TA   = f(6);

r    = h^2/mu/(1 + e*cos(TA)); %The radius
u    = w + TA;                 %Argument of latitude

%...Orbital element rates at time t (Equations 12.89):
hdot = -3/2*J2*mu*RE^2/r^3*sin(i)^2*sin(2*u);

edot = ...
      3/2*J2*mu*RE^2/h/r^3*(h^2/mu/r ...
      *(sin(u)*sin(i)^2*(3*sin(TA)*sin(u) - 2*cos(TA)*cos(u)) - sin(TA)) ...
      -sin(i)^2*sin(2*u)*(e + cos(TA)));

edot = 3/2*J2*mu*RE^2/h/r^3 ...
      *(h^2/mu/r*sin(TA)*(3*sin(i)^2*sin(u)^2 - 1) ...
      -sin(2*u)*sin(i)^2*((2+e*cos(TA))*cos(TA)+e));

TAdot = h/r^2 + 3/2*J2*mu*RE^2/e/h/r^3 ...
      *(h^2/mu/r*cos(TA)*(3*sin(i)^2*sin(u)^2 - 1) ...
      + sin(2*u)*sin(i)^2*sin(TA)*(h^2/mu/r + 1));

RADot = -3*J2*mu*RE^2/h/r^3*sin(u)^2*cos(i);

idot  = -3/4*J2*mu*RE^2/h/r^3*sin(2*u)*sin(2*i);

wdot  = 3/2*J2*mu*RE^2/e/h/r^3 ...
      *(-h^2/mu/r*cos(TA)*(3*sin(i)^2*sin(u)^2 - 1) ...
      - sin(2*u)*sin(i)^2*sin(TA)*(2 + e*cos(TA)) ...
      + 2*e*cos(i)^2*sin(u)^2);

%...Pass these rates back to ODE45 in the array dfdt:
dfdt = [hdot edot RADot idot wdot TAdot]';

end %rates
% ~~~~~

end %Example_12_6
% ~~~~~

```

D.45 Algorithm 12.2: Calculate the geocentric position of the sun at a given epoch

Function file: solar_position.m

```
% ~~~~~~
function [lamda eps r_S] = solar_position(jd)
%
% This function calculates the geocentric equatorial position vector
% of the sun, given the Julian date.
%
% User M-functions required: None
% -----
%...Astronomical unit (km):
AU    = 149597870.691;

%...Julian days since J2000:
n      = jd - 2451545;

%...Julian centuries since J2000:
cy     = n/36525;

%...Mean anomaly (deg):
M      = 357.528 + 0.9856003*n;
M      = mod(M,360);

%...Mean longitude (deg):
L      = 280.460 + 0.98564736*n;
L      = mod(L,360);

%...Apparent ecliptic longitude (deg):
lamda = L + 1.915*sind(M) + 0.020*sind(2*M);
lamda = mod(lamda,360);

%...Obliquity of the ecliptic (deg):
eps    = 23.439 - 0.0000004*n;

%...Unit vector from earth to sun:
u      = [cosd(lamda); sind(lamda)*cosd(eps); sind(lamda)*sind(eps)];

%...Distance from earth to sun (km):
r_S    = (1.00014 - 0.01671*cosd(M) - 0.000140*cosd(2*M))*AU;

%...Geocentric position vector (km):
r_S    = r_S*u;
end %solar_position
% ~~~~~~
```

D.46 Algorithm 12.3: Determine whether or not a satellite is in earth's shadow

Function file: los.m

```
% ~ ~ ~ ~ ~
function light_switch = los(r_sat, r_sun)
%
% This function uses the ECI position vectors of the satellite (r_sat)
% and the sun (r_sun) to determine whether the earth is in the line of
% sight between the two.
%
% User M-functions required: None
%-----
RE      = 6378;           %Earth's radius (km)
rsat    = norm(r_sat);
rsun    = norm(r_sun);

%...Angle between sun and satellite position vectors:
theta   = acosd(dot(r_sat, r_sun)/rsat/rsun);

%...Angle between the satellite position vector and the radial to the point
%   of tangency with the earth of a line from the satellite:
theta_sat = acosd(RE/rsat);

%...Angle between the sun position vector and the radial to the point
%   of tangency with the earth of a line from the sun:
theta_sun = acosd(RE/rsun);

%...Determine whether a line from the sun to the satellite
%   intersects the earth:
if theta_sat + theta_sun <= theta
    light_switch = 0; %yes
else
    light_switch = 1; %no
end

end %los
%-----
```

D.47 Example 12.9: Use the Gauss variational equations to determine the effect of solar radiation pressure on an earth satellite's orbital parameters

Function file: Example_12_09.m

```
% ~ ~ ~ ~ ~
```

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```
function Example_12_09
%
% This function solve Example 12.9 the Gauss planetary equations for
% solar radiation pressure (Equations 12.106).
%
% User M-functions required: sv_from_coe, los, solar_position
% User subfunctions required: rates
% The M-function rsmooth may be found in Garcia (2010).
% ~~~~~

global JD %Julian day

%...Preliminaries:
close all
clear all
clc

%...Conversion factors:
hours = 3600; %Hours to seconds
days = 24*hours; %Days to seconds
deg = pi/180; %Degrees to radians

%...Constants:
mu = 398600; %Gravitational parameter (km^3/s^2)
RE = 6378; %Earth's radius (km)
c = 2.998e8; %Speed of light (m/s)
S = 1367; %Solar constant (W/m^2)
Psr = S/c; %Solar pressure (Pa);

%...Satellite data:
CR = 2; %Radiation pressure coefficient
m = 100; %Mass (kg)
As = 200; %Frontal area (m^2);

%...Initial orbital parameters (given):
a0 = 10085.44; %Semimajor axis (km)
e0 = 0.025422; %eccentricity
incl0 = 88.3924*deg; %Inclination (radians)
RA0 = 45.38124*deg; %Right ascension of the node (radians)
TA0 = 343.4268*deg; %True anomaly (radians)
w0 = 227.493*deg; %Argument of perigee (radians)

%...Initial orbital parameters (inferred):
h0 = sqrt(mu*a0*(1-e0^2)); %angular momentum (km^2/s)
T0 = 2*pi/sqrt(mu)*a0^1.5; %Period (s)
rp0 = h0^2/mu/(1 + e0); %perigee radius (km)
ra0 = h0^2/mu/(1 - e0); %apogee radius (km)
```

```

%...Store initial orbital elements (from above) in the vector coe0:
coe0 = [h0 e0 RAO incl0 w0 TA0];

%...Use ODE45 to integrate Equations 12.106, the Gauss planetary equations
%   from t0 to tf:

JD0   = 2438400.5;           %Initial Julian date (6 January 1964 0 UT)
t0     = 0;                  %Initial time (s)
tf     = 3*365*days;        %final time (s)
y0     = coe0';              %Initial orbital elements
nout   = 4000;               %Number of solution points to output
tspan  = linspace(t0, tf, nout); %Integration time interval
options = odeset(...
    'reltol',    1.e-8, ...
    'abstol',    1.e-8, ...
    'initialstep', T0/1000);
[t,y] = ode45(@rates, tspan, y0, options);

%...Extract or compute the orbital elements' time histories from the
%   solution vector y:
h      = y(:,1);
e      = y(:,2);
RA     = y(:,3);
incl   = y(:,4);
w      = y(:,5);
TA     = y(:,6);
a      = h.^2/mu./(1 - e.^2);

%...Smooth the data to remove short period variations:
h      = rsmooth(h);
e      = rsmooth(e);
RA     = rsmooth(RA);
incl   = rsmooth(incl);
w      = rsmooth(w);
a      = rsmooth(a);

figure(2)
subplot(3,2,1)
plot(t/days,h - h0)
title('Angular Momentum (km^2/s)')
xlabel('days')
axis tight

subplot(3,2,2)
plot(t/days,e - e0)
title('Eccentricity')
xlabel('days')
axis tight

```

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```
subplot(3,2,4)
plot(t/days,(RA - RA0)/deg)
title('Right Ascension (deg)')
xlabel('days')
axis tight

subplot(3,2,5)
plot(t/days,(incl - incl0)/deg)
title('Inclination (deg)')
xlabel('days')
axis tight

subplot(3,2,6)
plot(t/days,(w - w0)/deg)
title('Argument of Perigee (deg)')
xlabel('days')
axis tight

subplot(3,2,3)
plot(t/days,a - a0)
title('Semimajor axis (km)')
xlabel('days')
axis tight

%...Subfunctions:
%~~~~~
function dfdt = rates(t,f)
%~~~~~
%...Update the Julian Date at time t:
JD      = JD0 + t/days;

%...Compute the apparent position vector of the sun:
[lamda eps r_sun] = solar_position(JD);

%...Convert the ecliptic latitude and the obliquity to radians:
lamda    = lamda*deg;
eps      = eps*deg;

%...Extract the orbital elements at time t
h        = f(1);
e        = f(2);
RA       = f(3);
i        = f(4);
w        = f(5);
TA       = f(6);

u        = w + TA; %Argument of latitude
```

```

%...Compute the state vector at time t:
coe    = [h e RA i w TA];
[R, V] = sv_from_coe(coe,mu);

%...Calculate the magnitude of the radius vector:
r      = norm(R);

%...Compute the shadow function and the solar radiation perturbation:
nu     = los(R, r_sun);
pSR    = nu*(S/c)*CR*As/m/1000;

%...Calculate the trig functions in Equations 12.105.
sl = sin(lamda);  cl = cos(lamda);
se = sin(eps);    ce = cos(eps);
sW = sin(RA);     cW = cos(RA);
si = sin(i);      ci = cos(i);
su = sin(u);      cu = cos(u);
sT = sin(TA);     cT = cos(TA);

%...Calculate the earth-sun unit vector components (Equations 12.105):
ur     = sl*ce*cW*ci*su + sl*ce*sW*cu - cl*sW*ci*su ...
        + cl*cW*cu + sl*se*si*su;

us     = sl*ce*cW*ci*cu - sl*ce*sW*su - cl*sW*ci*cu ...
        - cl*cW*su + sl*se*si*cu;

uw     = - sl*ce*cW*si + cl*sW*si + sl*se*ci;

%...Calculate the time rates of the osculating elements from
% Equations 12.106:

hdot    = -pSR*r*us;

edot    = -pSR*(h/mu*sT*ur ...
        + 1/mu/h*((h^2 + mu*r)*cT + mu*e*r)*us);

TAdot   = h/r^2 ...
        - pSR/e/h*(h^2/mu*cT*ur - (r + h^2/mu)*sT*us);

RADot   = -pSR*r/h/si*su*uw;

idot    = -pSR*r/h*cu*uw;

wdot    = -pSR*(-1/e/h*(h^2/mu*cT*ur - (r + h^2/mu)*sT*us) ...
        - r*su/h/si*ci*uw);

%...Return the rates to ode45:
dfdt    = [hdot edot RADot idot wdot TAdot]';

```

```

end %rates

end %Example_12_9
% ~~~~~

```

D.48 Algorithm 12.4: Calculate the geocentric position of the moon at a given epoch

Function file: lunar_position.m

```

% -----
function r_moon = lunar_position(jd)
%
%...Calculates the geocentric equatorial position vector of the moon
%   given the Julian day.
%
% User M-functions required: None
% -----

%...Earth's radius (km):
RE = 6378;

% ----- implementation -----
%...Time in centuries since J2000:
T = (jd - 2451545)/36525;

%...Ecliptic longitude (deg):
e_long = 218.32 + 481267.881*T ...
        + 6.29*sind(135.0 + 477198.87*T) - 1.27*sind(259.3 - 413335.36*T)...
        + 0.66*sind(235.7 + 890534.22*T) + 0.21*sind(269.9 + 954397.74*T)...
        - 0.19*sind(357.5 + 35999.05*T) - 0.11*sind(186.5 + 966404.03*T);
e_long = mod(e_long,360);

%...Ecliptic latitude (deg):
e_lat = 5.13*sind( 93.3 + 483202.02*T) + 0.28*sind(228.2 + 960400.89*T)...
        - 0.28*sind(318.3 + 6003.15*T) - 0.17*sind(217.6 - 407332.21*T);
e_lat = mod(e_lat,360);

%...Horizontal parallax (deg):
h_par = 0.9508 ...
        + 0.0518*cosd(135.0 + 477198.87*T) + 0.0095*cosd(259.3 - 413335.36*T)...
        + 0.0078*cosd(235.7 + 890534.22*T) + 0.0028*cosd(269.9 + 954397.74*T);
h_par = mod(h_par,360);

%...Angle between earth's orbit and its equator (deg):

```

```

obliquity = 23.439291 - 0.0130042*T;

%...Direction cosines of the moon's geocentric equatorial position vector:
l = cosd(e_lat) * cosd(e_long);
m = cosd(obliquity)*cosd(e_lat)*sind(e_long) - sind(obliquity)*sind(e_lat);
n = sind(obliquity)*cosd(e_lat)*sind(e_long) + cosd(obliquity)*sind(e_lat);

%...Earth-moon distance (km):
dist = RE/sind(h_par);

%...Moon's geocentric equatorial position vector (km):
r_moon = dist*[l m n];

end %lunar_position
% -----

```

D.49 Example 12.11: Use the Gauss variational equations to determine the effect of lunar gravity on an earth satellite's orbital parameters

Function file: Example_12_11.m

```

% ~~~~~
function Example_12_11
% This function solves Example 12.11 by using MATLAB's ode45 to integrate
% Equations 12.84, the Gauss variational equations, for a lunar
% gravitational perturbation.
%
% User M-functions required: sv_from_coe, lunar_position
% User subfunctions required: solveit rates
% -----

global JD %Julian day

%...Preliminaries:
close all
clear all
clc

%...Conversion factors:
hours = 3600; %Hours to seconds
days = 24*hours; %Days to seconds
deg = pi/180; %Degrees to radians

%...Constants:
mu = 398600; %Earth's gravitational parameter (km^3/s^2)
mu3 = 4903; %Moon's gravitational parameter (km^3/s^2)

```

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```
RE      = 6378;           %Earth's radius (km)

%...Initial data for each of the three given orbits:

type = {'GEO' 'HEO' 'LEO'};

%...GEO
n      = 1;
a0     = 42164;           %semimajor axis (km)
e0     = 0.0001;          %eccentricity
w0     = 0;               %argument of perigee (rad)
RA0    = 0;               %right ascension (rad)
i0     = 1*deg;           %inclination (rad)
TA0    = 0;               %true anomaly (rad)
JD0    = 2454283;         %Julian Day
solveit

%...HEO
n      = 2;
a0     = 26553.4;
e0     = 0.741;
w0     = 270;
RA0    = 0;
i0     = 63.4*deg;
TA0    = 0;
JD0    = 2454283;
solveit

%...LEO
n      = 3;
a0     = 6678.136;
e0     = 0.01;
w0     = 0;
RA0    = 0;
i0     = 28.5*deg;
TA0    = 0;
JD0    = 2454283;
solveit

%...Subfunctions:

%~~~~~
function solveit
%
% Calculations and plots common to all of the orbits
%
%-----
%
```

```

%...Initial orbital parameters (calculated from the given data):
h0 = sqrt(mu*a0*(1-e0^2)); %angular momentum (km^2/s)
T0 = 2*pi/sqrt(mu)*a0^1.5; %Period (s)
rp0 = h0^2/mu/(1 + e0); %perigee radius (km)
ra0 = h0^2/mu/(1 - e0); %apogee radius (km)

%...Store initial orbital elements (from above) in the vector coe0:
coe0 = [h0;e0;RA0;i0;w0;TA0];

%...Use ODE45 to integrate the Equations 12.84, the Gauss variational
% equations with lunar gravity as the perturbation, from t0 to tf:
t0 = 0;
tf = 60*days;
y0 = coe0; %Initial orbital elements
nout = 400; %Number of solution points to output
tspan = linspace(t0, tf, nout); %Integration time interval
options = odeset('reltol', 1.e-8, ...
                'abstol', 1.e-8);
[t,y] = ode45(@rates, tspan, y0, options);

%...Time histories of the right ascension, inclination and argument of
% perigee:
RA = y(:,3);
i = y(:,4);
w = y(:,5);

%...Smooth the data to eliminate short period variations:
RA = rsmooth(RA);
i = rsmooth(i);
w = rsmooth(w);

figure(n)
subplot(1,3,1)
plot(t/days,(RA - RA0)/deg)
title('Right Ascension vs Time')
xlabel('\itt (days)')
ylabel('\it\Omega (deg)')
axis tight

subplot(1,3,2)
plot(t/days,(i - i0)/deg)
title('Inclination vs Time')
xlabel('\itt (days)')
ylabel('\iti (deg)')
axis tight

subplot(1,3,3)

```

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```
plot(t/days,(w - w0)/deg)
title('Argument of Perigee vs Time')
xlabel('{\itt} (days)')
ylabel('{\it\omega} (deg)')
axis tight

drawnow

end %solveit
%~~~~~

%~~~~~
function dfdt = rates(t,f)
%~~~~~
%...The orbital elements at time t:
h      = f(1);
e      = f(2);
RA     = f(3);
i      = f(4);
w      = f(5);
TA     = f(6);
phi    = w + TA; %argument of latitude

%...Obtain the state vector at time t from Algorithm 4.5:
coe    = [h e RA i w TA];
[R, V] = sv_from_coe(coe,mu);

%...Obtain the unit vectors of the rsw system:
r      = norm(R);
ur     = R/r;          %radial
H      = cross(R,V);
uh     = H/norm(H);    %normal
s      = cross(uh, ur);
us     = s/norm(s);    %transverse

%...Update the Julian Day:
JD     = JD0 + t/days;

%...Find and normalize the position vector of the moon:
R_m    = lunar_position(JD);
r_m    = norm(R_m);

R_rel  = R_m - R;      %R_rel = position vector of moon wrt satellite
r_rel  = norm(R_rel);

%...See Appendix F:
q      = dot(R,(2*R_m - R))/r_m^2;
F      = (q^2 - 3*q + 3)*q/(1 + (1-q)^1.5);
```

```

%...Gravitational perturbation of the moon (Equation 12.117):
ap      = mu3/r_rel^3*(F*R_m - R);

%...Perturbation components in the rsw system:
apr     = dot(ap,ur);
aps     = dot(ap,us);
aph     = dot(ap,uH);

%...Gauss variational equations (Equations 12.84):
hdot    = r*aps;

edot    = h/mu*sin(TA)*apr ...
          + 1/mu/h*((h^2 + mu*r)*cos(TA) + mu*e*r)*aps;

RAdot   = r/h/sin(i)*sin(phi)*aph;

idot    = r/h*cos(phi)*aph;

wdot    = - h*cos(TA)/mu/e*apr ...
          + (h^2 + mu*r)/mu/e/h*sin(TA)*aps ...
          - r*sin(phi)/h/tan(i)*aph;

TAdot   = h/r^2 ...
          + 1/e/h*(h^2/mu*cos(TA)*apr - (r + h^2/mu)*sin(TA)*aps);

%...Return rates to ode45 in the array dfdt:
dfdt    = [hdot edot RAdot idot wdot TAdot]';

end %rates
%~~~~~

end %Example_12_11
% ~~~~~

```

D.50 Example 12.12: Use the Gauss variational equations to determine the effect of solar gravity on an earth satellite's orbital parameters

Function file: Example_12_12.m

```

% ~~~~~
function Example_12_12
% This function solves Example 12.12 by using MATLAB's ode45 to integrate
% Equations 12.84, the Gauss variational equations, for a solar
% gravitational perturbation.
%

```

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```
% User M-functions required: sv_from_coe, lunar_position
% User subfunctions required: solveit rates
% -----

global JD    %Julian day

%...Preliminaries:
close all
clear all
clc

%...Conversion factors:
hours    = 3600;           %Hours to seconds
days    = 24*hours;       %Days to seconds
deg       = pi/180;        %Degrees to radians

%...Constants:
mu        = 398600;         %Earth's gravitational parameter (km^3/s^2)
mu3       = 132.712e9;      %Sun's gravitational parameter (km^3/s^2)
RE        = 6378;          %Earth's radius (km)

%...Initial data for each of the three given orbits:

type = {'GEO' 'HEO' 'LEO'};

%...GEO
n      = 1;
a0     = 42164;             %semimajor axis (km)
e0     = 0.0001;           %eccentricity
w0     = 0;                %argument of perigee (rad)
RA0    = 0;                %right ascension (rad)
i0     = 1*deg;            %inclination (rad)
TA0    = 0;                %true anomaly (rad)
JD0    = 2454283;          %Julian Day
solveit

%...HEO
n      = 2;
a0     = 26553.4;
e0     = 0.741;
w0     = 270;
RA0    = 0;
i0     = 63.4*deg;
TA0    = 0;
JD0    = 2454283;
solveit

%...LEO
```

```

n      = 3;
a0     = 6678.136;
e0     = 0.01;
w0     = 0;
RA0    = 0;
i0     = 28.5*deg;
TA0    = 0;
JD0    = 2454283;
solveit

%...Subfunctions:

%~~~~~
function solveit
%
% Calculations and plots common to all of the orbits
%
%-----
%
%...Initial orbital parameters (calculated from the given data):
h0     = sqrt(mu*a0*(1-e0^2)); %angular momentum (km^2/s)
T0     = 2*pi/sqrt(mu)*a0^1.5; %Period (s)
rp0    = h0^2/mu/(1 + e0);    %perigee radius (km)
ra0    = h0^2/mu/(1 - e0);    %apogee radius (km)

%...Store initial orbital elements (from above) in the vector coe0:
coe0 = [h0;e0;RA0;i0;w0;TA0];

%...Use ODE45 to integrate the Equations 12.84, the Gauss variational
% equations with lunar gravity as the perturbation, from t0 to tf:
t0      = 0;
tf      = 720*days;
y0      = coe0; %Initial orbital elements
nout    = 400; %Number of solution points to output
tspan   = linspace(t0, tf, nout); %Integration time interval
options = odeset(...
    'reltol', 1.e-8, ...
    'abstol', 1.e-8);
[t,y]   = ode45(@rates, tspan, y0, options);

%...Time histories of the right ascension, inclination and argument of
% perigee:
RA = y(:,3);
i  = y(:,4);
w  = y(:,5);

%...Smooth the data to eliminate short period variations:
RA = rsmooth(RA);

```

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```
i = rsmooth(i);
w = rsmooth(w);

figure(n)
subplot(1,3,1)
plot(t/days,(RA - RA0)/deg)
title('Right Ascension vs Time')
xlabel('\itt (days)')
ylabel('\it\Omega (deg)')
axis tight

subplot(1,3,2)
plot(t/days,(i - i0)/deg)
title('Inclination vs Time')
xlabel('\itt (days)')
ylabel('\iti (deg)')
axis tight

subplot(1,3,3)
plot(t/days,(w - w0)/deg)
title('Argument of Perigee vs Time')
xlabel('\itt (days)')
ylabel('\it\omega (deg)')
axis tight

drawnow

end %solveit
%~~~~~

%~~~~~
function dfdt = rates(t,f)
%~~~~~
%...The orbital elements at time t:
h      = f(1);
e      = f(2);
RA     = f(3);
i      = f(4);
w      = f(5);
TA     = f(6);
phi    = w + TA; %argument of latitude

%...Obtain the state vector at time t from Algorithm 4.5:
coe    = [h e RA i w TA];
[R, V] = sv_from_coe(coe,mu);

%...Obtain the unit vectors of the rsw system:
r      = norm(R);
```

```

ur    = R/r;           %radial
H     = cross(R,V);
uh    = H/norm(H);     %normal
s     = cross(uh, ur);
us    = s/norm(s);     %transverse

%...Update the Julian Day:
JD     = JD0 + t/days;

%...Find and normalize the position vector of the sun:
[lamda eps R_S] = solar_position(JD);
r_S     = norm(R_S);

R_rel  = R_S' - R; %R_rel = position vector of sun wrt satellite
r_rel  = norm(R_rel);

%...See Appendix F:
q      = dot(R,(2*R_S' - R))/r_S^2;
F      = (q^2 - 3*q + 3)*q/(1 + (1-q)^1.5);

%...Gravitational perturbation of the sun (Equation 12.130):
ap     = mu3/r_rel^3*(F*R_S' - R);

%...Perturbation components in the rsw system:
apr    = dot(ap,ur);
aps    = dot(ap,us);
aph    = dot(ap,uh);

%...Gauss variational equations (Equations 12.84):
hdot   = r*aps;

edot   = h/mu*sin(TA)*apr ...
        + 1/mu/h*((h^2 + mu*r)*cos(TA) + mu*e*r)*aps;

RAdot  = r/h/sin(i)*sin(phi)*aph;

idot   = r/h*cos(phi)*aph;

wdot   = - h*cos(TA)/mu/e*apr ...
        + (h^2 + mu*r)/mu/e/h*sin(TA)*aps ...
        - r*sin(phi)/h/tan(i)*aph;

TAdot  = h/r^2 ...
        + 1/e/h*(h^2/mu*cos(TA)*apr - (r + h^2/mu)*sin(TA)*aps);

%...Return rates to ode45 in the array dfdt:
dfdt   = [hdot edot RAdot idot wdot TAdot]';

```

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```
end %rates
%~~~~~

end %Example_12_12
%
```