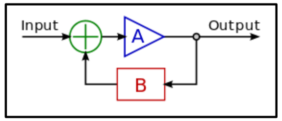
- ❑ General Robustness Problem
- ❑ Uncertainty Characterization and Robustness
- ❑ Robustness in MIMO Systems
- ❑ Shaping with Uncertainty
- ❑ Robustness of Optimal Controllers
- ❑ Uncertainty in a 2-Block Structure

❑ **Reference Material**

- Zhou, K., Essentials of Robust Control, Prentice Hall 1998, CH. 1-6
- Skogestad, S., Postlehwaite, I., Multivariable Feedback Control, Wiley 2010, Ch. 5, 6, 7, App. A
- Mackenroth, U., Robust Control Systems, Springer 2004, Ch. 1-7, App. A



Chapter 6: Topics

1. **General $\mathcal{H}_2$ Control Framework (**<u>loosely follow Skogestad Ch. 9, and Zhou Ch. 14</u>**)**

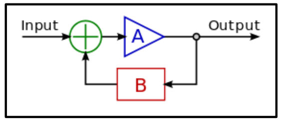
2. **Examples**

3. **General $\mathcal{H}_\infty$ Control Framework**

4. **Examples**

❑ To establish a structured, automated procedure for the design of a norm-based MIMO compensator capable of achieving nominal as well robust stability and performance.

- Linear matrix inequalities
- <u>Algebraic Riccati equations</u>

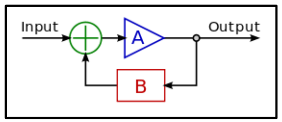


Doyle came to MIT in 1975 and fell in love with a science known as control theory. Control theorists, roughly speaking, try to understand how complicated things can run efficiently, quickly, and safely instead of crashing, exploding, or otherwise grinding to a halt. They analyze systems by modeling the variables that dictate how they will behave. But rather than checking through every possible combination of variables to see if, say, a plane will fly straight or stall when the wind picks up, control theorists look for underlying laws of control that can predict how something will behave using just a few key variables. "Control theory is at the center of modern technology," Doyle explains.

As technology has become more complex over the past century, researchers have had to find new ways to control airplanes, factories, computers, and the like. Much of that progress has come by brute-force tinkering, but a lot of it has come from a growing understanding of the basic laws of control. Doyle began developing his own innovative ideas in control theory as an undergraduate. By 1976 he was consulting for Honeywell. By age 32 he had been hired with immediate tenure at Caltech.

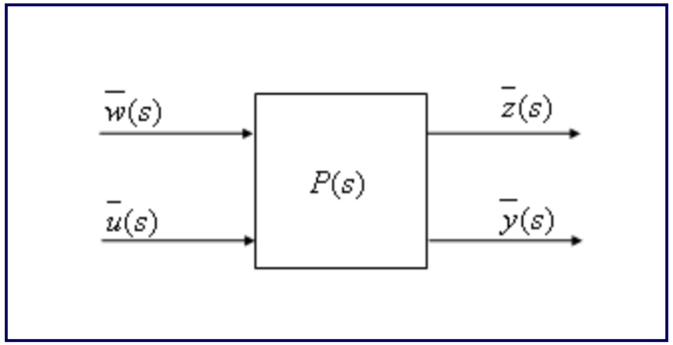
Doyle made his mark by figuring out how to prove a system is robust. In the early 1980s, NASA asked him to look at the space shuttle. Several shuttles had already flown, but the agency wanted reassurance about their behavior during reentry. Using wind tunnels and computer simulations, NASA had come up with apparently stable designs, but there were too many variables to test everything. "You had this obscenely large space of possibilities," Doyle says. "Somewhere lurking in there could be a crash, and you don't know."

Doyle looked at all the forces that might be exerted on a space shuttle due to atmospheric conditions, its velocity through the air, and so forth. NASA's engineers had plotted these forces in a so-called multidimensional space—say, a pitching torque along one axis and a longitudinal acceleration along another. By developing new mathematical tools, Doyle proved that there was a volume of this multidimensional space, inside of which every combination of forces was certainly safe. Outside that region lurked disaster. The space shuttle design was lodged comfortably inside the safe region.



http://www.cds.caltech.edu/~doyle/wiki/index.php?title=Main_Page

- General **Open Loop** System Configuration

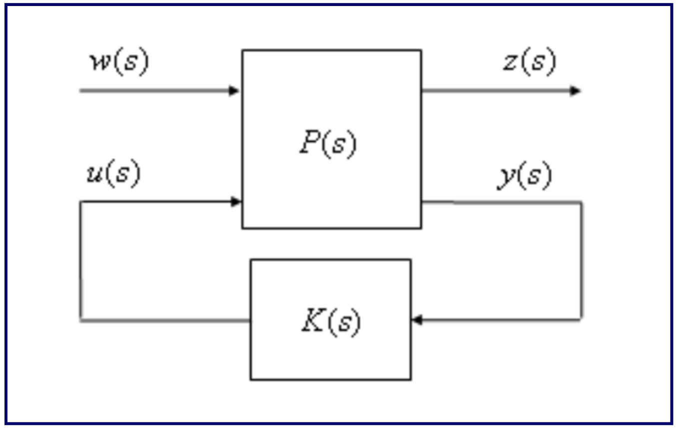


$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned}$$

$$P(s) = \left[\begin{array}{c:cc} A & B_1 & B_2 \\ \hdashline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \begin{bmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{bmatrix}$$

$$\begin{cases} z = P_{zw}(s)w + P_{zu}(s)u \\ y = P_{yw}(s)w + P_{yu}(s)u \end{cases}$$

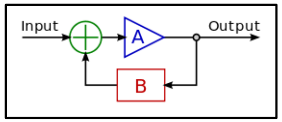
- General **Closed Loop** System Configuration



$$\begin{cases} y = (I - P_{yu}K)^{-1} P_{yw} w \\ z = \left\{ P_{zw} + P_{zu} K (I - P_{yu} K)^{-1} P_{yw} \right\} w = T_{zw} w \end{cases}$$

$$\begin{cases} \dot{q} = K_c q + Ly \\ u = Mq + Ny \end{cases}$$

$$K = K_{uy}(s) = M\left[sI - K_c\right]^{-1} L + N$$



- **Lemma:** The feedback system is well-Posed iff $I\text{-}D_{22}N$ is non singular.

- **Definition:** The feedback system is internally stable if the closed loop matrix A_c is stable.

$$A_c = \begin{bmatrix} A & B_2M \\ 0 & K_c \end{bmatrix} + \begin{bmatrix} B_2N \\ L \end{bmatrix} \left(I - D_{22}N\right)^{-1} \begin{bmatrix} C_2 & D_{22}M \end{bmatrix}$$

- **Note:** If the system is inter
the closed loop transfer m
a stable matrix

- **Definition:** The system $P(s)$ is stabilizable, If there is a proper controller $K(s)$ such that the Feedback system is well-posed and internally Stable. Such controller is said admissible.

- **Theorem:**
1. An admissible controller exists, if and only if $(A,\ B_2)$ is stabilizable and $(C_2,\ A)$ detectable.
2. Suppose $(A,\ B_2)$ is stabilizable and $(C_2,\ A)$
t K_1 and K_2 such that $A+B_2K_1$,
stable,
sible controller $K(s)$ is:

$$+ B_2K_1 + K_2C_2 + K_2D_{22}K_1 \;\big|\; -K_2 \quad\quad 0$$

- If $D_{22} = 0$, the controller has the classical expression of a LQG compensator

$$K(s) = \left[\begin{array}{c|c} A + B_2K_C + K_FC_{21} & -K_F \\ \hline K_C & 0 \end{array}\right]$$

- **A full observer structure sa**

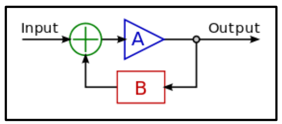
$$\begin{cases} \boldsymbol{u} = K_1\boldsymbol{q} \\ \dot{\boldsymbol{q}} = A\boldsymbol{q} + K_2\left(C_2\boldsymbol{q} + D_{22}\boldsymbol{u} - \boldsymbol{y}\right) + B_2\boldsymbol{u} \end{cases}$$

$$\begin{cases} \boldsymbol{u} = K_1\boldsymbol{q} \\ \dot{\boldsymbol{q}} = \left(A + B_2K_1 + K_2C_2 + K_2D_{22}K_1\right)\boldsymbol{q} - K_2\boldsymbol{y} \end{cases}$$

$$K_c\boldsymbol{q} + L\boldsymbol{y}$$

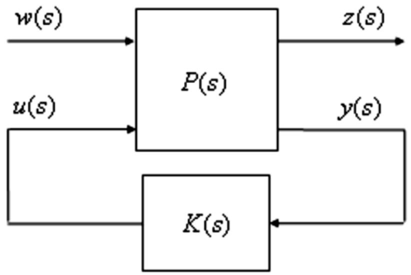
$$M\boldsymbol{q} + N\boldsymbol{y}$$

$$\begin{cases} K_c = \left(A + B_2K_1 + K_2C_2 + K_2D_{22}K_1\right) \\ L = 0 \\ M = K_1 \\ N = -K_2 \end{cases}$$



❑ $\mathcal{H}_2$ Optimal Control Problem: General Statement

- Consider the control system in Figure.



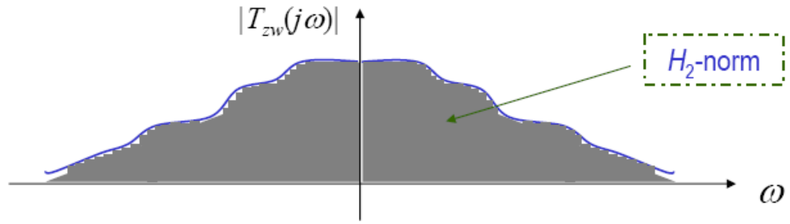
$$P(s) = \begin{bmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{bmatrix}$$

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x + D_{11} w + D_{12} u$$
$$y = C_2 x + D_{21} w + D_{22} u$$

- the closed-loop system transfer matrix is:

$$\mathrm{F}\left[P(s), K(s)\right] = T_{zw}(\mathrm{s}) =$$
$$= \left\{P_{zw} + P_{zu} K (I - P_{yu} K)^{-1} P_{yw}\right\}$$

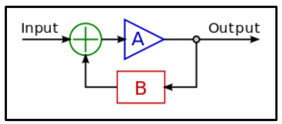
Graphically,



Note: The H_2-norm is the total energy corresponding to the impulse response of $T_{zw}(s)$. Thus, minimization of the H_2-norm of $T_{zw}(s)$ is equivalent to the minimization of the total energy from the disturbance w to the controlled output z.

- The $\mathcal{H}_2$-optimal control problem consists of finding a causal controller $K(s)$, which stabilizes the plant $P(s)$ and minimizes the function:

$$\min_{K(s)} \left\| \mathrm{F}\left[P(s), K(s)\right] \right\|_2^2 = \min_{K(s)} \left\| T_{zw}(s) \right\|_2^2 ; T_{zw}(s) \in RH_\infty$$

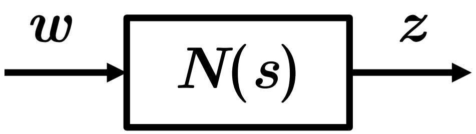


$\mathcal{H}_2$ norm

- Useful nominal performance measure.
- Linear quadratic (LQ) design methods use this norm.
- Minimizes "average" errors.

$\|N(s)\|_{\mathcal{H}_2} < 1$ implies:

- If $w(t) = \delta(t)$, then $\|z(t)\|_2 < 1$.
- If $\|w(t)\|_2 < 1$, then $\max\limits_t |z(t)| < 1$.
- If $w(t)$ is unit variance white noise, the $\mathrm{var}(z(t)) < 1$.



$N(s) = T_{zw}(s)$

- Note that $||.||_2$ is the $\mathcal{H}_2$ – norm, thus (use squares to avoid square roots)

$$\left\|T_{zw}(s)\right\|_2^2 = \left[\int_{-\infty}^{\infty} tr\left[g_{zw}^*(t)g(t)\right]dt\right] = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} tr\left[T_{zw}^*(j\omega)T_{zw}(j\omega)\right]d\omega\right]$$

$$\left\|T_{zw}\right\|_2^2 = tr\left[B_2^* L_0 B_2\right] = tr\left[C_2 L_C C_2^*\right]$$

- Recall results from Ch. 3

> • **Theorem:** The *$\mathcal{H}_2$-norm* is finite *if and only if* the TFM $G(s)$ is strictly proper and stable, which means:
> $$\lim_{s\to\infty} G(s) = G(\infty) = 0 \Leftrightarrow D = 0$$

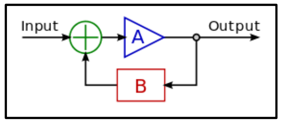
Observability Gramian $$L_0 = \left[\int_0^\infty e^{A^T t} C^T C e^{At} dt\right]$$

Controllability Gramian $$L_C = \left[\int_0^\infty e^{At} B B^T e^{A^T t} dt\right]$$

$$\begin{cases} A^T L_0 + L_0 A + C^T C = 0 \\ A L_C + L_C A^T + B B^T = 0 \end{cases}$$

- **Note:** The $\mathcal{H}_2$ norm can also be used to measure the energy of stochastic signals

- **Note on the matrix D:**
  - Assume $D_{22} = 0$ (since the input signal $\boldsymbol{u}(t)$ is known)
  - Assume $D_{11} = 0$ (otherwise the closed loop system is not strictly proper)
  - So it must be: $T_{zw}(\infty) = D_{11} + D_{12} D_{22} D_{21} = 0$

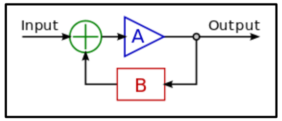


- Consider a stable, strictly causal CT LTI system with state-space model $(A, B, C, 0)$, transfer function $G(s)$, and impulse response $G(t)$.
- Consider a hypothetical stochastic input signal u such that $\mathbb{E}[u(t)] = 0$, and $\mathbb{E}[u(t)u(t+\tau)'] = I\delta(\tau)$. This is called white noise, and is just a mathematical abstraction, since it is a signal with infinite power.
- The $\mathcal{H}_2$ norm of G measures:
  - C) The (expected) power of the response to white noise:

$$
\begin{aligned}
&\mathbb{E}\left[\lim_{T\to+\infty}\frac{1}{T}\mathrm{Tr}\left(\int_0^T y(t)y(t)'\,dt\right)\right] \\
&= \lim_{T\to+\infty}\frac{1}{T}\mathrm{Tr}\left(\int_0^T \mathbb{E}\left[\int_0^t\int_0^t G(t-\tau_1)u(\tau_1)u(\tau_2)'G(t-\tau_2)'\,d\tau_1 d\tau_2\right]\,dt\right) \\
&= \lim_{T\to+\infty}\frac{1}{T}\mathrm{Tr}\left(\int_0^T\int_0^t G(t-\tau)G(t-\tau)'\,d\tau\,dt\right) \\
&= -\lim_{T\to+\infty}\mathrm{Tr}\left[\int_0^T G(T-\tau)G(T-\tau)'\,d(T-\tau)\right] = \|G\|_{\mathcal{L}_2}^2 = \|G\|_{\mathcal{H}_2}^2.
\end{aligned}
$$

Note on the matrix D:

- Assume $D_{22} = 0$ (since the input signal $u(t)$ is known)
- Assume $D_{11} = 0$ (otherwise the closed loop system is not strictly proper)
- So it must be: $T_{zw}(\infty) = D_{11} + D_{12}D_{22}D_{21} = 0$



- In order to compute an $\mathcal{H}_2$ optimal controller, some assumptions are needed:

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x + D_{11} w + D_{12} u$$
$$y = C_2 x + D_{21} w + D_{22} u$$

- Find a causal admissible controller $K(s)$, which stabilizes the plant $P(s)$ and minimizes the function:

$$\min_{K(s)} \left\| \mathrm{F}\left[P(s), K(s) \right] \right\|_2^2 = \min_{K(s)} \left\| T_{zw}(s) \right\|_2^2 ; T_{zw}(s) \in RH_\infty$$

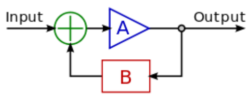
(A1) (A, B_2, C_2) is stabilizable and detectable.

(A2) D_{12} and D_{21} have full rank.

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω.

(A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

(A5) $D_{11} = 0$ and $D_{22} = 0$.



$$\text{(A1)} \quad (A, B_2, C_2) \text{ is stabilizable and detectable.}$$

- Necessary condition for the existence of a stabilizing controller $K(s)$

$\mathcal{H}_2$ $\mathcal{H}_\infty$

$$\text{(A2)} \quad D_{12} \text{ and } D_{21} \text{ have full rank.}$$

- Sufficient Condition to have proper and realizable controllers (feedback system well – posed).

$$\text{(A3)} \quad \begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} \text{ has full column rank for all } \omega.$$

$$\text{(A4)} \quad \begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} \text{ has full row rank for all } \omega.$$

- The conditions ensure that the controller does not cancel poles/zeros on the imaginary axis, which would result in closed loop instability (The Hamiltonian matrices **would not belong** to dom (Ric))

$$\text{(A5)} \quad D_{11} = 0 \text{ and } D_{22} = 0.$$

- Conventional assumption in $\mathcal{H}_2$ optimization. $D_{11} = 0$ implies that P_{11} is strictly proper, since is required for the computation of $\mathcal{H}_2$. $D_{22} = 0$ simplifies computation, making P_{22} strictly proper

$$\text{(A6)} \quad D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix} \text{ and } D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}.$$

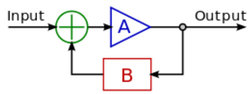
- Provides scaling and better conditioning of signals w and z

$$\text{(A7)} \quad D_{12}^T C_1 = 0 \text{ and } B_1 D_{21}^T = 0.$$

- **Additional**. In LQG setting implies no cross terms in the cost function, and uncorrelated noises.

$$\text{(A8)} \quad (A, B_1) \text{ is stabilizable and } (A, C_1) \text{ is detectable.}$$

- **Additional**. In LQG setting if (A7) holds, (A8) replaces (A3)+(A4).



LQR as $\mathcal{H}_2$ Optimization

The LQR problem is a special case of $\mathcal{H}_2$ synthesis in which we assume:

- $C_2 = \mathrm{I}$
- $\boldsymbol{w} = 0$

$$\begin{aligned}\dot{\boldsymbol{x}} &= A\boldsymbol{x} + B_1\boldsymbol{w} + B_2\boldsymbol{u}\\ \boldsymbol{z} &= C_1\boldsymbol{x} + D_{11}\boldsymbol{w} + D_{12}\boldsymbol{u}\\ \boldsymbol{y} &= C_2\boldsymbol{x} + D_{21}\boldsymbol{w} + D_{22}\boldsymbol{u}\end{aligned} \quad\Rightarrow\quad \begin{aligned}\dot{\boldsymbol{x}} &= A\boldsymbol{x} + B_2\boldsymbol{u}\\ \boldsymbol{z} &= C_1\boldsymbol{x} + D_{12}\boldsymbol{u}\\ \boldsymbol{y} &= I\boldsymbol{x}\end{aligned}$$

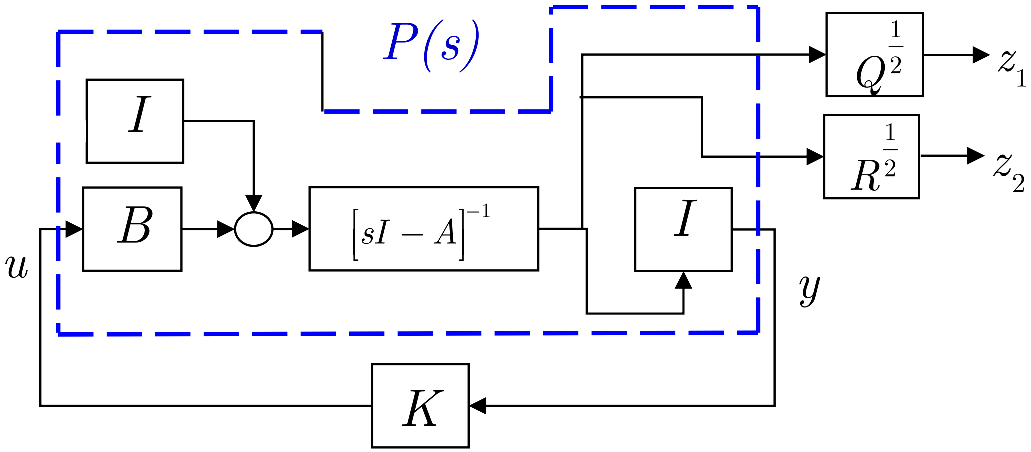
- **Objective**: find a feedback control signal $\boldsymbol{u}(t) \in \mathcal{L}_2$ that minimizes:

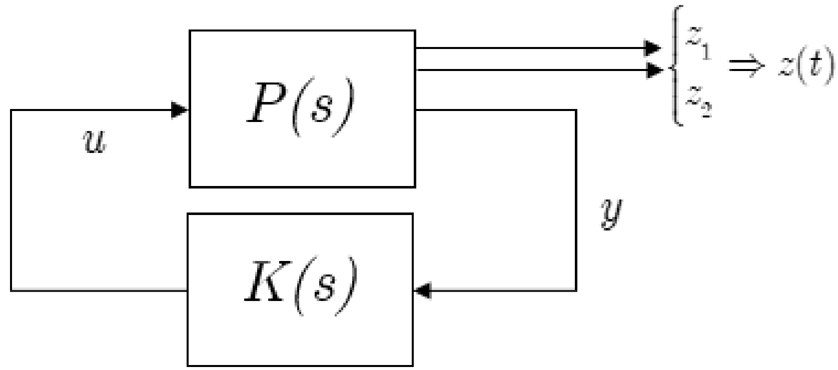
$$\left\|\boldsymbol{z}\right\|_2^2 = \int_0^{\infty} \left\|C_1\boldsymbol{x} + D_{12}\boldsymbol{u}\right\|_2^2 dt;\ \boldsymbol{x}(0) = \boldsymbol{x}_0$$

- For: $C_1 = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix}, D_{12} = \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix}$

$$\boxed{\left\|\boldsymbol{z}\right\|_2^2 = \int_0^{\infty} [\boldsymbol{x}^T Q\boldsymbol{x} + \boldsymbol{u}^T R\boldsymbol{u}]dt,\ \boldsymbol{x}(0) = \boldsymbol{x}_0.}$$

- 2 – Block P – K format





$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + B_2\boldsymbol{u} \\ \boldsymbol{z}_1 = Q^{\frac{1}{2}}\boldsymbol{x} \\ \boldsymbol{z}_2 = R^{\frac{1}{2}}\boldsymbol{u} \\ \boldsymbol{y} = I\boldsymbol{x} \end{cases}$$

$$P := \left[\begin{array}{c|cc} A & 0 & B \\ \hline Q^{\frac{1}{2}}C & 0 & 0 \\ 0 & 0 & R^{\frac{1}{2}} \\ I & 0 & 0 \end{array}\right]$$

Main Result

- **Theorem:** There exists a unique static state state-feedback controller $u(t) = K_{opt}x(t)$ that minimizes

$$\min_{K_{opt}(s)} J_2 \Leftrightarrow \min_{K_{opt}(s)} \left\| T_{zw}(s) \right\|_2^2; \quad K_{opt} = -(D_{12}^T D_{12})^{-1} B_2^T S$$

and where S is the unique symmetric positive (semi)definite solution to the algebraic Riccati equation (ARE)

$$A^T S + SA - SB_2(D_{12}^T D_{12})^{-1} B_2^T S + C_1^T C_1 = 0$$

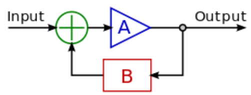
such that the matrix $A + B_2 K_{opt}$ *is stable, i.e. all its eigenvalues have negative real parts.*

- Consider a <u>stabilizing</u> control of the form $\boldsymbol{u} = F\boldsymbol{x}$. By assumption, $A_F = A + B_2 F$ is a stability matrix. Then:

$$\left\| \boldsymbol{z} \right\|_2^2 = \int_0^\infty \boldsymbol{x}_0^T [e^{A_F^T t} C_2^T C_2 e^{A_F t} + e^{A_F^T t} F^T D_{12}^T D_{12} e^{A_F t}] \boldsymbol{x}_0 dt = x_0^T X_F x_0$$

- X_F is the observability Gramian of the pair ($C_1 + D_{12}F, \ A_F$) solution of:

$$A_F^T X_F + X_F A_F = -(C_1 + D_{12}F)^T (C_1 + D_{12}F)$$



- Since we know that the closed-loop is stable, we can also rewrite the above equation as:

$$\|\boldsymbol{z}\|_2^2 = \int_0^\infty \left[\frac{d}{dt}\left(\boldsymbol{x}^T X_F \boldsymbol{x}\right)\right] dt$$

- The integrand can be written as: $\boldsymbol{x}^T[A^T X_F + F^T B_2^T X_F + X_F A + X_F B_2 F]\boldsymbol{x}$

- Assume there is a matrix S such that $F = SX_F$. Then, the integrand becomes:

$$\begin{aligned}&\boldsymbol{x}^T[A^T X_F + X_F S^T B_2^T X_F + X_F A + X_F B_2 F X_F]\boldsymbol{x}\\ &= -\boldsymbol{x}^T[C_1^T C_1 + X_F S^T D_{12}^T D_{12} S X_F]\boldsymbol{x}\end{aligned}$$

- Which means:

$$A^T X_F + X_F A + X_F S^T B_2^T X_F + X_F B_2 F X_F + C_1^T C_1 + X_F S^T D_{12}^T D_{12} S X_F = 0$$

- Set $S = -(D_{12}^T D_{12})^{-1} B_2^T$. Then, X_F must satisfy:

$$\begin{cases} A^T X_F + X_F A + X_F B_2 X_F + X_F B_2 (D_{12}^T D_{12})^{-1} B_2^T X_F + C_1^T C_1 = 0 \\ F = -(D_{12}^T D_{12})^{-1} B_2^T X_F \end{cases}$$

- Recall: $D_{12} = \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} \Rightarrow \boxed{F = -R^{-1} B_2^T X_F}$

Is this “solution” indeed stabilizing/optimal?

❑ The proof is based on:

- The relationship between Riccati equations and Hamiltonian matrices in Chapter 2

$$\begin{cases} H = dom(Ric) \\ X = Ric(H) \end{cases}$$

No eigenvalues of H on the imaginary axis

X form a basis of $\mathcal{X}_-(H)$ =

- The assumptions (A1) – (A4) made earlier

(A1) (A, B_2, C_2) is stabilizable and detectable.

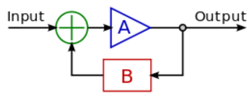
(A2) D_{12} and D_{21} have full rank.

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω.

(A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

A1-A3) ensure that the Riccati equation admits a solution X that is positive semi-definite. In particular, A1) is obviously necessary, A2) ensures that any unstable mode of A will be detected by the performance output, and A3) ensures that the control effort is penalized at all frequencies (this is an additional technical condition ensuring that the Hamiltonian does not have purely imaginary eigenvalues). A4) is just for convenience.

- **Kang - Zhi Liu, Yu Yao - Robust Control_ Theory and Applications-John Wiley & Sons, Inc.**
- **Zhou, Doyle, Clover, Robust and Optimal Control, Prentice Hall**



❑ KBF as $\mathcal{H}_2$ Optimization

- The Kalman Bucy Filter (KBF) was derived as the optimal state estimator that minimizes the square of the expected value of the estimation error for a system driven by white noise with sensors corrupted also by a white noise process. (See Chapter 2)

- This optimal estimation problem can also be solved as a $\mathcal{H}_2$ optimization

- In the estimation problem, $\boldsymbol{u}$ takes the role of the estimator update. The problem is therefore to find an update signal $\boldsymbol{u}(t,\tilde{x}) \in \mathcal{L}_2$ that minimizes the power in the error signal due to white noise disturbance $\boldsymbol{w}$.

- Note that the disturbance enters the system in two places:

$$\dot{\tilde{\boldsymbol{x}}} = A\tilde{\boldsymbol{x}} + B_1\boldsymbol{w}$$
$$\boldsymbol{y} = C_2\tilde{\boldsymbol{x}} + D_{21}\boldsymbol{w}$$

- For:

$$C_2 = \begin{bmatrix} W^{1/2} \\ 0 \end{bmatrix}, D_{21} = \begin{bmatrix} 0 \\ V^{1/2} \end{bmatrix}$$

- The noises are uncorrelated and we have the standard form of the KBF problem

- The $\mathcal{H}_2$ estimator optimization procedure is the **dual** of $\mathcal{H}_2$ LQR control.

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned} \longrightarrow \begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + Iu \\ y &= C_2 x + D_{21} w \end{aligned}$$

- Consider a <u>stabilizing</u> update $\boldsymbol{u}$ of the form below, that yields a stability matrix $\boldsymbol{A_L}$.

$$\begin{aligned} \boldsymbol{u} &= L(C_2 \tilde{\boldsymbol{x}} + D_{21} \boldsymbol{w}) \\ A_L &= A + LC_2 \end{aligned}$$

- The power of the error, under white noise disturbance, that is the $\mathcal{H}_2$ norm to be minimized is given by (see earlier results) is:

$$P_z = Tr[\int_0^\infty (B_1 + LD_{21})^T e^{A_L^T t} e^{A_L t} (B_1 + LD_{21}) dt]$$

- This yields $\boldsymbol{P_z = tr\,[Y_L]}$, where $\boldsymbol{Y_L}$ is the controllability Gramian of the pair $(B_1 + \mathrm{L}D_{21},\ A_L)$ solution of:

$$A_L Y_L + Y_L A_L^T = -(B_1 + LD_{21})(B_1 + LD_{21})^T$$

- Assume there is a matrix S such that $L = Y_L S$. Then Y_L must satisfy:

$$A\,Y_L + LC_2 Y_L + Y_L A^T + Y_L C_2^T L^T + B_1 B_1^T + LD_{21} D_{21}^T L^T = 0$$

- Set $S = -C_2^T (D_{21} D_{21}^T)^{-1}$. Then, Y_L must satisfy:

$$\begin{cases} A\,Y_L + Y_L A^T - Y_L C_2^T (D_{21} D_{21}^T)^{-1} C_2 Y_L + B_1 B_1^T = 0 \\ L = -Y_L C_2^T (D_{21} D_{21}^T)^{-1} \end{cases}$$

- Recall: $D_{21} = \begin{bmatrix} 0 \\ V^{1/2} \end{bmatrix} \Rightarrow \boxed{L = -V^{-1} C_2 Y_L}$

❑ **Is this “solution” indeed stabilizing/optimal?**

❑ The proof is based on:

- The relationship between Riccati equations and Hamiltonian matrices in Chapter 2

$$\begin{cases} H = dom(Ric) \\ X = Ric(H) \end{cases}$$

No eigenvalues of H on the imaginary axis

X form a basis of $\mathcal{X}_-(H)$ =

- The assumptions (B1) – (B4):

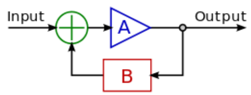
(B1) $(C_2,\ A)$ detectable

(B2) $(A,\ B_1)$ stabilizable

(B3) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full rank for all frequencies

(B4) $D_{21}{D_{21}}^T = V$, invertible (i.e. D_{21} Has full rank).

B1-B3) ensure that the Riccati equation admits a solution Y that is positive semi-definite. In particular, B1) is obviously necessary, B2) ensures that any unstable mode of A can be excited by the disturbance, and B3) ensures that errors are penalized at all frequencies (this is an additional technical condition ensuring that the Hamiltonian does not have purely imaginary eigenvalues). B4) is just for convenience.



LQG as $\mathcal{H}_2$ Optimization

- The general version of the problem can be seen as a combination of the LQR problem and of the KBF problem. This is also called the LQG problem

Main Result

H_2-optimal control problem. *Consider the system (4.5). Suppose that the assumptions (A1)–(A4) hold. Then the controller $u = Ky$ which minimizes the H_2 cost (4.6) is given by the equations*

$$\dot{\hat{x}}(t) = (A + B_2K_{opt})\hat{x}(t) + L[y(t) - C_2\hat{x}(t)]$$
$$u(t) = K_{opt}\hat{x}(t) \tag{4.52}$$

where

$$K_{opt} = -(D_{12}^T D_{12})^{-1} B_2^T S \tag{4.53}$$

and

$$L = PC_2^T(D_{21}D_{21}^T)^{-1} \tag{4.54}$$

and S and P are the symmetric positive (semi)definite solutions of the algebraic Riccati equations (4.13) and (4.27), respectively.

Moreover, the minimum value of the cost achieved by the controller (4.52) is

$$\min_K J_2(K) = \mathrm{tr}(D_{12}K_{opt}PK_{opt}^T D_{12}^T) + \mathrm{tr}(B_1^T S B_1) \tag{4.55}$$

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B_1\boldsymbol{w} + B_2\boldsymbol{u}$$
$$\boldsymbol{z} = C_1\boldsymbol{x} + D_{12}\boldsymbol{u}$$
$$\boldsymbol{y} = C_2\boldsymbol{x} + D_{21}\boldsymbol{u}$$

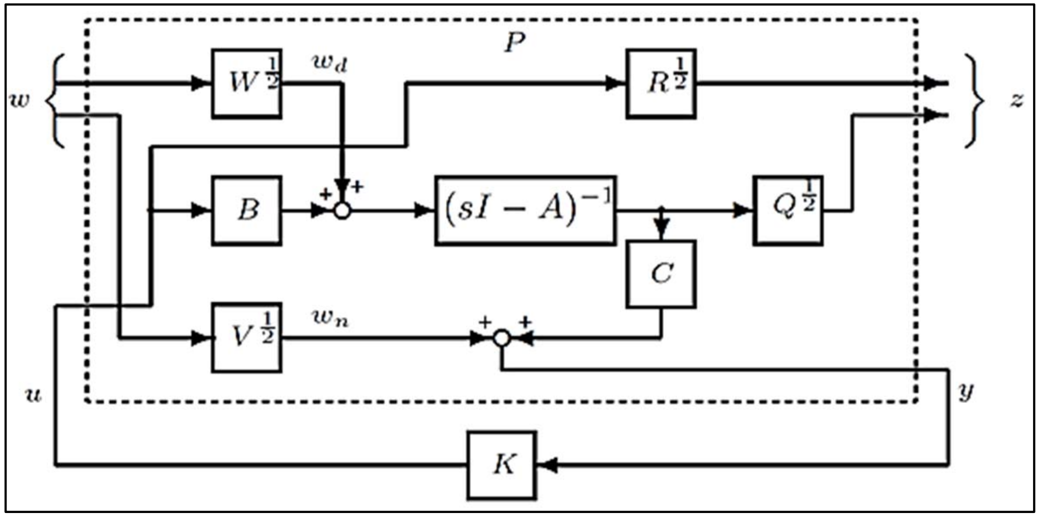
$$\left\|\boldsymbol{z}\right\|_2^2 = \int_0^\infty \left[\boldsymbol{z}^T(t)\boldsymbol{z}(t)\right]dt$$

- Controller/Observer transfer matrix models:

$$G_C = \begin{bmatrix} A + B_2 F & I \\ C_1 + D_{12} F & 0 \end{bmatrix} \qquad G_F = \begin{bmatrix} A + L C_2 & B_1 + L D_{21} \\ I & 0 \end{bmatrix}$$

- The state-space model of the optimal controller is then given by:

$$K(s) = \begin{bmatrix} A + B_2 F + L C_2 & -L & B_2 \\ F & 0 & I \\ -C_2 & I & 0 \end{bmatrix}$$



$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \overset{s}{=} \left[\begin{array}{c|cc|c} A & W^{\frac{1}{2}} & 0 & B \\ \hline Q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & R^{\frac{1}{2}} \\ \hline C & 0 & V^{\frac{1}{2}} & 0 \end{array}\right]$$

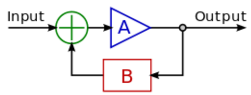
- The assumptions for LQG / $\mathcal{H}_2$ optimization are known:

- The optimal controller gain F below is a function of X, the stabilizing solution of the associated ARE:

$$\begin{cases} (A - B_2 R^{-1} D_{12}^T C_1)^T X + X(A - B_2 R^{-1} D_{12}^T C_1) - X B_2 R^{-1} B_2^T X + + C_1^T (I - D_{12} R^{-1} D_{12}^T) C_1 = 0 \\ F = -R^{-1}(B_2^T X + D_{12} C_1) \end{cases}$$

- The optimal observer gain L below is a function of Y, the stabilizing solution of the associated ARE:

$$\begin{cases} (A - B_1 D_{21}^T V^{-1} C_2) Y + Y(A - B_1 D_{21}^T V^{-1} C_2)^T - Y C_2^T V^{-1} C_2 Y + B_1 (I - D_{21} V^{-1} D_{21}^T) B_1^T = 0 \\ L = -(Y C_2^T + B_1 D_{21}^T) V^{-1} \end{cases}$$



Alternate Statement from Mackenroth, "Robust Control Systems", Springer Verlag, 2004.

Theorem ($\mathcal{H}_2$ – Optimal Control from Mackenroth's book):

1. $(A,\ B_2)$ is stabilizable and $(C_2,\ A)$ detectable.
2. Suppose $(A,\ B_1 W^{1/2})$ has no uncontrollable eigenvalues on the imaginary axis.
3. Suppose $(Q^{1/2},\ A)$ has no unobservable eigenvalues on the imaginary axis.
4. $R,\ V$ are positive definite and $Q,\ W$ are positive semidefinite.

- Let $X_2,\ Y_2$ solutions of:

$$\begin{cases} A^T X_2 + X_2 A - X_2 B_2 R^{-1} B_2^T X_2 + Q = 0 \\ Y_2 A^T + A Y_2 - Y_2 C_2^T V^{-1} C_2 + B_1 W B_1^T = 0 \end{cases}$$

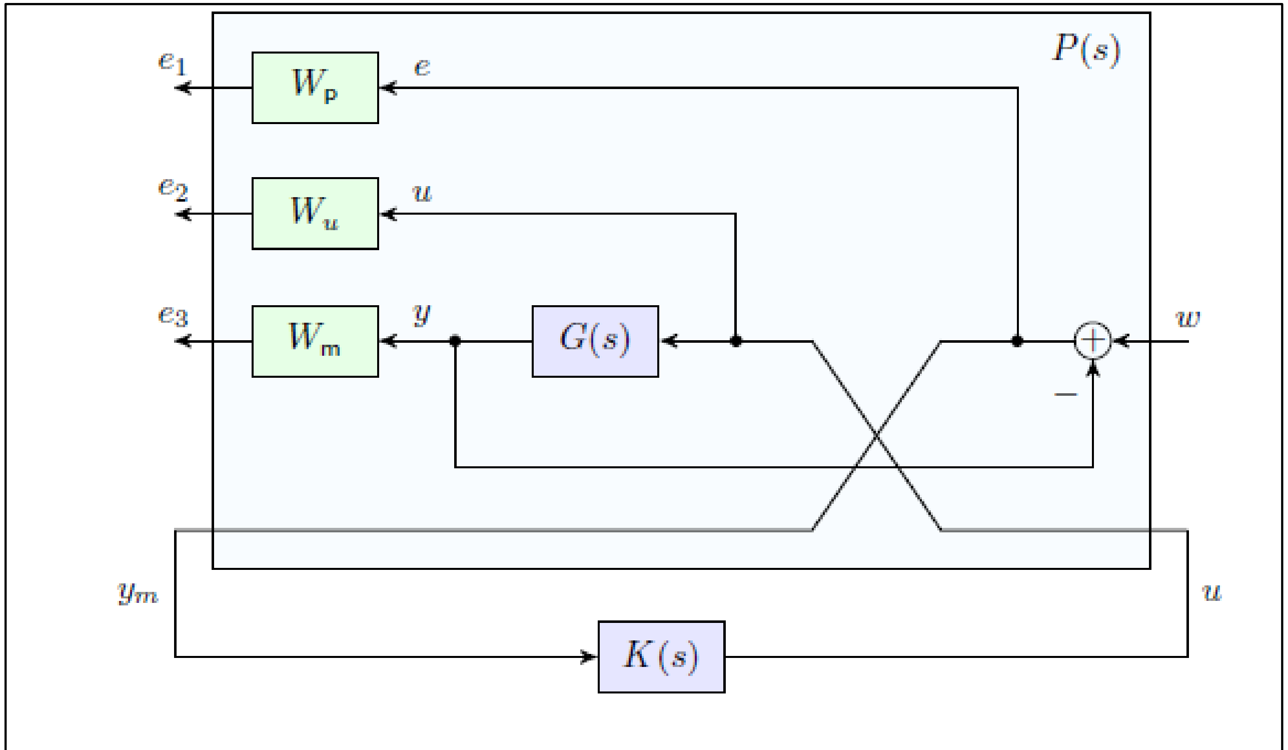
- Define:

$$\begin{cases} F_2 = -R^{-1} B_2^T X_2 \\ L_2 = -Y_2 C_2^T V^{-1} \end{cases}$$

- The optimal solution is unique and given by:

$$\begin{cases} u = F_2 q \\ \dot{q} = \left(A + B_2 F_2 + L_2 C_2 + L_2 D_{22} F_2\right) q - L_2 y \end{cases}$$

- **Frequency shaping interconnection, weighted $\mathcal{H}_2$ norm minimization (Also defined as mixed sensitivity optimization)**



$N(s) = \mathcal{F}_l\left(P(s), K(s)\right)$

$$N(j\omega) = \left\| \begin{bmatrix} W_u K(j\omega) S(j\omega) \\ W_P S(j\omega) \\ W_m T(j\omega) \end{bmatrix} \right\|_2$$

$$\min_K \left\| N(j\omega) \right\|_2^2$$

TRAJECTORY TRACKING $\mathcal{H}_2$ CONTROLLER FOR AUTONOMOUS HELICOPTERS: AN APPLICATION TO INDUSTRIAL CHIMNEY INSPECTION

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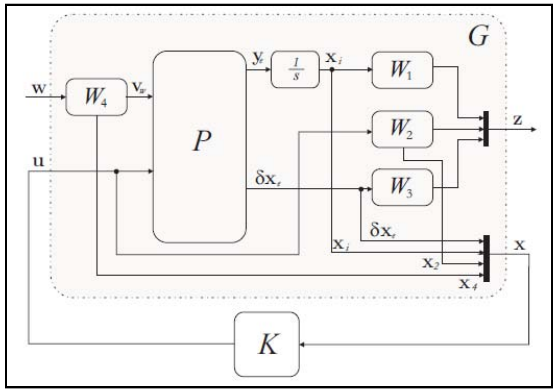
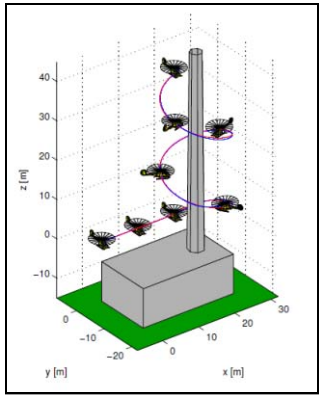
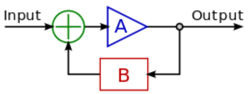
Abstract: This paper addresses the problem of industrial chimney inspection using autonomous helicopters. The importance of using these platforms is evidenced by the maintenance cost and risk reduction stemming from the replacement of the standard procedures and the improved detection of structural defects. The approach presented relies on the definition of a trajectory-dependent error space to express the dynamic model of the vehicle, and adopts a Linear Parameter Varying (LPV) representation with piecewise affine dependence on the parameters to model the error dynamics over a set of predefined operating regions. The synthesis problem is stated as a continuous-time $\mathcal{H}_2$ control problem, solved using Linear Matrix Inequalities (LMIs) and implemented within the scope of gain-scheduling control theory. The effectiveness of the proposed controller is assessed in simulation using the full nonlinear model of a small-scale helicopter.







- The controller is designed as a gain scheduler over a number of equilibrium points around which the dynamic equations are linearized

- Choice of Weights:

$$W_1(s) = 0.1 \cdot I_4$$

- Weights on integrators' gains

$$W_3(s) = 0.1 \cdot I_{12}$$

- Weights on perturbed state variables

$$W_2(s) = \left[\begin{array}{ccc:c} \frac{13}{10}\frac{s+10}{s+13} & 0 & .. & 0 \\ 0 & \frac{13}{10}\frac{s+10}{s+13} & 0 & .. \\ 0 & 0 & \frac{13}{10}\frac{s+10}{s+13} & 0 \\ \hdashline 0 & .. & 0 & \frac{10}{7}\frac{s+7}{s+10} \end{array}\right]$$

- Weights on actuators

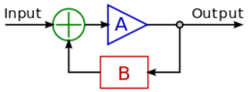
- Weights on wind disturbance

$$W_4(s) = \begin{bmatrix} W_{4u}(s) & 0 & 0 \\ 0 & W_{4v}(s) & 0 \\ 0 & 0 & W_{4w}(s) \end{bmatrix}$$

$$W_{4u}(s) = \frac{\sigma_u \sqrt{\frac{2}{\pi}\frac{L_u}{V}}\left(1 + 0.25\,\frac{L_u}{V}\,s\right)}{1 + 1.357\,\frac{L_u}{V}\,s + 0.199\,(\frac{L_u}{V}\,s)^2}$$

$$W_{4v}(s) = \frac{\sigma_v \sqrt{\frac{1}{\pi}\frac{L_v}{V}}\left(1 + 2.748\,\frac{L_v}{V}\,s + 0.34\,(\frac{L_v}{V}\,s)^2\right)}{1 + 2.996\,\frac{L_v}{V}\,s + 1.975\,(\frac{L_v}{V}\,s)^2 + 0.154\,(\frac{L_v}{V}\,s)^3}$$

$$W_{4w}(s) = \frac{\sigma_w \sqrt{\frac{1}{\pi}\frac{L_w}{V}}\left(1 + 2.748\,\frac{L_w}{V}\,s + 0.34\,(\frac{L_w}{V}\,s)^2\right)}{1 + 2.996\,\frac{L_w}{V}\,s + 1.975\,(\frac{L_w}{V}\,s)^2 + 0.154\,(\frac{L_w}{V}\,s)^3}$$



- The full model becomes:

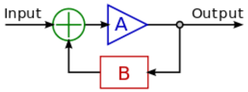
$$\begin{cases} \dot{\mathbf{x}} = A(\boldsymbol{\xi})\mathbf{x} + B_w(\boldsymbol{\xi})\mathbf{w} + B(\boldsymbol{\xi})\mathbf{u} \\ \mathbf{z} = C\mathbf{x} + D\mathbf{w} + E\mathbf{u} \end{cases} \qquad \mathbf{x} = \left[\, \delta\mathbf{x}_e^T \;\; \mathbf{x}_i^T \;\; \mathbf{x}_2^T \;\; \mathbf{x}_4^T \,\right]^T \in \mathbb{R}^{12+4+4+8}$$

$$A(\boldsymbol{\xi}) = \begin{bmatrix} A_e(\boldsymbol{\xi}) & 0 & 0 & B_{w_e}(\boldsymbol{\xi})\, C_{w_4} \\ C_e & 0 & 0 & D_e\, C_{w_4} \\ 0 & 0 & A_{w_2} & 0 \\ 0 & 0 & 0 & A_{w_4} \end{bmatrix},$$

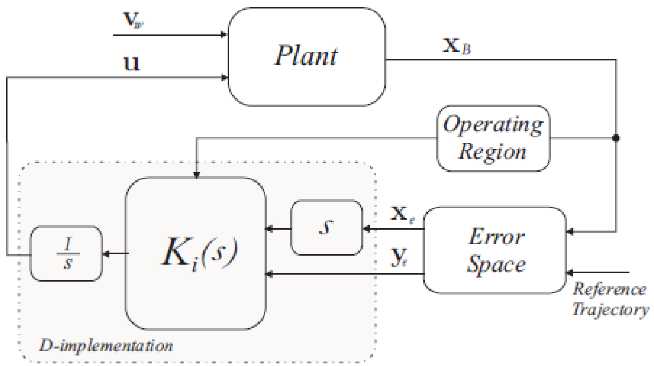
$$B_w(\boldsymbol{\xi}) = \begin{bmatrix} B_{w_e}(\boldsymbol{\xi})\, D_{w_4} \\ D_e\, D_{w_4} \\ 0 \\ B_{w_4} \end{bmatrix}, \quad B(\boldsymbol{\xi}) = \begin{bmatrix} B_e(\boldsymbol{\xi}) \\ E_e \\ B_{w_2} \\ 0 \end{bmatrix}$$

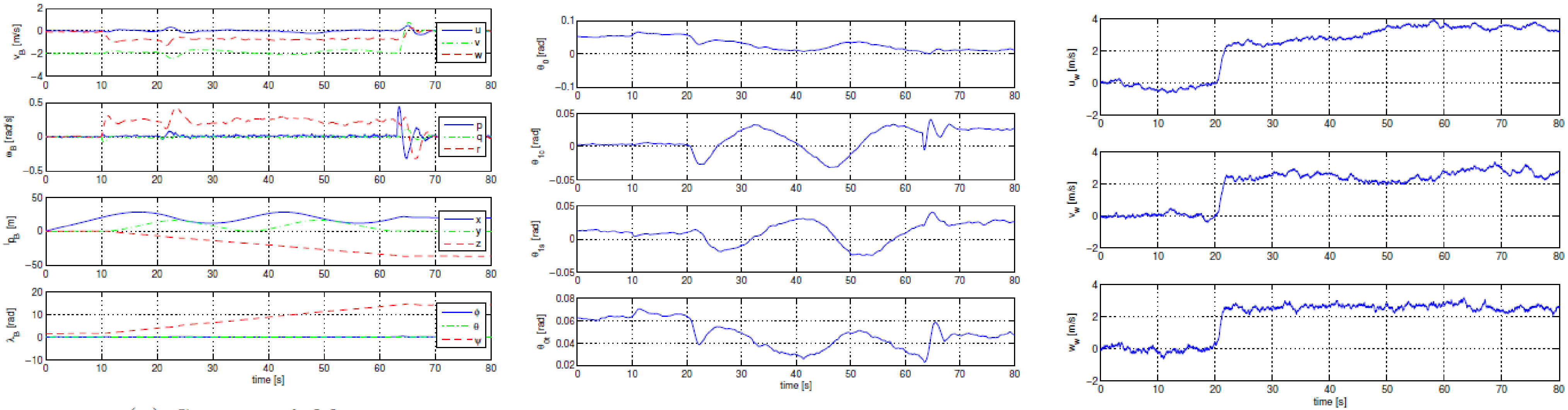
$$C = \begin{bmatrix} 0 & W_1 & 0 & 0 \\ 0 & 0 & C_{w_2} & 0 \\ W_3 & 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ D_{w_2} \\ 0 \end{bmatrix},$$

For that purpose the vehicle is required to track the following trajectory: *(i)* a straight line moving sideways ($V_r = 2m/s$, $\psi_c = \pi/2rad$ and $\dot{\psi}_r = \gamma_r = 0$); *(ii)* a helix keeping the camera axis pointing in the direction of the surface under inspection ($V_r = 2m/s$, $\psi_c = \pi/2rad$, $\dot{\psi}_r = 0.24rad/s$ and $\gamma_r = 0.34rad$); and *(iii)* finally a hover at a prespecified point. In addition to the noise generated using the Von Karman disturbance model a discrete wind gust with amplitude $2.5m/s$ and rising time of $1s$ is applied at time $t = 20s$.



Param.	Intervals	Unit
V_c	$[-3, 3]$	$[m/s]$
ψ_c	$[-0.2, 0.2]$	$[rad/s]$
γ_c	$[-100, -85]$, $[-90, -60]$, $[-65, -35]$, $[-40, -10]$, $[-15, 15]$, $[10, 40]$, $[35, 65]$, $[60, 90]$, $[85, 100]$	$[deg]$
ψ_0	$[-190, -160]$, $[-165, -135]$, $[-140, -110]$, $[-115, -85]$, $[-90, -60]$, $[-65, -35]$, $[-40, -10]$, $[-15, 15]$, $[10, 40]$, $[35, 65]$, $[60, 90]$, $[85, 115]$, $[110, 140]$ $[135, 165]$, $[160, 190]$	$[deg]$





(a) State variables

(b) Actuation

(c) Wind velocity disturbance

https://it.mathworks.com/help/robust/ref/lti.h2syn.html

Some Historical Notes

While linear quadratic (H_2) optimal control was applied successfully during the 1960's and 1970's, mainly to aerospace problems, its failure to explicitly address robustness issues was critized by a number of workers. As robustness is most naturally treated in a frequency domain framework, these researchers continued to develop frequency-domain methods during that time.

The H_∞ control problem, and its connection to robustness, which will be discussed in Chapter 5, was introduced in the seminal work by George Zames in the late 1970's (Zames 1981). First the theory was presented entirely in the frequency domain, and the computation of H_∞-optimal controllers was based on analytic function theory and operator-theoretic methods. These methods were quite complicated, and they gave only a limited insight into the structure of the solutions. For example, before the introduction of the state-space solution in 1988 it was not known what order a controller was required to have to achieve an H_∞-norm bound. Also, the common folklore regarding the difficulty of H_∞ control goes back to this period. An excellent exposition of theses methods is given in the book by Francis (1987).

The connection between the H_∞ problem and linear quadratic game theory was first observed by Petersen (1987), and the stationary state feedback H_∞ problem was then presented in Khargonekar, Petersen and Rotea (1988) and Zhou and Khargonekar (1988). The general state-space solution to the H_∞-optimal control problem was first given by Glover and Doyle (1988), and developed in full in the classic paper by Doyle, Glover, Khargonekar and Francis (1989), and the preliminary conference version in Doyle *et al.* (1988).

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State-Space Solutions to Standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control Problems

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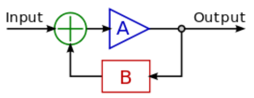
Abstract—**Simple state-space formulas are derived for all controllers solving a standard $\mathcal{H}_\infty$ problem: for a given number $\gamma > 0$, find all controllers such that the $\mathcal{H}_\infty$ norm of the closed-loop transfer function is (strictly) less than γ. A controller exists if and only if the unique stabilizing solutions to two algebraic Riccati equations are positive definite and the spectral radius of their product is less than γ^2. Under these conditions, a parametrization of all controllers solving the problem is given as a linear fractional transformation (LFT) on a contractive, stable free parameter. The state dimension of the coefficient matrix for the LFT, constructed using these same two Riccati solutions, equals that of the plant, and has a separation structure reminiscent of classical LQG (i.e., $\mathcal{H}_2$) theory. This paper is also intended to be of tutorial value, so a standard $\mathcal{H}_2$ solution is developed in parallel.**



The state-space solution revolutionized the practical numerical computation of H_∞-optimal controllers. Whereas the calculation of an H_∞-optimal controller before 1988 had been an extremely demanding task, the state-space solution has approximately the same complexity as the standard linear quadratic control problem. However, the original papers (Doyle *et al.* 1988, 1989) did not do much to increase the general understanding of H_∞ control, and it is fair to say that even not many researchers in the field can completely read the original papers by Doyle *et al.* (1988, 1989). In fact, the paper by Doyle *et al.* (1989) has been called 'the most important unreadable paper in the history of control science'. Since then, increased understanding of the problem has made it possible to present the H_∞-optimal controller along the significantly simpler lines as has been done in this chapter. This approach has been developed gradually by many people during the 1990's.

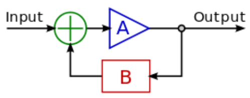
Somewhat ironically, the H_∞-optimal control problem, which was originally introduced in the frequency domain to deal with the robustness issues which the then prevailing time-domain linear quadratic control methods failed to address, is today solved by algorithms which have a close resemblance to the procedures used to solve precisely the very same linear quadratic control problems.

Linear quadratic game theory, with which the state-space H_∞ solution is closely related, was largely developed in the 1960's. The standard reference in this field is the book by Başar and Olsder (1982). After the connection between problem and dynamic game theory was first realized, it became clear that much of the required theory was already available in the game-theory literature. In particular, the optimal state-feedback result (Theorem 4.1) was fully developed in classical linear-quadratic game theory. A thorough presentation of the game-theory approach to H_∞ control can be found in the book by Başar and Bernhard (1991).

The solution of the H_∞-optimal estimation problem has been given in the (hard-to-read) paper by Nagpal and Khargonekar (1991). An alternative, and simpler, derivation is given by Banavar and Speyer (1991).

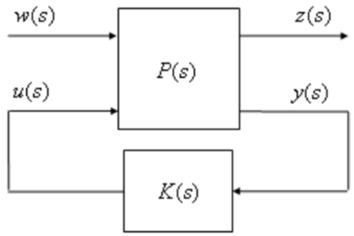
$\mathcal{H}_2$ Optimization 1960 / 1970	No formal Robustness issues Addressed (except for Kalman's inequality) Robustness was sought in the frequency domain
$\mathcal{H}_\infty$ Optimization 1 Zames 1981 McGill Univ.	Developed completely in the frequency domain Very complex analytic functions theory No physical insight
$\mathcal{H}_\infty$ Optimization 2 1988 Glover Cambridge 1981 Doyle CalTech	Use state space approach Algorithms similar to LQ techniques Only for full state feedback
$\mathcal{H}_\infty$ Optimization 3 1991 Basar, Univ. Illinois	Connection with LQ game theory
$\mathcal{H}_\infty$ with fixed Structure 2004 Apkarian, CNES	Synthesis of controllers with fixed structure

https://www.youtube.com/watch?v=Oq4qbp-vKt0&t=682s
https://www.youtube.com/watch?v=kRt7H0k8A4k Brian Douglas RCp4



Statement of the general $\mathcal{H}_\infty$ Optimal Control Problem

- Consider the control system in Figure. Introduce the partition of $P(s)$ according to



$$P(s) = \begin{bmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{bmatrix}$$

the closed-loop system is given by the transfer matrix: $\boldsymbol{z}(s) = F_L\left[P(s), K(s)\right]\boldsymbol{w}(s)$

$$F_L\left[P(s), K(s)\right] = T_{zw}(\mathrm{s}) = \left\{P_{zw} + P_{zu}K(I\text{-}P_{yu}K)^{-1}P_{yw}\right\}$$

- The $\mathcal{H}_\infty$-optimal control problem consists in finding a <u>causal admissible</u> controller $K(s)$ which stabilizes the plant $P(s)$ and which minimizes the cost function:

$$\boxed{J_\infty(K) = \left\| F_L\left[P(s), K(s)\right] \right\|_\infty}$$

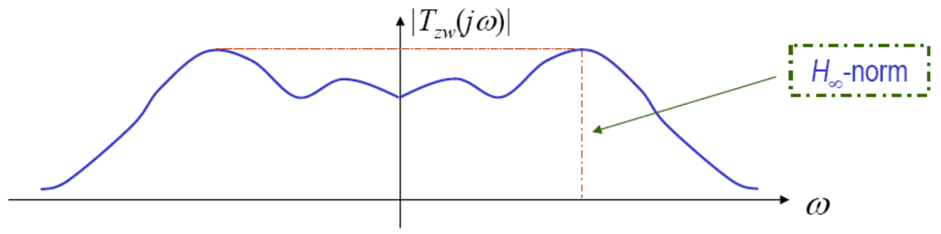
- Where $||.||_\infty$ is the $\mathcal{H}_\infty$ - norm $\left\|T_{zw}(s)\right\|_\infty := \sup_{\omega \in \Re} \sigma_{\max}\left[T_{zw}(j\omega)\right]$

- Or the Induced worst case 2-Norm $\left\|T_{zw}(s)\right\|_\infty = \sup_{w(t)\neq 0} \dfrac{\left\|\boldsymbol{z}(t)\right\|_2}{\left\|\boldsymbol{w}(t)\right\|_2}$

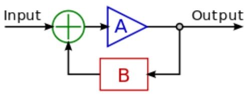
$$\boxed{\left\|\boldsymbol{z}(t)\right\|_2 = \left[\int_0^\infty \boldsymbol{z}^*(t)\boldsymbol{z}(t)dt\right]^{\frac{1}{2}} = \sqrt{\int_0^\infty \sum_i \left|z_i(t)\right|^2}}$$

❑ Comments:

- The computation of a $\mathcal{H}_\infty$ Controller is performed using several methods:
  1. Riccati Equations Approach
  2. Linear Matrix Inequalities Approach
  3. Zero – Sum Differential Game Approach
- The resulting $\mathcal{H}_\infty$ Controller assumes the following feedback forms:
  1. Full state feedback
  2. Dynamic output feedback (with the computation of an appropriate "estimator"
  3. The separation principle valid for the $\mathcal{H}_2$ controller, is no longer applicable (i.e. the relevant ARE are now coupled)
- The $\mathcal{H}_\infty$ Optimal control solution requires a numerical optimization of the $\mathcal{H}_\infty$ norm. A common alternative is to compute a "suboptimal" solution



Note: The H_∞-norm is the worst case gain in $T_{zw}(s)$. Thus, minimization of the H_∞-norm of $T_{zw}(s)$ is equivalent to the minimization of the worst case (gain) situation on the effect from the disturbance w to the controlled output z.



$\mathcal{H}_\infty$-suboptimal control problem

- Direct minimization of $J_\infty(K)$ is a very hard problem, therefore conditions have been found to ensure the existence of a stabilizing controller for which the following $\mathcal{H}_\infty$-norm bound holds for a given $\gamma > \gamma_{\min} > 0$ (recall the BRL):

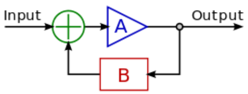
$$J_\infty(K) < \gamma; \text{where} \left\|T_{zw}(s)\right\|_\infty := \sup_{\omega\in\Re} \sigma_{\max}\left[T_{zw}(j\omega)\right] = \gamma_{\min}$$

- The minimum of $J_\infty(K)$ can be derived to any degree of accuracy by means of a numerical procedure called ***γ-iteration***

$\mathcal{H}_\infty$-norm as worst case gain in Game Theory

- Recall: $J_\infty(K) = \sup\left\{\dfrac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{w}\|_2} : w \neq 0\right\} = \dfrac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{w}\|_2} < \gamma$ $\qquad L(\boldsymbol{u}, \boldsymbol{w}) = \|\boldsymbol{z}\|_2^2 - \gamma^2 \|\boldsymbol{w}\|_2^2 < 0 : w \neq 0$

$$L(\boldsymbol{u}, \boldsymbol{w}) = \int_0^\infty \left[\boldsymbol{z}(t)^T \boldsymbol{z}(t) - \gamma^2 \boldsymbol{w}(t)^T \boldsymbol{w}(t)\right] dt < 0 : w \neq 0$$



- **Game theory approach**: assume that the controller $K(s)$ and the disturbance w are playing a zero-sum game, in which the cost is :

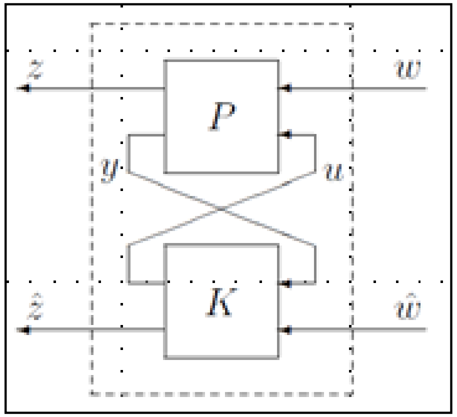
$$L(\boldsymbol{u},\boldsymbol{w}) = \int_0^\infty \left[\boldsymbol{z}(t)^T \boldsymbol{z}(t) - \gamma^2 \boldsymbol{w}(t)^T \boldsymbol{w}(t)\right] dt < 0 : w \neq 0$$

  - what is the smallest γ such that the controller can win the game (i.e., achieve a negative cost)?

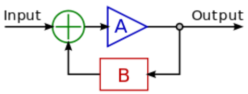
❑ Numerical toolbox for closed loop computation:

- Assume the the suboptimal problem has been solved for an admissible and stabilizing $K(s)$, the closed loop system can be found using the RedHeffer star product

$$\left\|T_{zw}(s)\right\|_\infty = \left\|P * K\right\|_\infty$$



$$P \star K := \begin{bmatrix} F_l\left(P, K_{11}\right) & P_{12}\left(I - K_{11}P_{22}\right)^{-1} K_{12} \\ K_{21}\left(I - P_{22}K_{11}\right)^{-1} P_{21} & F_u\left(K, P_{22}\right) \end{bmatrix}$$



❑ $\gamma-$ iteration procedure

❑ **Theorem** (Zhou, "Essentials of Robust Control"): The $\mathcal{H}_\infty$-norm of a stable **proper** system is bounded by a positive scalar $\gamma > 0$, that is:

$$\left\|G(s)\right\|_\infty < \gamma, G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

if and only if:

$$\sigma_{\max}(D) < \gamma$$

and the 2nx2n Hamiltonian Matrix $\mathcal{H}$ has no eigenvalues on the imaginary axis.

$$\begin{cases} \mathcal{H} := \left[\begin{array}{c|c} A + BR^{-1}D^T C & BR^{-1}B^T \\ \hline -C^T(I + DR^{-1}D^T)C & -A^T - C^T DR^{-1}B^T \end{array}\right] \\ R = (\gamma^2 I - D^T D) \end{cases}$$

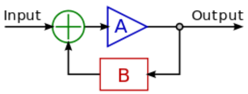
$$\sigma_{MAX}(D) < \gamma$$

$$X = X^T \geq 0$$

$$(A + BR^{-1}D^T C)^T X + X(A + BR^{-1}D^T C) + XBR^{-1}B^T X + C^T(I + DR^{-1}D^T)C = 0$$

$(A + BR^{-1}D^T C + BR^{-1}B^T X)$ **Has no eigenvalues on the imaginary axis**

- The optimal performance γ^* can be approximated arbitrarily well, by a bisection method, maintaining lower and upper bounds $\gamma_- < \gamma^* < \gamma_+$:
  - Init to, e.g., $\gamma_- = 0$, $\gamma_+ =$ the $\mathcal{H}_\infty$ norm of the $\mathcal{H}_2$ optimal design. Let K_+ be the optimal $\mathcal{H}_2$ controller.
  - Let $\gamma \leftarrow (\gamma_- + \gamma_+)/2$. Check whether a controller exists such that $\|T_{zw}\|_{\mathcal{H}_\infty} < \gamma$.
  - If yes, set $\gamma_+ \leftarrow \gamma$, and set K_+ to the controller just designed. Otherwise, set $\gamma_- \leftarrow \gamma$.
  - Repeat from step 2 until $\gamma_+ - \gamma_- < \epsilon$.
  - Return K_+.



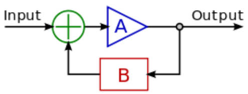
❑ $\mathcal{H}_\infty$-Mixed Sensitivity Approach

- Assume the the suboptimal problem has been solved, this might be too conservative in that for all admissible and stabilizing $K(s)$, the following inequality holds for all frequencies:

$$\left\| T_{zw}(s) \right\|_\infty \leq \gamma$$

- Mixed sensitivity design provides tool for incorporating control requirements in the form of <u>frequency response shaping</u> and to reduce the above conservatism. Recall:
  - Sensitivity: $S = (I+GK)^{-1}$
  - Complementary Sensitivity: $T = I - S$
  - Control Effort: KS

- We can introduce appropriate weighting matrices to represent performance objectives:

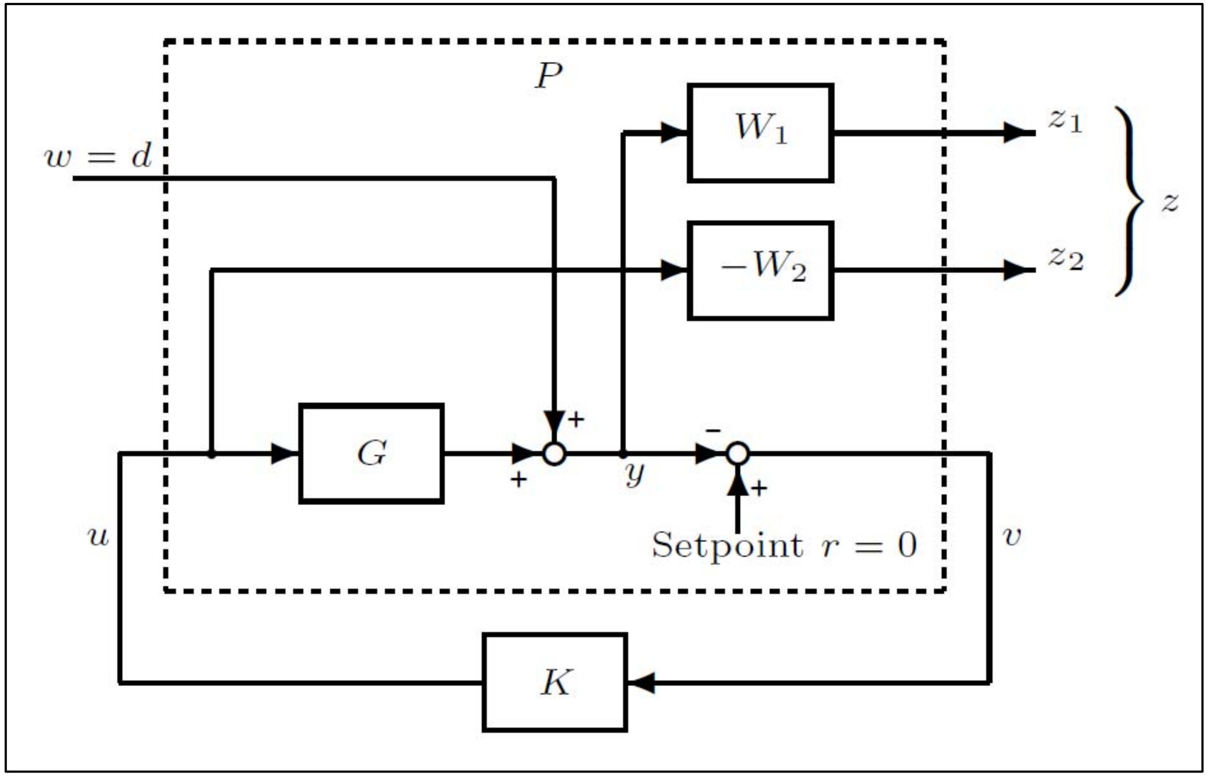


- **Example 1:** Design a regulator for disturbance rejection with limited control effort

To optimize performance, minimize $\|w_1 S\|_\infty$,

to minimize control inputs, minimize $\|w_2 K S\|_\infty$.

Compromise: $\left\| \begin{bmatrix} w_1 S \\ w_2 KS \end{bmatrix} \right\|_\infty \leq \gamma$



$$w_1(s) = W_1(s) w$$
$$w_2(s) = W_2(s) w$$
$$\|w\|_\infty \leq 1$$

$$\boxed{\begin{aligned} z_1 &= W_1 S w \\ z_2 &= W_2 K S w \end{aligned}}$$

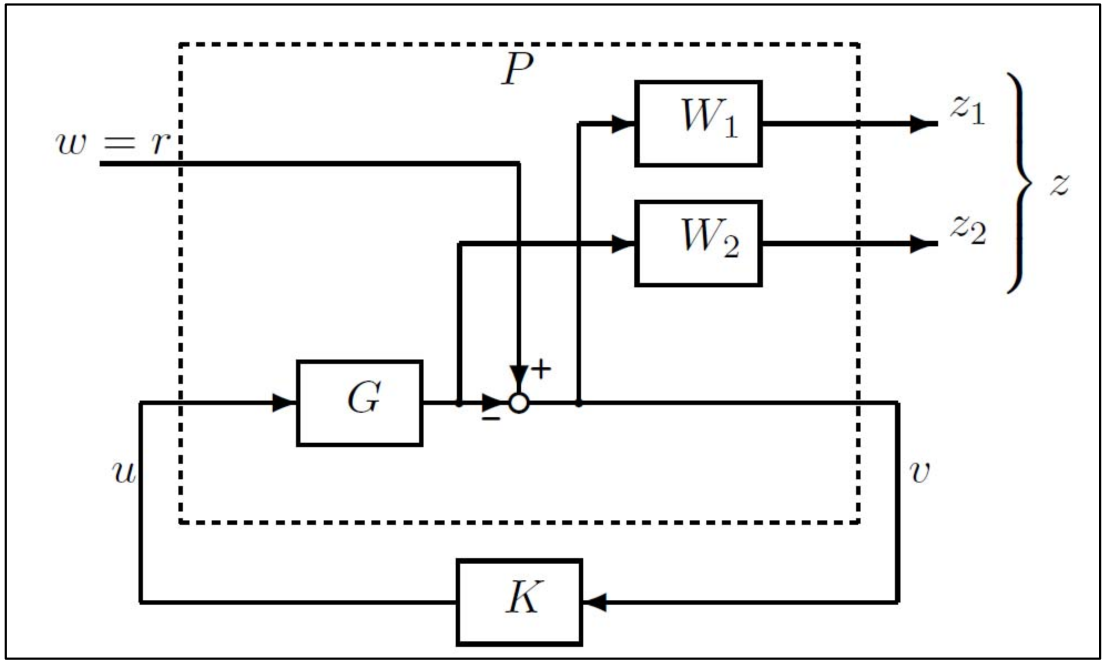
$$P_{11} = \begin{bmatrix} W_1 \\ 0 \end{bmatrix} \quad P_{12} = \begin{bmatrix} W_1 G \\ -W_2 \end{bmatrix}$$

$$P_{21} = -I \qquad P_{22} = -G$$

$$\begin{bmatrix} z_1 \\ z_2 \\ \hdashline v \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

$$F_l(P,K) = \begin{bmatrix} W_1 S \\ W_2 K S \end{bmatrix}$$

- **Example 2:** Design a tracking controller (servomechanism) with limited control effort



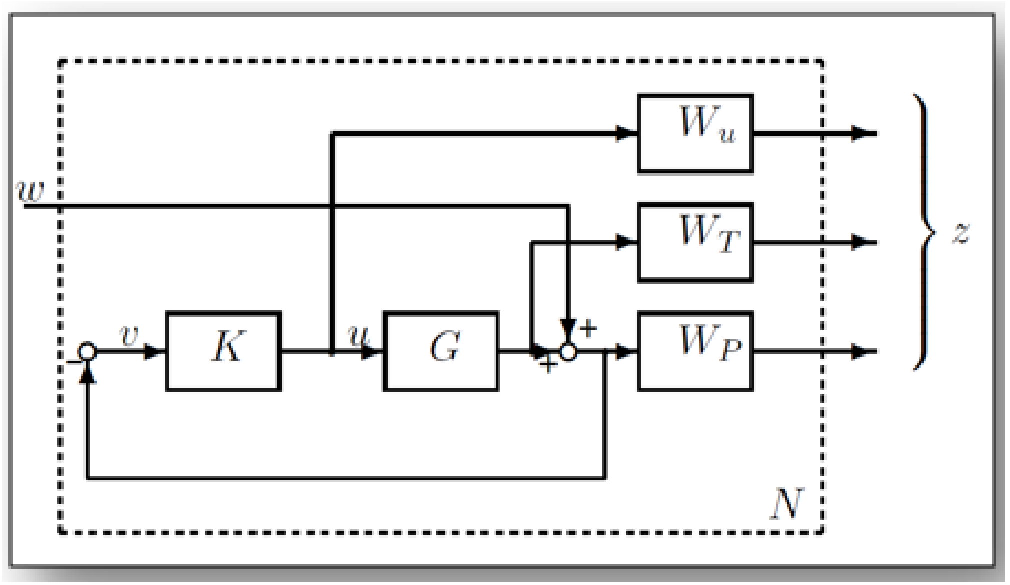
$$F_l(P,K) = \left\| \begin{bmatrix} W_1 S \\ W_2 T \end{bmatrix} \right\|_\infty$$

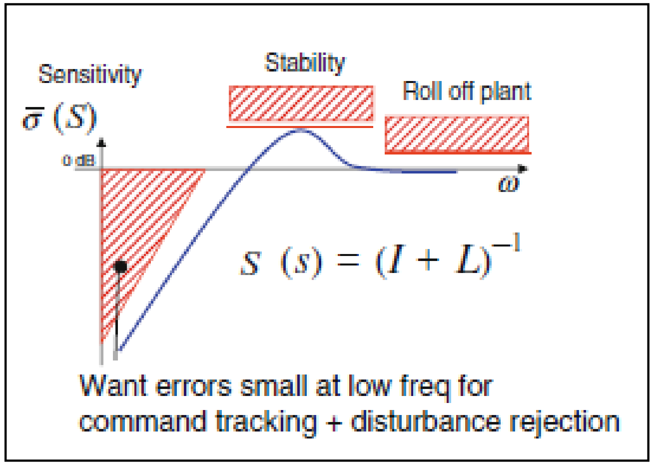
$$P_{11} = \begin{bmatrix} W_1 \\ 0 \end{bmatrix} \quad P_{12} = \begin{bmatrix} -W_1 G \\ W_2 G \end{bmatrix}$$

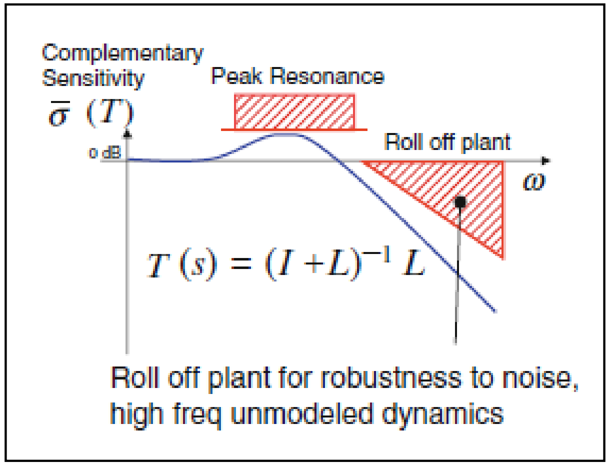
$$P_{21} = I \qquad P_{22} = -G$$

- **Example 3:** General stacked Mixed Sensitivity Problem

Consider an $\mathcal{H}_\infty$ problem where we want to bound $\bar{\sigma}(S)$ (for performance), $\bar{\sigma}(T)$ (for robustness and to avoid sensitivity to noise) and $\bar{\sigma}(KS)$ (to penalize large inputs). These requirements may be combined into a stacked $\mathcal{H}_\infty$ problem



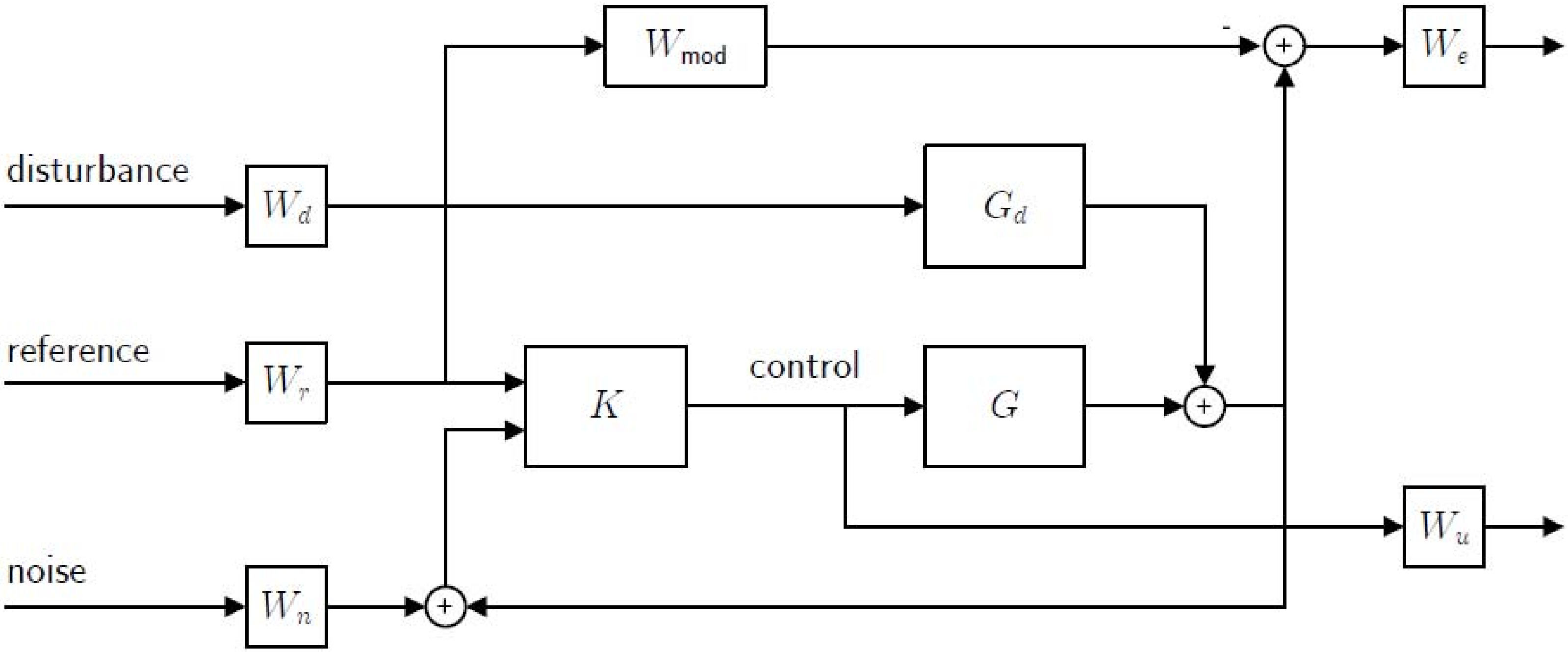




$$\min_K \|N(K)\|_\infty, \quad N = \begin{bmatrix} W_u K S \\ W_T T \\ W_P S \end{bmatrix}$$

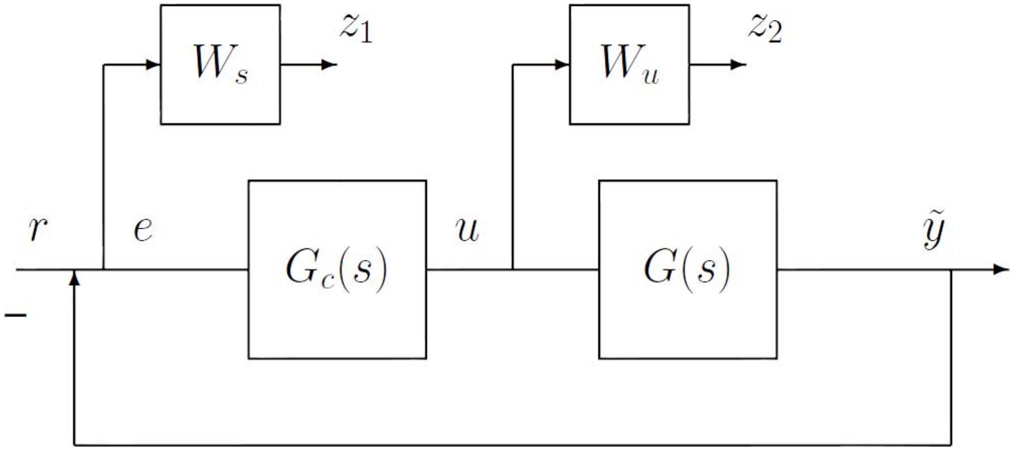
$$P = \left[\begin{array}{c|c} 0 & W_u I \\ 0 & W_T G \\ W_P I & W_P G \\ \hline -I & -G \end{array}\right]$$

The general structure



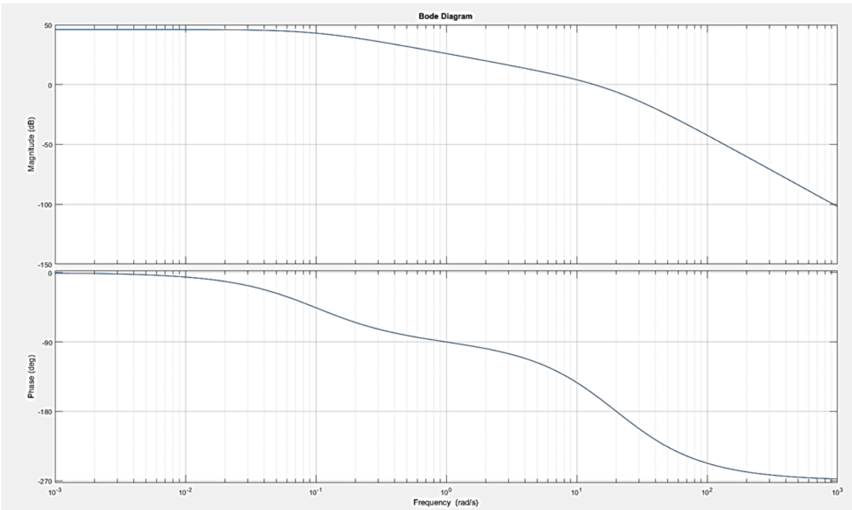
Filters W_d, W_r, W_n shape frequency contents of disturbance, reference, noise. The weight W_{mod} is an ideal desired model for the channel from reference to error, which should be matched over a frequency band defined by W_e. W_u specifies a bound on the control action over frequency.

❑ Example (Frazzoli 2007)



$$G = \frac{200}{(0.05s+1)^2(10s+1)}$$

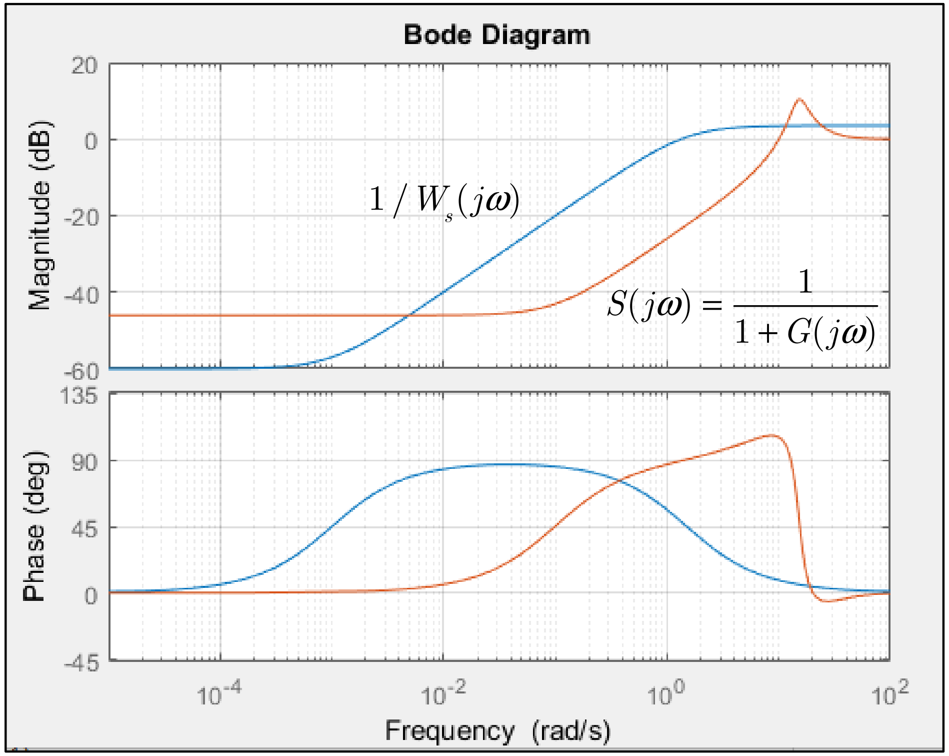
Note there is 1 input (r) and two performance outputs - one that penalizes the sensitivity $S(s)$ of the system, and the other that penalizes the control effort used.

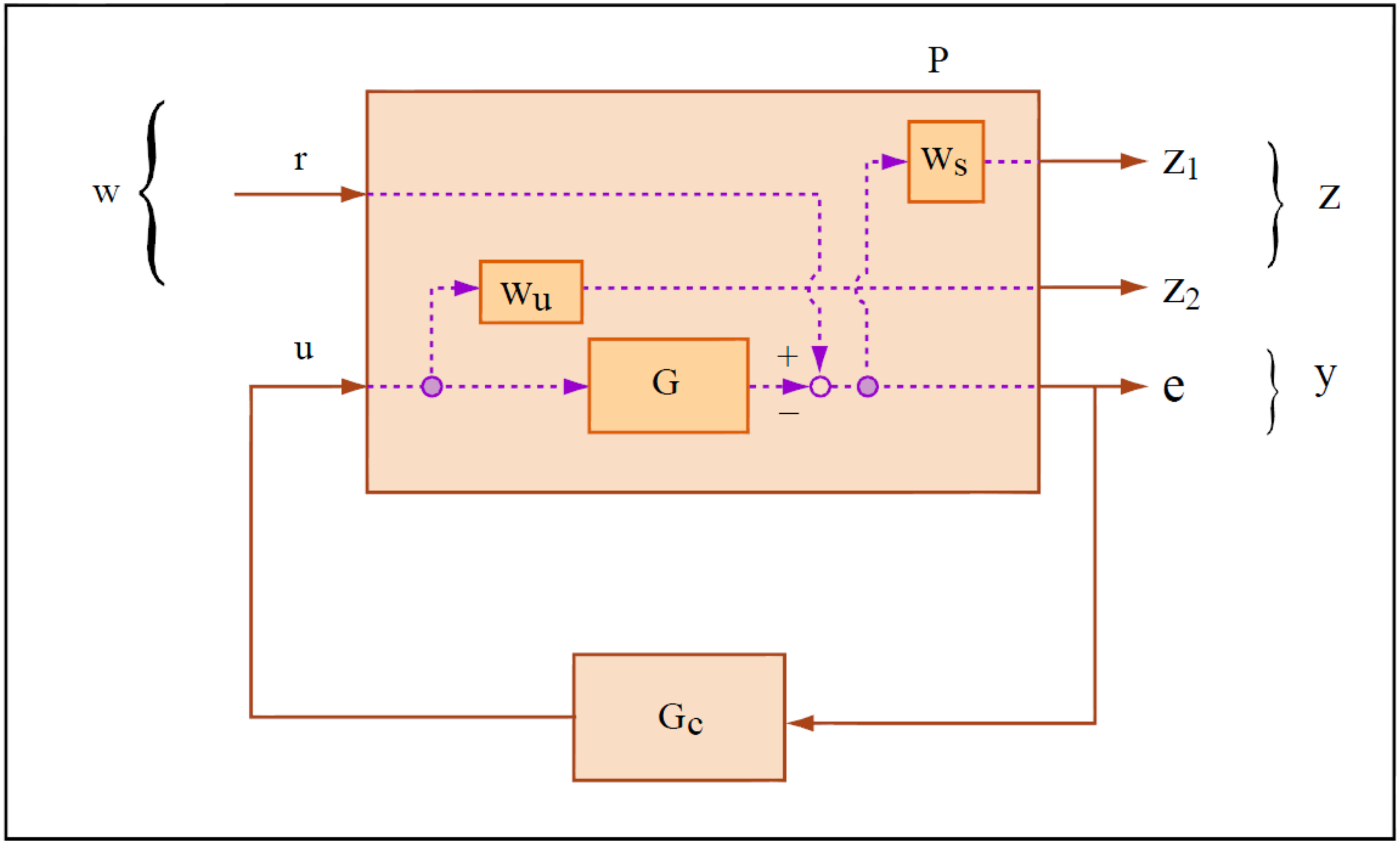


- Performance: low frequency tracking and a crossover frequency of about 10 rad/sec, limited control effort

$$W_s = \frac{s/1.5 + 10}{s + (10)\cdot(0.0001)} \qquad W_u = 1$$

Typically treat Wu as a constant (similar role as ρ as the LQR control control cost)





- The closed-loop transfer function matrix $T_{zw}(s)$ is is:

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} W_s S \\ W_u G_c S \end{bmatrix} r$$

The signal $\boldsymbol{r}$ acts as “exogenous input” $\boldsymbol{w}$ to the generalized system

$$
\begin{aligned}
z_1 &= W_s(s)(r - Gu) \\
z_2 &= W_u u \\
e &= r - Gu \\
u &= G_c e
\end{aligned}
\qquad
\begin{aligned}
P(s) &= \left[\begin{array}{c|c} W_s(s) & -W_s(s)G(s) \\ 0 & W_u(s) \\ \hline 1 & -G(s) \end{array}\right] \\
&= \begin{bmatrix} P_{\mathrm{zw}}(s) & P_{\mathrm{zu}}(s) \\ P_{\mathrm{yw}}(s) & P_{\mathrm{yu}}(s) \end{bmatrix}
\end{aligned}
$$

Using the lower linear fractional transformation:

$$
F_L\left[P(s), G_c(s)\right] = T_{zw}(\mathrm{s}) = \left\{P_{zw} + P_{zu} G_c (I\text{-}P_{yu} G_c)^{-1} P_{yw}\right\}
$$

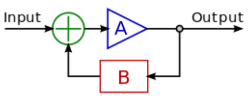
$$
\begin{aligned}
P_{CL} &= F_l(P, G_c) \\
&= \begin{bmatrix} W_s \\ 0 \end{bmatrix} + \begin{bmatrix} -W_s G \\ W_u \end{bmatrix} G_c (I + GG_c)^{-1} \\
&= \begin{bmatrix} W_s - W_s G G_c S \\ W_u G_c S \end{bmatrix} = \begin{bmatrix} W_s S \\ W_u G_c S \end{bmatrix}
\end{aligned}
$$

In state space form, let:

$$G(s) := \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \qquad W_s(s) := \left[\begin{array}{c|c} A_w & B_{\mathrm{w}} \\ \hline C_w & D_w \end{array}\right] \qquad W_u = 1$$

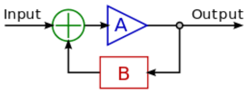
$$\begin{aligned}
\dot{x} &= Ax + Bu \\
\dot{x}_w &= A_w x_w + B_{\mathrm{w}} e = A_w x_w + B_{\mathrm{w}} r - B_{\mathrm{w}} C x \\
z_1 &= C_w x_w + D_w e = C_w x_w + D_w r - D_w C x \\
z_2 &= W_u u \\
e &= r - Cx
\end{aligned}$$

$$P(s) := \left[\begin{array}{cc|c|c} A & 0 & 0 & B \\ -B_{\mathrm{w}} C & A_w & B_{\mathrm{w}} & 0 \\ \hline -D_w C & C_w & D_w & 0 \\ 0 & 0 & 0 & W_u \\ \hline -C & 0 & 1 & 0 \end{array}\right]$$



Use Matlab to solve the problem:

```
Wu=1;
% define plant
[Ag,Bg,Cg,Dg]=tf2ss(200,conv(conv([0.05 1],[0.05 1]),[10 1]));
Gol=ss(Ag,Bg,Cg,Dg);
% define sensitivity weight
M=1.5;wB=10;A=1e-4;
[Asw,Bsw,Csw,Dsw]=tf2ss([1/M wB],[1 wB*A]);
Ws=ss(Asw,Bsw,Csw,Dsw);
% form augmented P dynamics
n1=size(Ag,1);
n2=size(Asw,1);
A=[Ag zeros(n1,n2);-Bsw*Cg Asw];
Bw=[zeros(n1,1);Bsw];
Bu=[Bg;zeros(n2,1)];
Cz=[-Dsw*Cg Csw;zeros(1,n1+n2)];
Cy=[-Cg zeros(1,n2)];
Dzw=[Dsw;0];
Dzu=[0;Wu];
Dyw=[1];
Dyu=0;
P=pck(A,[Bw Bu],[Cz;Cy],[Dzw Dzu;Dyw Dyu]);
```

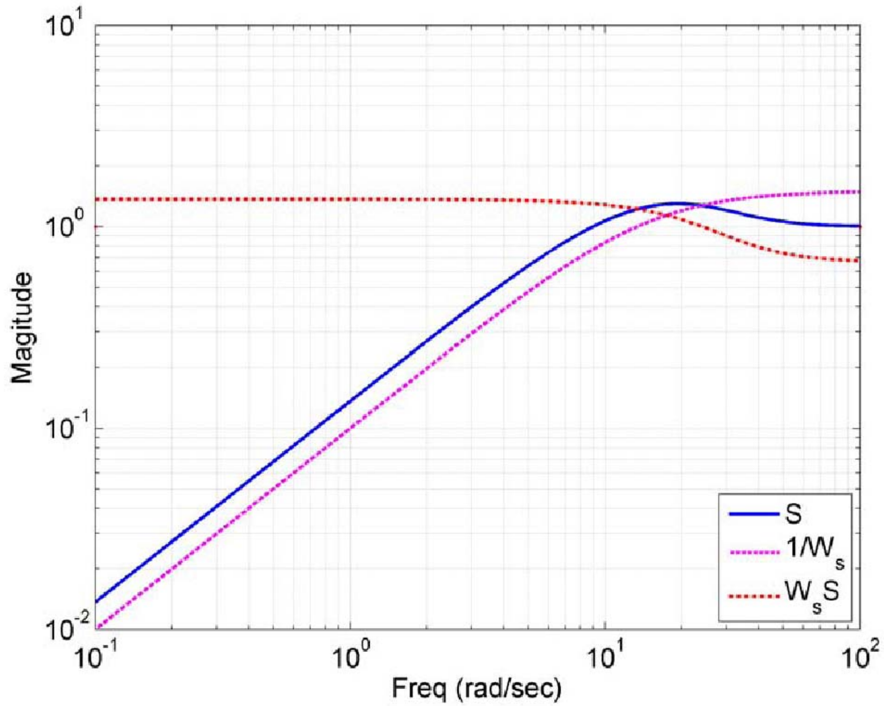


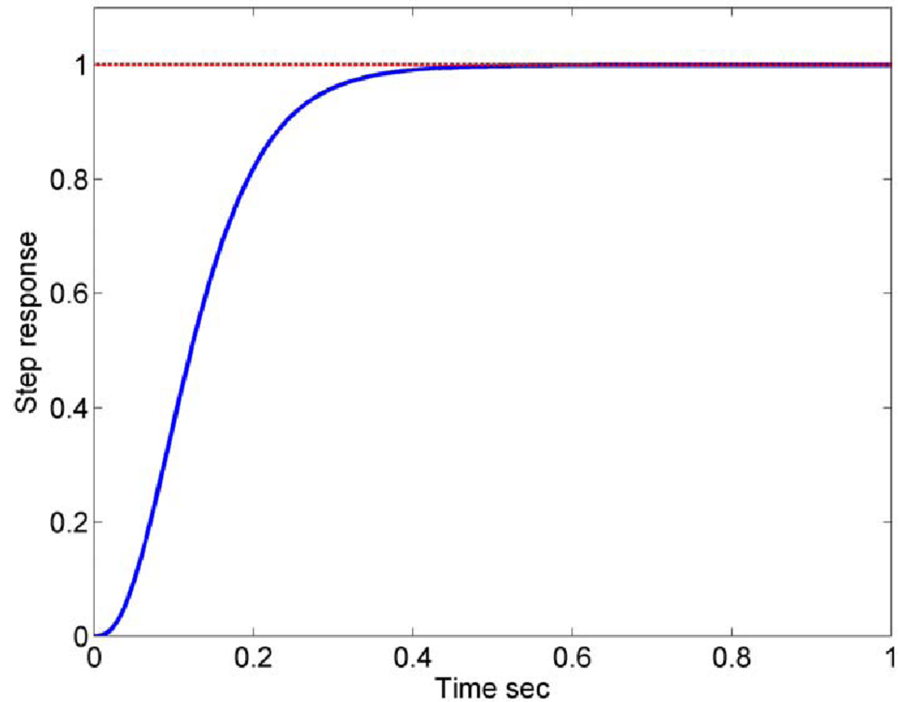

```
A=[Ag zeros(n1,n2);-Bsw*Cg Asw];
Bw=[zeros(n1,1);Bsw];
Bu=[Bg;zeros(n2,1)];
Cz=[-Dsw*Cg Csw;zeros(1,n1+n2)];
Cy=[-Cg zeros(1,n2)];
Dzw=[Dsw;0];
Dzu=[0;1];
Dyw=[1];
Dyu=0;
P=pck(A,[Bw Bu],[Cz;Cy],[Dzw Dzu;Dyw Dyu]);
% call hinf to find Gc (mu toolbox)
[Gc,G,gamma]=hinfsyn(P,1,1,0.1,20,.001);

[ac,bc,cc,dc]=unpck(Gc);
ev=max(real(eig(ac)/2/pi));
PP=ss(A,[Bw Bu],[Cz;Cy],[Dzw Dzu;Dyw Dyu]);
GGc=ss(ac,bc,cc,dc);
CLsys = feedback(PP,GGc,[2],[3],1);
[acl,bcl,ccl,dcl]=ssdata(CLsys);
% reduce closed-loop system so that it only has
% 1 input and 2 outputs
bcl=bcl(:,1);ccl=ccl([1 2],:);dcl=dcl([1 2],1);
CLsys=ss(acl,bcl,ccl,dcl);
f=logspace(-1,2,400);
Pcl=freqresp(CLsys,f);
CLWS=squeeze(Pcl(1,1,:)); % closed loop weighted sens
WS=freqresp(Ws,f); % sens weight
SensW=squeeze(WS(1,1,:));
Sens=CLWS./SensW; % divide out weight to get closed-loop sens
```

Results from the γ-iteration showing whether the various X, Y , $\rho(XY)$ tests are passed/failed during the bisection searching over γ, starting at the initial bound of 20.

```
Resetting value of Gamma min based on D_11, D_12, D_21 terms

Test bounds:      0.6667 <  gamma  <=     20.0000

    gamma    hamx_eig  xinf_eig  hamy_eig   yinf_eig   nrho_xy   p/f
    20.000   9.6e+000  6.2e-008  1.0e-003   0.0e+000   0.0000     p
    10.333   9.6e+000  6.3e-008  1.0e-003   0.0e+000   0.0000     p
     5.500   9.5e+000  6.3e-008  1.0e-003   0.0e+000   0.0000     p
     3.083   9.5e+000  6.5e-008  1.0e-003   0.0e+000   0.0000     p
     1.875   9.4e+000  6.9e-008  1.0e-003   0.0e+000   0.0000     p
>>   1.271   9.1e+000  -1.2e+004# 1.0e-003   -4.5e-010   0.0000    f
     1.573   9.3e+000  7.3e-008  1.0e-003   0.0e+000   0.0000     p
     1.422   9.2e+000  7.6e-008  1.0e-003   0.0e+000   0.0000     p
>>   1.346   9.2e+000  -6.4e+004# 1.0e-003   0.0e+000   0.0000    f
     1.384   9.2e+000  7.7e-008  1.0e-003   0.0e+000   0.0000     p
>>   1.365   9.2e+000  -1.9e+006# 1.0e-003   0.0e+000   0.0000    f
     1.375   9.2e+000  7.7e-008  1.0e-003   -4.5e-010   0.0000    p
     1.370   9.2e+000  7.7e-008  1.0e-003   0.0e+000   0.0000     p
     1.368   9.2e+000  7.7e-008  1.0e-003   0.0e+000   0.0000     p
     1.366   9.2e+000  7.7e-008  1.0e-003   0.0e+000   0.0000     p
>>   1.366   9.2e+000  -1.3e+007# 1.0e-003   0.0e+000   0.0000    f

 Gamma value achieved:     1.3664
```

- Since $\gamma_{\min} = 1.3664$, this indicates that the controller **does not** meet the desired goal of $|S| < 1/|W_s|$ (can only say that $|S| < 1.3664/|W_s|$).





Note that, even though this design fails the sensitivity weight, the controller still gets pretty good performance. For performance problems, we can think of the objective of getting $\gamma_{\min} < 1$ as a “design goal” ; it is “not crucial”. Use W_u to tune the control design

- **Design of a $\mathcal{H}_\infty$ controller follows the same sequence seen for the $\mathcal{H}_2$ problem:**
  1. Full – State Feedback Control
  2. Output Feedback with Observer – Based Estimator
- **As mentioned before, the theory behind control synthesis is based on:**
  1. Riccati Equations approach and Bounded Real Lemma (see Ch. 3)
  2. LMI Approach (followed in the robust control toolbox)

Theorem (Zhou, "Essentials of Robust Control"): The $\mathcal{H}_\infty$-norm of a stable **proper** system is bounded by a positive scalar $\gamma > 0$, that is:

$$\|G(s)\|_\infty < \gamma, G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

if and only if:

$$\sigma_{max}(D) < \gamma$$

and the 2nx2n Hamiltonian Matrix $\mathcal{H}$ has no eigenvalues on the imaginary axis.

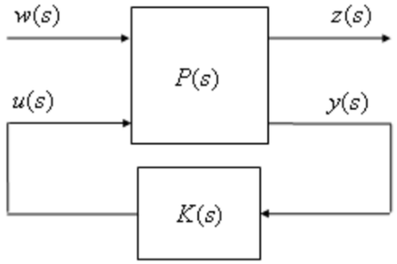
$$\begin{cases} \mathcal{H} := \left[\begin{array}{c|c} A + BR^{-1}D^T C & BR^{-1}B^T \\ \hline -C^T(I + DR^{-1}D^T)C & -A^T - C^T DR^{-1}B^T \end{array}\right] \\ R = (\gamma^2 I - D^T D) \end{cases}$$

- Assume the same Hypotheses required for $\mathcal{H}_2$ Control

$$\boxed{\text{(A5)} \;\; D_{11} = 0 \text{ and } D_{22} = 0.}$$

- These assumptions are more restrictive for $\mathcal{H}_\infty$ than for $\mathcal{H}_2$.

$$\begin{aligned}\dot{\boldsymbol{x}} &= A\boldsymbol{x} + B_1\boldsymbol{w} + B_2\boldsymbol{u}\\ \boldsymbol{z} &= C_1\boldsymbol{x} + D_{11}\boldsymbol{w} + D_{12}\boldsymbol{u}\\ \boldsymbol{y} &= C_2\boldsymbol{x} + D_{21}\boldsymbol{w} + D_{22}\boldsymbol{u}\end{aligned}$$



$$P(s) = \begin{bmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{bmatrix}$$

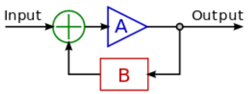
❑ $\mathcal{H}_\infty$-(sub) optimal state-feedback problem

- Consider the system **(5.1)** below.

$$\begin{aligned}\dot{\boldsymbol{x}} &= A\boldsymbol{x} + B_1\boldsymbol{w} + B_2\boldsymbol{u}\\ \boldsymbol{z} &= C_1\boldsymbol{x} + D_{12}\boldsymbol{u} \qquad \textbf{(5.1)}\\ \boldsymbol{y} &= C_2\boldsymbol{x} + D_{21}\boldsymbol{w}\end{aligned}$$

$$J_\infty(K) < \gamma := \left\| T_{zw}(s) \right\|_\infty := \sup_{\omega \in \Re} \sigma_{\max}\left[T_{zw}(j\omega) \right] < \gamma \qquad \textbf{(5.2)}$$

- Suppose that the assumptions (A1) - (A4) hold. Assume that the control signal $\boldsymbol{u}(t)$ has access to the present and past values of the state (no estimator needed).
- Then there exists a state-feedback controller such that the inequality **(5.2)** holds for all $\boldsymbol{w}$, <u>if and only if</u> there exists a positive (semi)definite solution to the algebraic Riccati equation:



$$A^T X + XA - XB_2(D_{12}^T D_{12})^{-1} B_2^T X + \gamma^{-2} XB_1 B_1^T X + C_1^T C_1 = 0$$

- Such that the matrix **(5.3)** is a stability matrix (All eigenvalues have negative real part)

$$A - B_2(D_{12}^T D_{12})^{-1} B_2^T X + \gamma^{-2} B_1 B_1^T X \qquad \textbf{(5.3)}$$

- When the above conditions hold, a controller which satisfies **(5.2)** is given by:

$$\boldsymbol{u}(t) = K_\infty \boldsymbol{x}(t)$$
$$K_\infty = -(D_{12}^T D_{12})^{-1} X$$

- The proof follows the lines used for the similar $\mathcal{H}_2$ state feedback problem

❑ $\mathcal{H}_\infty$-(sub) optimal estimation problem

- Consider the system:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B_1\boldsymbol{w} + B_2\boldsymbol{u}$$
$$\boldsymbol{z} = C_1\boldsymbol{x}$$
$$\boldsymbol{y} = C_2\boldsymbol{x} + D_{21}\boldsymbol{w}$$

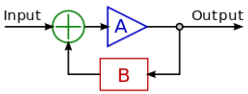
$$\boxed{\left\|T_{zw}(s)\right\|_\infty = \sup_{w(t)\neq 0} \frac{\left\|\boldsymbol{z}(t)\right\|_2}{\left\|\boldsymbol{w}(t)\right\|_2} < \gamma}$$

- Recall the equivalence to the worst-case induced 2 norm of input output signals, and the fact that the regulated output is now an estimated vector $\hat{\boldsymbol{z}}(t)$

- Find a causal stable estimator $F(s)$ of the output $\boldsymbol{z}$ based on the measured output $\boldsymbol{y}$ such that $\hat{z}(s) = F(s)y(s)$ and the following cost is minimized (suboptimal):

$$J_{e,\infty}(F) = \sup\left\{\frac{\left\|\boldsymbol{z} - \hat{\boldsymbol{z}}\right\|_2}{\left\|\boldsymbol{w}\right\|_2} : w \neq 0\right\} < \gamma \qquad \textbf{(5.4)}$$

- Suppose that the assumptions (A1) - (A4) hold. Then there exists a stable estimator $F(s)$ which achieves the bound **(5.4)** <u>if and only if</u> there exists a positive (semi)definite solution to the algebraic Riccati equation:



$$AY + YA^T - YC_2^T(D_{21}D_{21}^T)^{-1}C_2Y + \gamma^{-2}YC_1^TC_1Y + B_1B_1^T = 0$$

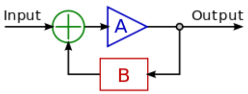
- Such that the matrix **(5.5)** is a stability matrix (All eigenvalues have negative real part)

$$A - YC_2^T(D_{21}D_{21}^T)^{-1}C_2 + \gamma^{-2}YC_1^TC_1 \quad \textbf{(5.5)}$$

- Moreover, when these conditions are satisfied, an estimator which achieves the bound **(5.4)** is given by:

$$\dot{\hat{x}} = A\hat{x} + L_\infty[y - C_2\hat{x}], \hat{x}(0) = 0$$
$$\hat{z} = C_1\hat{x}$$
$$L_\infty = YC_2^T(D_{21}D_{21}^T)^{-1}$$

- **Note:** The $\mathcal{H}_\infty$-optimal estimator has a structure similar to the $\mathcal{H}_2$-optimal estimator, but, in contrast to it, it also depends on the output z to be estimated.



❑ $\mathcal{H}_\infty$-(sub) optimal output-feedback problem

- Consider the system **(5.1)**. Suppose that the assumptions (A1) - (A8) hold.

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{12} u \qquad\qquad \textbf{(5.1)} \\ y &= C_2 x + D_{21} w \end{aligned}$$

- Then there exists a controller $\boldsymbol{u} = K\boldsymbol{y}$, which achieves the $\mathcal{H}_\infty$-norm bound (or equivalently the inequality below):

$$J_\infty(K) < \gamma := \left\| T_{zw}(s) \right\|_\infty := \sup_{\omega \in \Re} \sigma_{\max}\left[T_{zw}(j\omega) \right] < \gamma$$

$$:= \left\| F_l(P,K) \right\|_\infty = \max_{\boldsymbol{w}(t) \neq 0} \frac{\left\| \boldsymbol{z}(t) \right\|_2}{\left\| \boldsymbol{w}(t) \right\|_2} < \gamma$$

- <u>if and only if</u> the following conditions are satisfied:

a) There exists a symmetric positive definite or semidefinite solution $\boldsymbol{X}$ to the Riccati equation

$$A^T X + XA - XB_2(D_{12}^T D_{12})^{-1} B_2^T X + \gamma^{-2} XB_1 B_1^T X + C_1^T C_1 = 0$$

such that the matrix below in is stable (all eigenvalues with negative real part).

$$A - B_2(D_{12}^T D_{12})^{-1} B_2^T X + \gamma^{-2} B_1 B_1^T X$$

b. There exists a symmetric positive definite or semidefinite solution $\boldsymbol{Y}$ to the Riccati equation:

$$AY + YA^T - YC_2^T(D_{21}D_{21}^T)^{-1} C_2 Y + \gamma^{-2} YC_1^T C_1 Y + B_1 B_1^T = 0$$

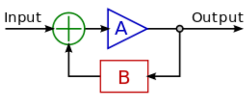
such that the matrix:

$$A - YC_2^T(D_{21}D_{21}^T)^{-1} C_2 + \gamma^{-2} YC_1^T C_1$$

is stable.

c. $\rho(XY) = \max_i \left|\lambda_i(XY)\right| < \gamma^2.$ $\qquad \rho(A) = \max\{|\lambda_1|, \ldots, |\lambda_n|\}.$

- Spectral radius inequality is required to satisfy BRL conditions



d. When these conditions are satisfied, such a controller is:

$$K_{sub}(s) = \left(\begin{array}{c|c} A_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right)$$

where:

$$A_\infty = A + \gamma^{-2} B_1 B_1^T X + B_2 (D_{12}^T D_{12})^{-1} F_\infty - Z_\infty L_\infty C_2$$

$$\begin{aligned} F_\infty &= -(D_{12}^T D_{12})^{-1} B_2^T X \\ L_\infty &= -Y C_2^T (\tilde{D}_{21} \tilde{D}_{21}^T)^{-1} \\ Z_\infty &= (I - \gamma^{-2} XY)^{-1} \end{aligned}$$

Note: It is possible to write the compensator in a more standard controller – observer format:

- **Observer**

$$\dot{\hat{\boldsymbol{x}}} = A\hat{\boldsymbol{x}} + B_1 \gamma^{-2} B_1^T X \hat{\boldsymbol{x}} + B_2 (D_{12}^T D_{12})^{-1} \boldsymbol{u} + Z_\infty L_\infty (C_2 \hat{\boldsymbol{x}} - \boldsymbol{y})$$

- The term $\hat{\boldsymbol{w}}_{worst} = B_1\gamma^{-2}B_1^T X\hat{\boldsymbol{x}}$ can be interpreted as an estimate of the worst disturbance.

- **Controller:** $\boldsymbol{u} = F_\infty \hat{\boldsymbol{x}}$

- **Note:** we can rewrite the suboptimal compensator:

$$K_{sub}(s) = \left(\begin{array}{c|c} A_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right) \Rightarrow K_{sub}(s) = -F_\infty (sI - A_\infty)^{-1} Z_\infty L_\infty$$

❑ Outline of Proof:

1. Mackenroth: "Robust Control Systems", Springer. 2004, pp. 252 – 258.
2. Skogestad, Postlethwaite: "Multivariable Feedback Control", Wiley, 2005, pp. 357 – 380.

❑ **Theorem:** The $\mathcal{H}_\infty$-norm of a stable proper system (A, B, C, D) is bounded by a positive scalar $\gamma > 0$ that is:

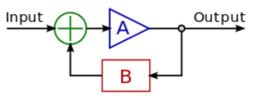
$$\left\|G(s)\right\|_\infty < \gamma, G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

if and only if:

$$\sigma_{\max}(D) < \gamma$$

$$\begin{cases} \mathcal{H} := \left[\begin{array}{c|c} A + BR^{-1}D^TC & BR^{-1}B^T \\ \hline -C^T(I + DR^{-1}D^T)C & -A^T - C^TDR^{-1}B^T \end{array}\right] \\ R = (\gamma^2 I - D^T D) \end{cases}$$

and $\mathcal{H}$ has no eigenvalues on the imaginary axis.



- Conditions on the existence of the solution of the Riccati equations in the $\mathcal{H}_\infty$-(sub) optimal output-feedback problem

$$A^T X + XA - XB_2(D_{12}^T D_{12})^{-1} B_2^T X + \gamma^{-2} XB_1 B_1^T X + C_1^T C_1 = 0$$

$$H = \left(\begin{array}{c|c} A & \gamma^{-2} B_1 B_1^T - B_2 B_2^T \\ \hline -C_1^T C_1 & -A^T \end{array}\right)$$

$$H \in dom(Ric), X = Ric(H) \geq 0$$

$$AY + YA^T - YC_2^T (D_{21} D_{21}^T)^{-1} C_2 Y + \gamma^{-2} YC_1^T C_1 Y + B_1 B_1^T = 0$$

$$J = \left(\begin{array}{c|c} A^T & \gamma^{-2} C_1^T C_1 - C_2^T C_2 \\ \hline -B_1 B_1^T & -A \end{array}\right)$$

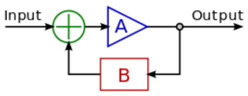
$$J \in dom(Ric), Y = Ric(J) \geq 0$$

- The actual computation of an $\mathcal{H}_\infty$ controller is performed via γ - iteration

  Hinf Computation using Matlab   hinfsyn

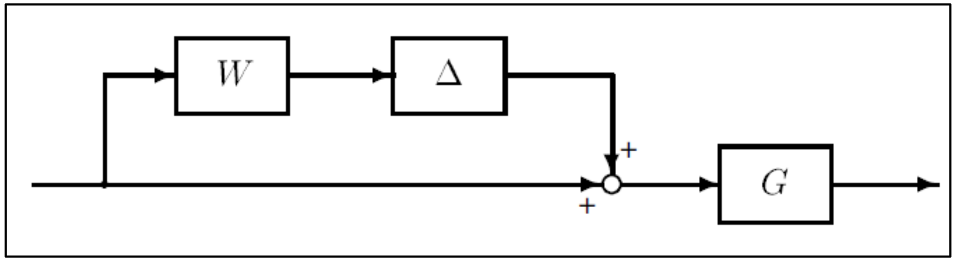
- **Concluding Remarks:**

  - A general $\mathcal{H}_\infty$ output feedback controller is a dynamic compensator with size equal to that of the original system (<u>including weights</u>).
  - The design can be performed as a mixed – sensitivity or a loop shaping approach.
  - A control system is robust if it remains stable and achieves certain performance criteria in the presence of possible. The robust design is to find a controller, for a given system, such that the closed-loop system is robust.
  - The $\mathcal{H}_\infty$ optimization approach and its related, have been shown to be effective and efficient robust design methods for linear, time-invariant control systems.
  - In this context, the Small Gain Theorem plays an important role in the $\mathcal{H}_\infty$ optimization methods.
  - As detailed in the next chapter, The $\mathcal{H}_\infty$ optimization usually yields a controller capable of providing nominal stability, nominal performance, and robust stability. Robust performance may or may not satisfied by an $\mathcal{H}_\infty$ controller.



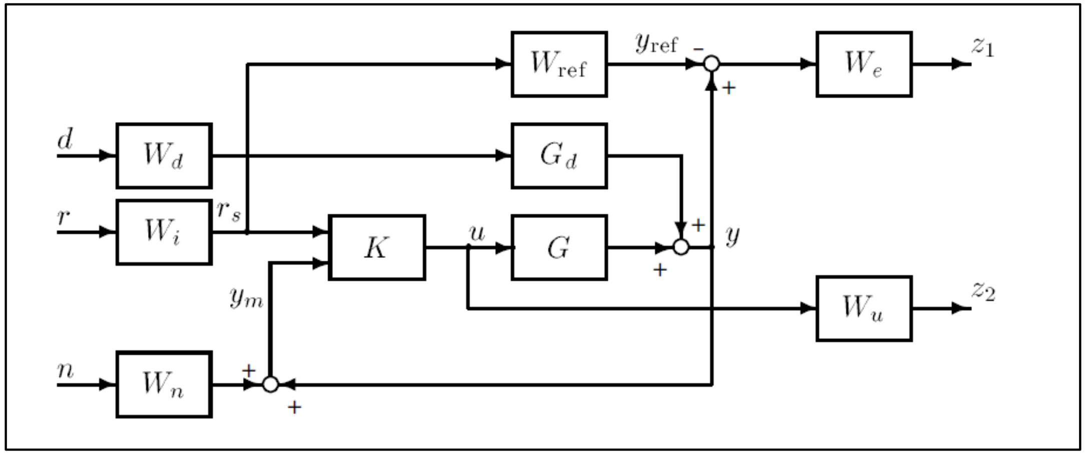
❑ $\mathcal{H}_\infty$ Signal – based Control

- The signal-based approach to $\mathcal{H}_\infty$ controller design is very general and is appropriate for multivariable problems in which several objectives must be taken into account simultaneously.
- In this approach, we define the plant and possibly the model uncertainty, we define the class of external signals affecting the system and we define the norm of the error signals we want to keep small.
- The focus of attention has moved to the size of signals and away from the size and bandwidth of selected closed-loop transfer functions.
- Weights are used to describe the expected or known frequency content of exogenous signals and the desired frequency content of error signals. Weights are also used if a perturbation is used to model uncertainty, as the example below:



Where: $\Delta(s) = W(s) \cdot \Delta,\ \|\Delta\|_\infty < 1$

Note: LQG control is a simple example of the signal-based approach, in which the exogenous signals are assumed to be stochastic (or alternatively impulses in a deterministic setting) and the error signals are measured in terms of the 2-norm. The weights Q and R are constant, but LQG can be generalized to include frequency dependent weights on the signals leading to what is called $\mathcal{H}_2$ control.

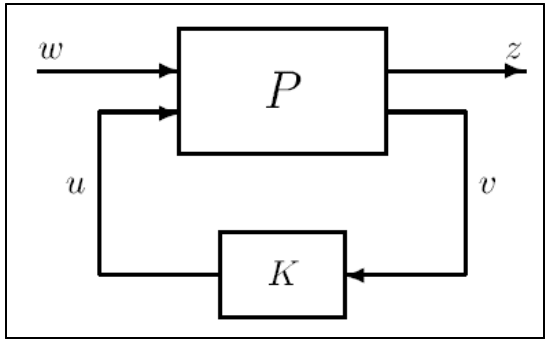


- G and Gd are nominal models of the plant and disturbance dynamics, and K is the controller to be designed.
- The weights Wd, Wi and Wn describe the relative importance and/or frequency content of the disturbances, set points, and noise signals.
- The weight Wref is a desired closed-loop transfer function between the weighted set point rs and the actual output y.
- The weights We and Wu reflect the desired frequency content of the error (y - yref) and the control signals u, respectively.

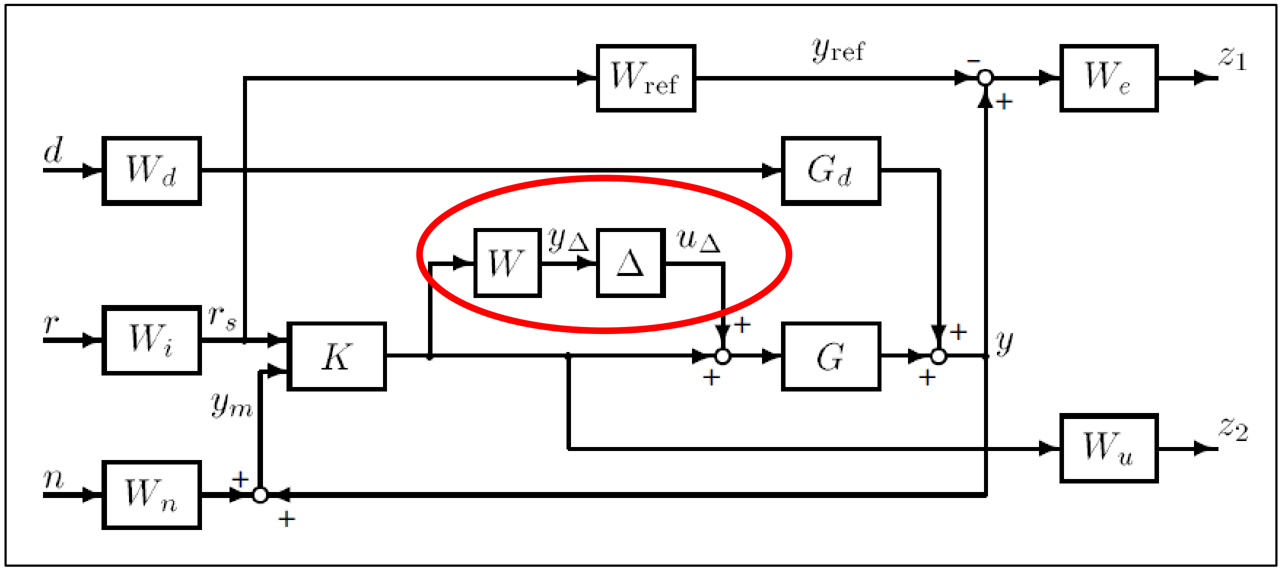
- The problem can be cast as a standard $\mathcal{H}_\infty$ optimization in the general control configuration especially if there is uncertainty involved by defining

$$w = \begin{bmatrix} d \\ r \\ n \end{bmatrix} \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$v = \begin{bmatrix} r_s \\ y_m \end{bmatrix} \quad u = u$$



- Structure in the presence of an input multiplicative uncertainty:



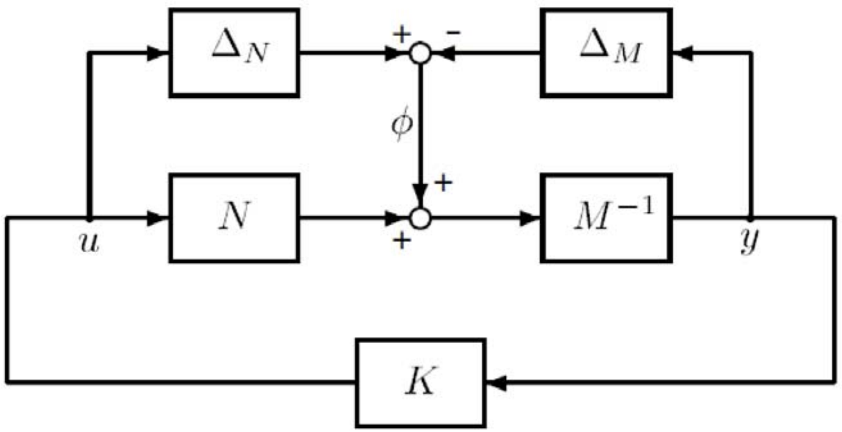
❑ $\mathcal{H}_\infty$ Loop Shaping Control

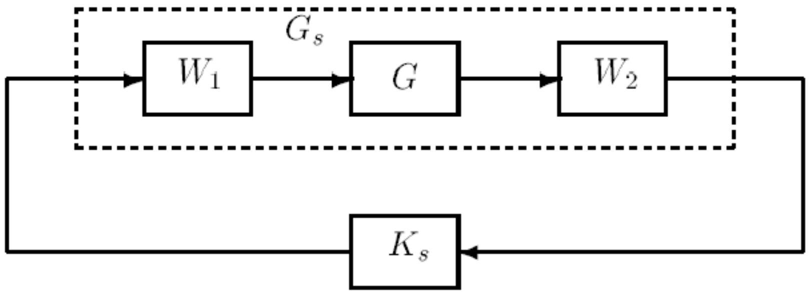
- The loop-shaping design procedure is based on $\mathcal{H}_\infty$ robust stabilization combined with classical loop shaping, as proposed by McFarlane and Glover (1990). It is essentially a two stage design process.
- First, the open-loop plant is augmented by pre and post-compensators to give a desired shape to the singular values of the open-loop frequency response.
- Then the resulting shaped plant is robustly stabilized with respect to coprime factor uncertainty using $\mathcal{H}_\infty$ optimization. An important advantage is that no problem-dependent uncertainty modelling, or weight selection, is required in this second step.

- **Step 1:** Define the desired loop transfer matrix frequency response:

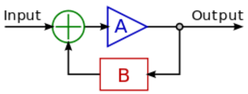
$$G = M^{-1}N$$

$$G_p = (M + \Delta_M)^{-1}(N + \Delta_N)$$





$$G_S(s) = W_2(s)G(s)W_1(s)$$



- **Step 2:** Synthesize the controller (for instance see Skogestad pp. 364 – 368.

$$
\begin{aligned}
K &\stackrel{s}{=} \left[\begin{array}{c|c} A + BF + \gamma^2 (L^T)^{-1} Z C^T (C + DF) & \gamma^2 (L^T)^{-1} Z C^T \\ \hline B^T X & -D^T \end{array}\right] \\
F &= -S^{-1}(D^T C + B^T X) \\
L &= (1 - \gamma^2) I + XZ.
\end{aligned}
$$

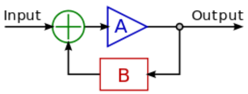
- **Step 3:** Incorporate the controller in the closed loop system

$$K_{final}(s) = W_1(s) K(s) W_2(s)$$

$$K_s(0) W_2(0) = \lim_{s \to 0} K_s(s) W_2(s)$$

- The MATLAB function ***coprimeunc***, can be used to generate the controller. It is important to emphasize that since we can compute γ_{MIN} from maximum stability margins Glover, McFarlane (1989) we get an explicit solution by solving just the two Riccati equations (aresolv) for the general central problem, without performing γ – iteration.

- A similar procedure can be found in the reference text: Gu, Petkov, Konstantinov, "Robust Control Design with MATLAB® , Second Edition, Springer 2013, Chapter 11.1. using the command *loopsyn*.



- **Question:** Can we design an $\mathcal{H}_\infty$ controller with a specific structure that could resemble a known structure?
  - PID, observer –based etc…

- Standard controller structure

$$\begin{aligned}\dot{\boldsymbol{x}} &= A\boldsymbol{x} + B_1\boldsymbol{w} + B_2\boldsymbol{u}\\ \boldsymbol{z} &= C_1\boldsymbol{x} + D_{11}\boldsymbol{w} + D_{12}\boldsymbol{u}\\ \boldsymbol{y} &= C_2\boldsymbol{x} + D_{21}\boldsymbol{w} + D_{22}\boldsymbol{u}\end{aligned}$$

$$K_{H\inf}(s) = \left(\begin{array}{c|c} A_\infty & B_\infty \\ \hline C_\infty & D_\infty \end{array}\right) \Rightarrow \begin{cases}\dot{\boldsymbol{q}} = A_\infty \boldsymbol{q} + B_\infty \boldsymbol{y} \\ \boldsymbol{u} = C_\infty \boldsymbol{q} + D_\infty \boldsymbol{y}\end{cases}; \boldsymbol{q} \in \Re^n$$

- Enforce the controller to have a specific structure given by

$$K_{STR}(s) = \left(\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right)$$

- Solve an $\mathcal{H}_\infty$ controller problem assuming a given $\boldsymbol{K_{STR}}$.

- $\mathcal{H}_\infty$ structured controller design:

$$F_L\left[P(s),K_{STR}(s)\right]=T_{zw}(\mathrm{s})=\left\{P_{zw}+P_{zu}K_{STR}(I\text{-}P_{yu}K_{STR})^{-1}P_{yw}\right\}$$

- Find the optimal controller $K^{*}_{STR}(s)=\left(\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right)$ such that

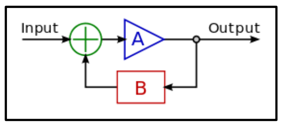
$$\left\|F_L\left[P(s),K^{*}_{STR}(s,\mathscr{k})\right]\right\|_\infty \leq \left\|F_L\left[P(s),K_{STR}(s,\mathscr{k})\right]\right\|_\infty$$

- The set of parameters $\mathscr{k}$ used in the optimization depends on the type of structure used.

- **Example:** select the controller to be a PID.

$$K(s)=k_p+\frac{k_i}{s}+\frac{k_d s}{1+T_f s}=d_K+\frac{r_i}{s}+\frac{r_d}{s+\tau}$$

$$d_K=k_p+k_d/T_f,\ \tau=1/T_f,\ r_i=k_i,\ r_d=-k_d T_f^2$$



- In state space form

$$K_{STR}^{PID}(k) = \left[\begin{array}{cc|c} 0 & 0 & r_i \\ 0 & -\tau & r_d \\ \hline 1 & 1 & d_k \end{array}\right]; k = [r_i, r_d, \tau, d_k] \qquad \left\| F_L\left[P(s), K^{*PID}_{STR}(s,k)\right]\right\|_\infty \le \left\| F_L\left[P(s), K^{PID}_{STR}(s,k)\right]\right\|_\infty$$

- The algorithmic procedure is a non-convex non-smooth optimization problem [1,2] implemented in Matlab with the commands: hinfstruct, and systune.

https://it.mathworks.com/help/robust/gs/fixed-structure-h-infinity-synthesis-with-hinfstruct.html

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 51, NO. 1, JANUARY 2006

Nonsmooth H_∞ Synthesis

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Conference Paper *in* IFAC Proceedings Volumes (IFAC Papers-OnLine) · August 2011

DOI: 10.3182/20110828-6-IT-1002.00708

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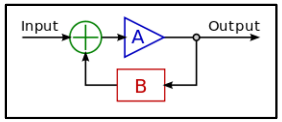
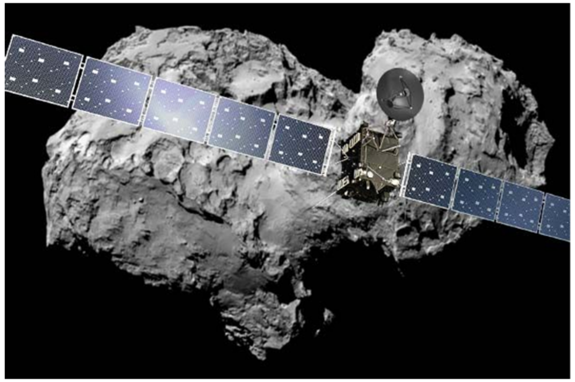
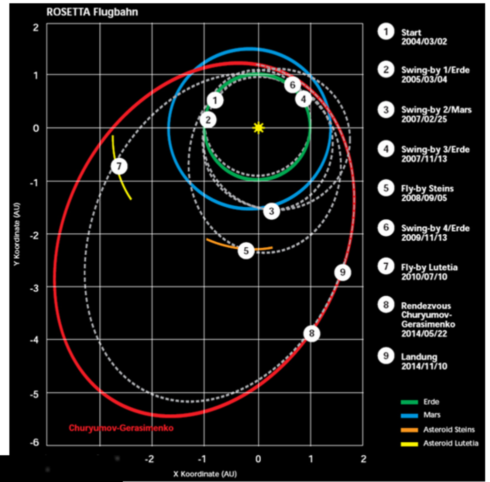
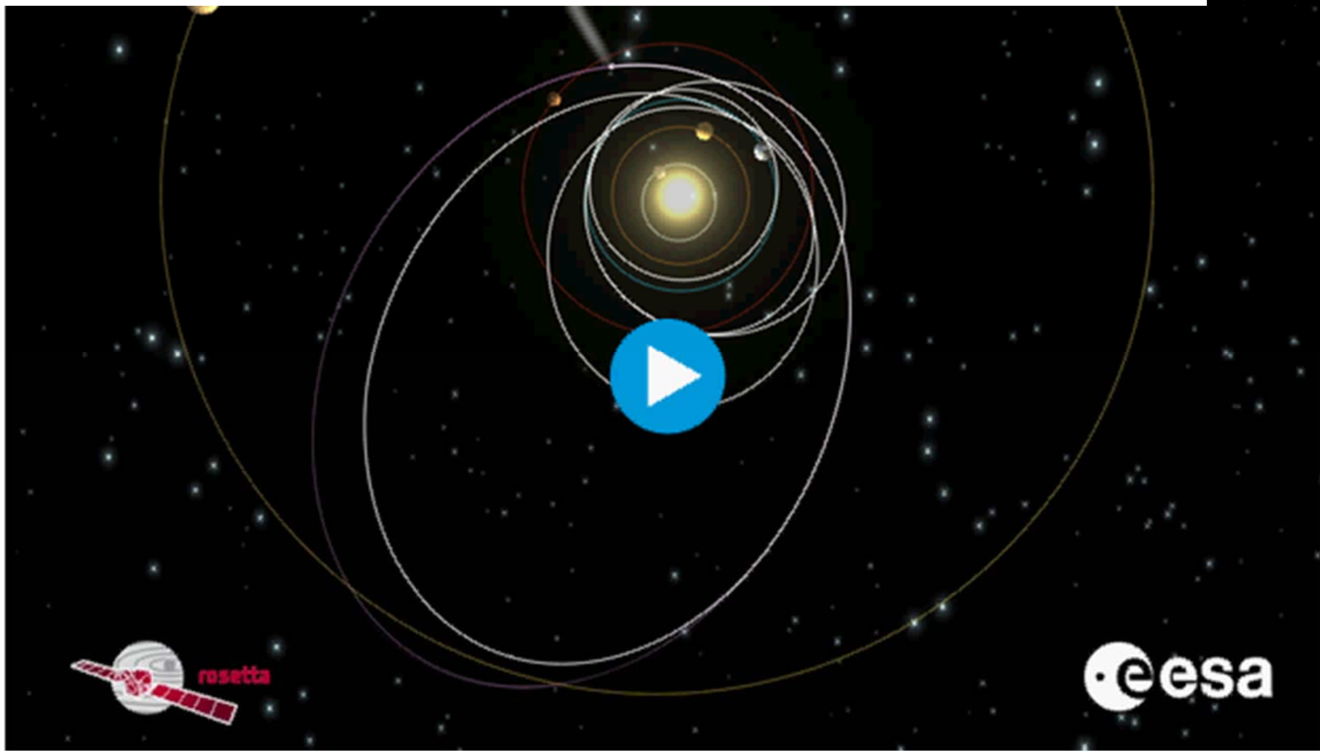
Navarro-Tapia, D., Marcos, A., Simplicio, P., Bennani, S., & Roux, C. (2019). Legacy recovery and robust augmentation structured design for the VEGA launcher. *International Journal of Robust and Nonlinear Control*, *29*(11), 3363-3388. https://doi.org/10.1002/rnc.4557



HOW WE SAVED THE ROSETTA MISSION





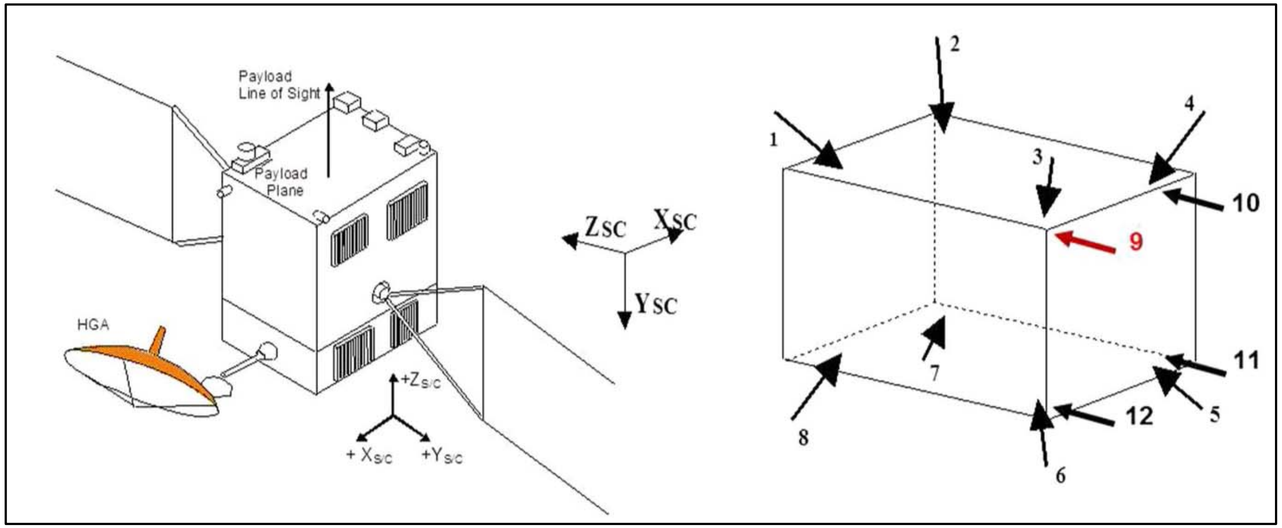


- Sequence of Events:

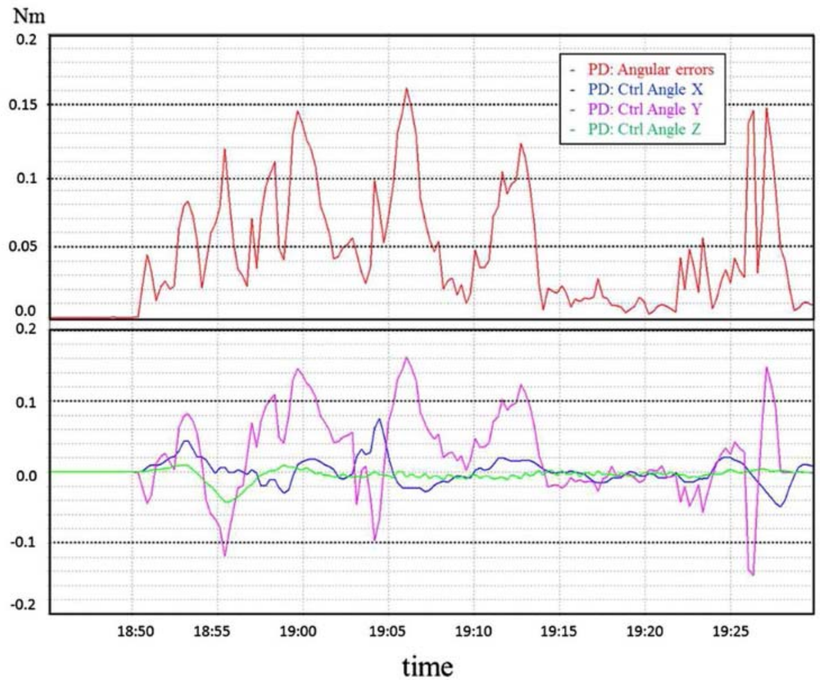
  - After a large set of deep-space manoeuvres, the spacecraft entered into hibernation between June 2011 and January 2014

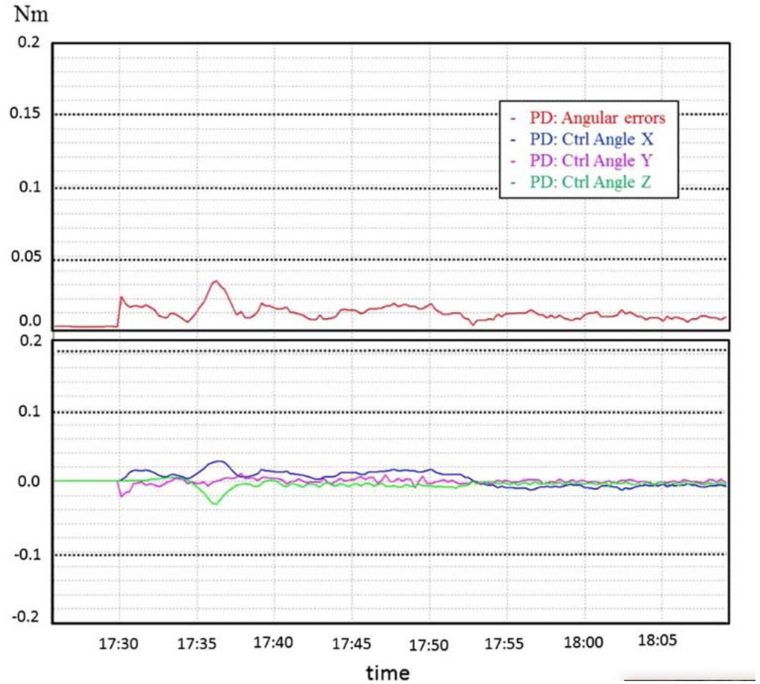
  - Just before this 31-month deep space hibernation mode, a rendezvous manoeuvre led to an unexpected satellite off-pointing about the Y-axis spacecraft axis triggering attitude off-pointing FDIR (Fault Detection Isolation and Recovery) and leading the space probe in safe mode.

  - In-depth on-ground analysis of the anomaly revealed a Loss-Of-Efficiency (LOS) of one thruster [7.4% (1σ) on Thruster 9] leading to strong parasite torques which could not be compensated by the initially designed orbit controller (AIRBUS Design).



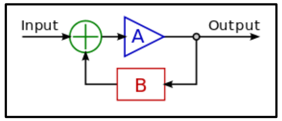
- The attitude controller had a size limitation given by three PID in each channel:
- An Hinfstruct controller was implemented and uploaded on March 2014 prior to the final maneuver





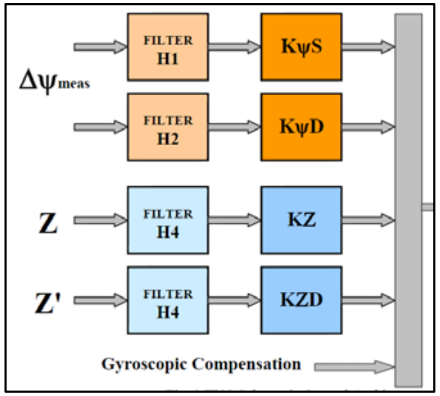
On the Y axis: a significant reduction of the pointing errors is observed with respect to the in-flight tuning by almost a factor 5. The response time of the controller has been reduced by at least a factor 2

On the X-axis: the new controller tuning has brought some reduction of the pointing error. The response time of the controller has been reduced by a factor 2.
On the Z-axis, the new controller tuning has brought a slight reduction of the pointing error. The response time of the controller has been reduced by a factor 2.

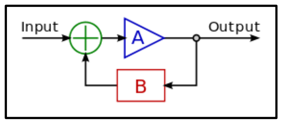


Example: Yaw controller of the VEGA launcher

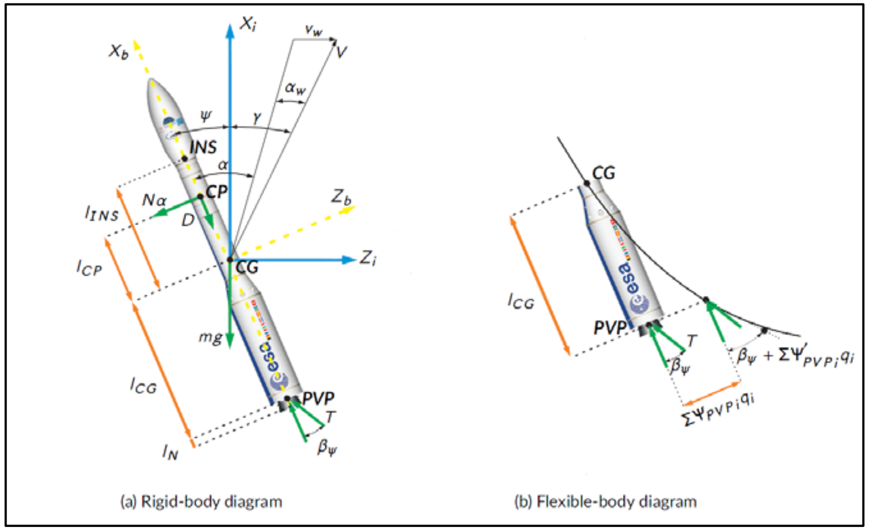
- The design of the ascent-flight control system of a launch vehicle for the atmospheric phase is still a challenging task. Along the first phase of the mission, any launch vehicle encounters undesired events such as wind disturbances, high aerodynamic pressure and dramatic dynamic changes.



The global controller (TVC) consists of a proportional-derivative (PD) controller in attitude, two filters H1 and H2 to phase the rigid mode, a filter H3 to attenuate the 1st bending mode and notch upper modes. It also includes a lateral control to maintain the launcher on the reference trajectory as well as to improve the response to wind gust. All gains and filters coefficients are non stationary and are interpolated versus non gravitational velocity (VNG) in steady state phase and versus longitudinal acceleration in tail off phase.

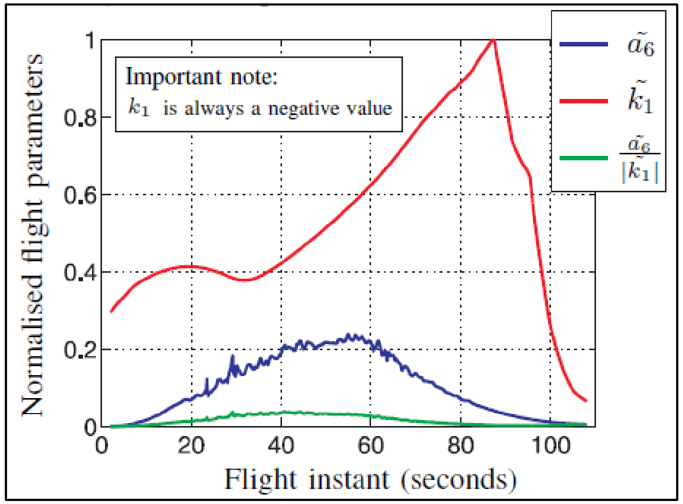


- Simplified yaw motion of the rigid body

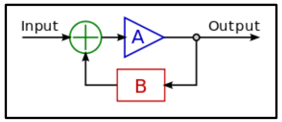


$$\begin{bmatrix}\dot{\psi}\\ \ddot{\psi}\end{bmatrix} = \begin{bmatrix}0 & 1\\ a_6 & 0\end{bmatrix}\begin{bmatrix}\psi\\ \dot{\psi}\end{bmatrix} + \begin{bmatrix}0\\ k_1\end{bmatrix}\beta_\psi$$

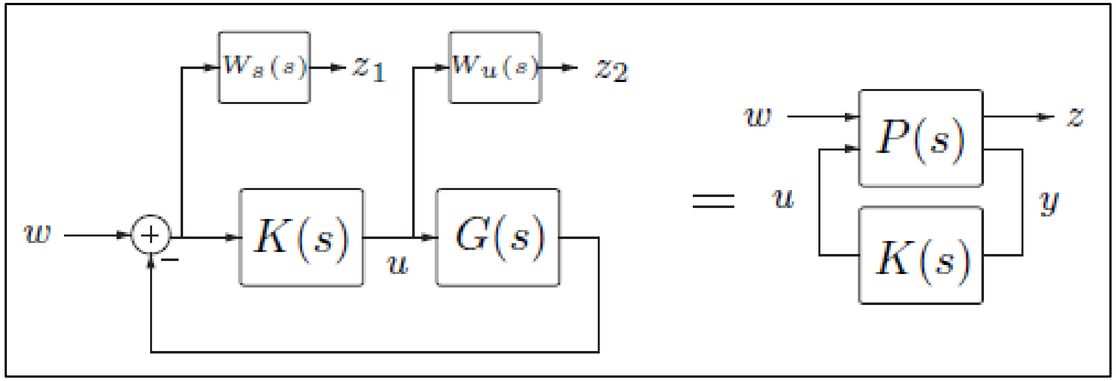
$$\psi = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\psi\\ \dot{\psi}\end{bmatrix} + [0]\,\beta_\psi$$



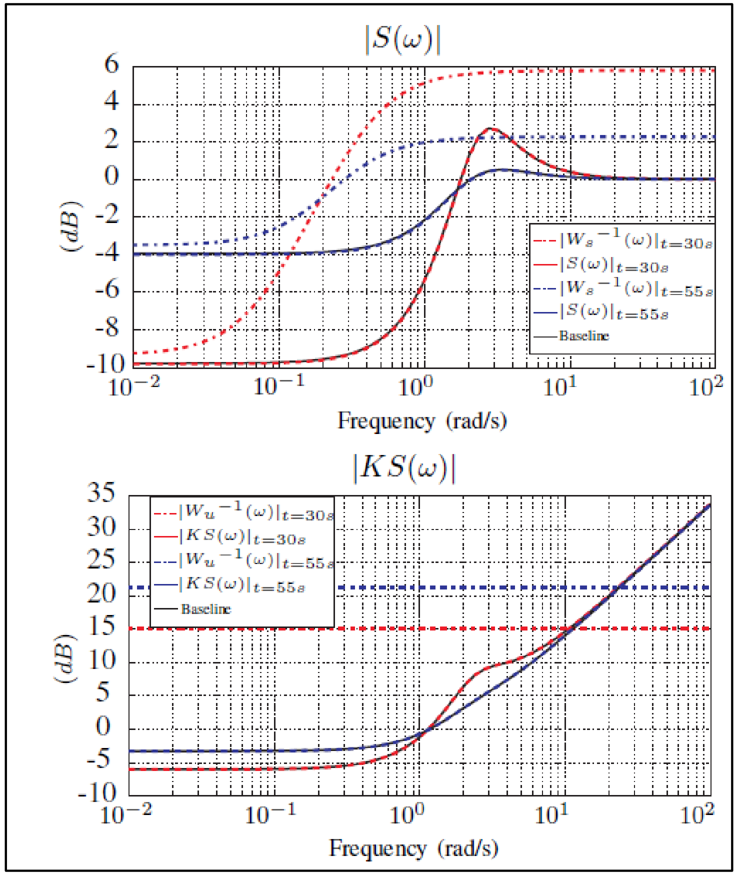
- The VEGA TVC control architecture is based on a PD controller to stabilize the launcher's attitude, a lateral control feedback to reduce the angle of attack and to minimize the drift of the vehicle and a set of bending filters to attenuate the bending modes.
- A gain-scheduling approach is used for the PD and lateral controllers, using the time variation of a6 and k1 to focus on critical points along the flight trajectory (e.g. pitch-over, maximum dynamic pressure, maximum acceleration prior to stage separation).
- Here we only consider the rigid-motion PD controller



- Application of a structured $\mathcal{H}_\infty$ controller design using a mixed sensitivity approach



$K(s)$ is the tunable structured controller:
$K = ltiblock.pid(\text{'K'},\text{'PD'})$.



$$W_s(s) = \frac{\frac{s}{h_s} + \omega_s}{s + \omega_s l_s}$$

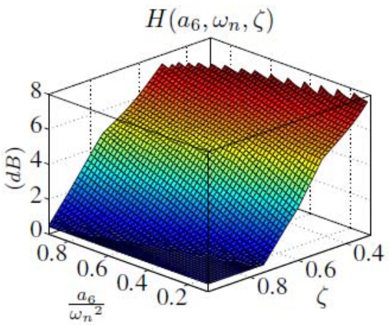
$$W_u(s) = 1/A_u$$

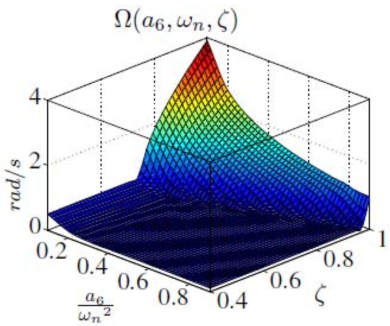
$$l_s = \frac{a_6}{{\omega_n}^2} 10^{\frac{0.45}{20}}$$

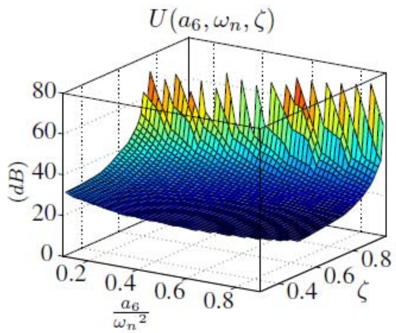
$$h_s = \begin{cases} 10^{\frac{0.45}{20}} & \text{if } |S(\omega_{peak})| = 1 \\ 10^{\frac{H(a_6,\omega_n,\zeta)}{20}} & \text{other} \end{cases}$$

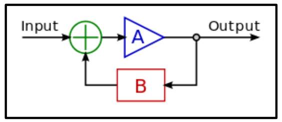
$$\omega_s = \Omega(a_6, \omega_n, \zeta)$$

$$A_u = \begin{cases} 10^{\frac{53}{20}} & \text{if } |S(\omega_{peak})| = 1 \\ |\frac{a_6}{k_1}| 10^{\frac{U(a_6,\omega_n,\zeta)}{20}} & \text{other} \end{cases}$$









- Application of a structured $\mathcal{H}_\infty$ controller uses the following commands:

hinfstruct

H_∞ tuning of fixed-structure controllers

Syntax

```
CL = hinfstruct(CL0)
[CL,gamma,info] = hinfstruct(CL0)
[CL,gamma,info] = hinfstruct(CL0,options)
[C,gamma,info] = hinfstruct(P,C0,options)
```

Description

The `hinfstruct` command extends classical H_∞ synthesis (see `hinfsyn`) to fixed-structure control systems. If you are unfamiliar with constructing weighting functions to capture design requirements for H_∞ synthesis, use `systune` or `looptune` instead.

systune

Tune fixed-structure control systems modeled in MATLAB

Syntax

```
[CL,fSoft] = systune(CL0,SoftReqs)
[CL,fSoft,gHard] = systune(CL0,SoftReqs,HardReqs)
[CL,fSoft,gHard] = systune(CL0,SoftReqs,HardReqs,options)
[CL,fSoft,gHard,info] = systune( __ )
```