Chapter 5: Review of Chapter 4

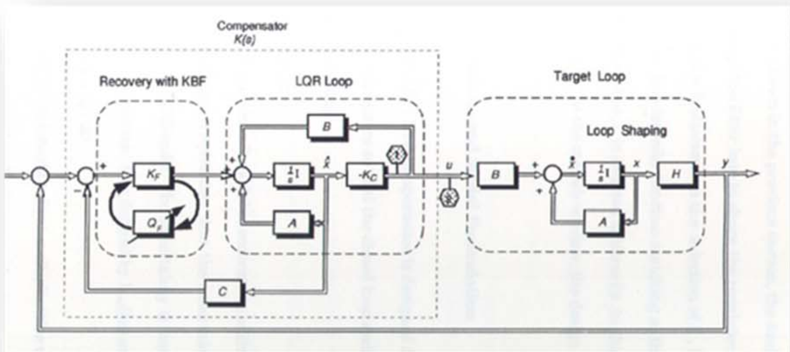
- ❑General Robustness Problem
- ❑Uncertainty Characterization and Robustness
- ❑Robustness in MIMO Systems
- ❑Shaping with Uncertainty
- ❑Robustness of Optimal Controllers
- ❑Uncertainty in a 2-Block Structure

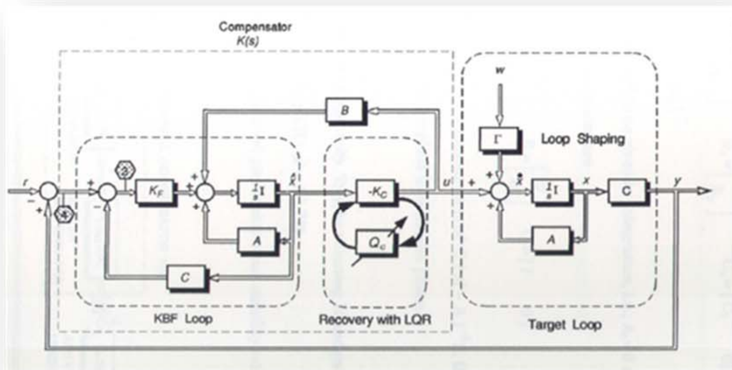
❑ Reference Material

- Zhou, K., Essentials of Robust Control, Prentice Hall 1998, CH. 1-6
- Skogestad, S., Postlehwaite, I., Multivariable Feedback Control, Wiley 2010, Ch. 5, 6, 7, App. A
- Mackenroth, U., Robust Control Systems, Springer 2004, Ch. 1-7, App. A

Chapter 5: Topics

1. **LTR-General Concepts**
2. **Loop Transfer Recovery**
3. **Frequency Weighted Loop Shaping**
4. **Examples**





Main Concepts

- The loop transfer recovery method (LTR) is a design methodology for <u>an observer based controller</u>. It deals with the design of controller parameters such that frequency domain specifications are satisfied.
  - Nominal system frequency shaping.
  - Robust stability and performance frequency shaping

- The method was first developed as a modification of the LQG compensator, where design parameters are changed in order to achieve robustness similar to the LQR (at input) or KBF (at output). The method can also be seen as a $\mathcal{H}_2$ Optimization.

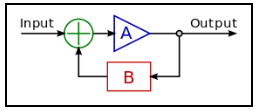
  - Recall MIMO stability margins

<u>Corollary 1 (Th. 4)</u>: If $\Phi_{CL}(j\omega)$ has no CRHP zeros, and $\underline{\sigma}\left[I+G^{-1}\right]>\alpha$ for $\omega\in\Omega_R$, then the stability margins are limited by:

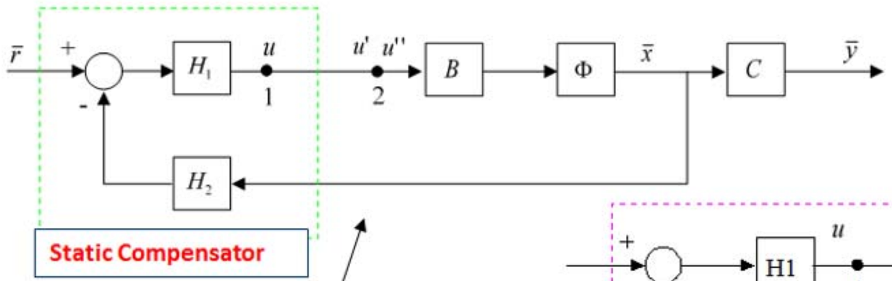
$$GM\in\left[1-\alpha,1+\alpha\right]$$
$$PM\in\left[-2\sin^{-1}\frac{\alpha}{2},2\sin^{-1}\frac{\alpha}{2}\right]$$

<u>Corollary n° 2</u> (Th. 6): If $\Phi_{CL}(j\omega)$ has no CRHP zeros, and $\underline{\sigma}\left[I+G\right]>\alpha$ for $\omega\in\Omega_R$, then the stability margins are limited by:

$$GM\in\left[\frac{1}{1-\alpha},\frac{1}{1+\alpha}\right]$$
$$PM\in\left[-2\sin^{-1}\frac{\alpha}{2},2\sin^{-1}\frac{\alpha}{2}\right]$$

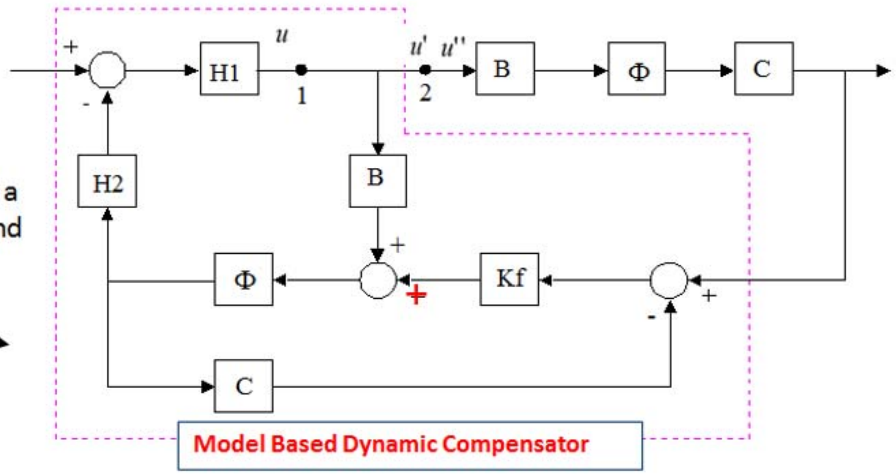


- From Kalman's inequality
- LQR stability margins: $GM = [\frac{1}{2}, \infty],\ PM = \pm 60^0$
- KBF stability margins: $GM = [\frac{1}{2}, \infty],\ PM = \pm 60^0$
- LQG stability margins: **There are NONE**



$$\Phi(s) = (sI - A)^{-1}$$

- Consider now two structures, the first uses a static gain (LQR for instance), and the second a model based dynamic compensator:



- The two controller structures are given by:

  - Static Compensator $\boldsymbol{u}(s) = H_1(s)\boldsymbol{r}(s) - H_1(s)H_2(s)\boldsymbol{x}(s)$

  - Model based compensator $\begin{cases} \boldsymbol{u}(s) = H_1(s)\boldsymbol{r}(s) - H_1(s)H_2(s)\hat{\boldsymbol{x}}(s) \\ \hat{\boldsymbol{x}}(s) = (sI - A)^{-1}\left\{B\boldsymbol{u}(s) + KF\left[\boldsymbol{y} - C\hat{\boldsymbol{x}}(s)\right]\right\} \end{cases}$

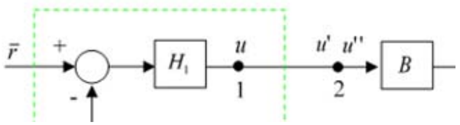
❑ Evaluate the loop transfer function matrices

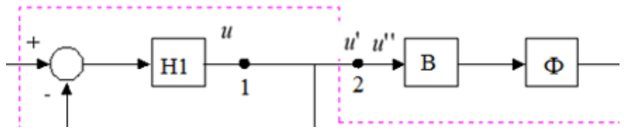
  - The closed loop transfer matrix is the same for both cases (try it):

$$\boldsymbol{x}(s) = \underbrace{\Phi(s)B\left[I + H_1H_2\Phi(s)B\right]^{-1}H_1}_{TFM}\boldsymbol{r}(s); \boldsymbol{y}(s) = C\boldsymbol{x}(s)$$

  - where we assumed that the inverses exist and $H_1H_2 = H_2H_1$.

- Compare open loop transfer matrices at points 1 and 2 (same signal).



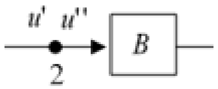


- The loop TFM at point 1 (between u' and u) is the same in both cases $\boldsymbol{u}(s) = -H_1 H_2 \Phi(s) B \boldsymbol{u}'(s)$

- The loop TFM at point 2 (between u'' and u') is different in general

CASE Static Compensator:

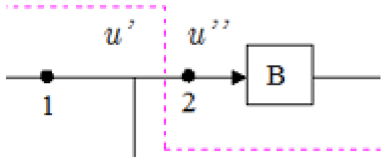
$$\boldsymbol{u}' = -H_1 H_2 \Phi(s) B \boldsymbol{u}''$$

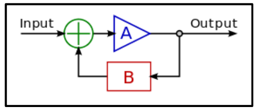


CASE MBC:

$$\begin{cases} \boldsymbol{u}' = -H_1 H_2 \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{x}} = \Phi(s)\left[B\left(C\Phi B\right)^{-1} - K_F\left(I + C\Phi K_F\right)^{-1}\right] C\Phi B \boldsymbol{u}' \\ +\Phi\left[K_F\left(I + C\Phi K_F\right)^{-1}\right] C\Phi B \boldsymbol{u}'' \end{cases}$$

- Note that existence of the inverse of $C\Phi B$, implies that the system must be **minimum phase** **and square (or well – posed)**





❑ Summary:

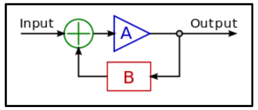
- Robustness is maintained if the loop is open at point 1 and the controller is a LQR or a controller with guaranteed stability margins (from MIMO Nyquist)

- Nothing can be said at point 2, since the transfer matrices are **<u>not</u>** the same.

- The objective of the "recovery" is therefore to <u>recover the robustness properties of the LQR at point 2</u>, which is the plant input. This implies of course that both loop transfer matrices be the same.

- For the loop transfer matrices to be the same the following identity is required (Doyle & Stein Condition, 1979) $B\left(C\Phi B\right)^{-1} = K_F\left(I + C\Phi K_F\right)^{-1}$ **(1)**

$$\begin{cases} \boldsymbol{u}' = -H_1 H_2 \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{x}} = \Phi(s)\left[B\left(C\Phi B\right)^{-1} - K_F\left(I + C\Phi K_F\right)^{-1}\right] C\Phi B\boldsymbol{u}' \\ +\Phi\left[K_F\left(I + C\Phi K_F\right)^{-1}\right] C\Phi B\boldsymbol{u}'' \end{cases} \Rightarrow \quad \boldsymbol{u}' = -H_1 H_2 \Phi(s) B\boldsymbol{u}''$$



❑ Do we remember the LQG compensator ?:

- Plant Dynamics

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} + \boldsymbol{w} \\ \boldsymbol{y} = C\boldsymbol{x} + \boldsymbol{v} \end{cases}$$

- Compensator Dynamics

$$\begin{cases} \boldsymbol{u} = -K_C\hat{\boldsymbol{x}} \\ \dot{\hat{\boldsymbol{x}}} = A\hat{\boldsymbol{x}} + B\boldsymbol{u} + K_F\left(\boldsymbol{y} - C\hat{\boldsymbol{x}}\right) \\ = \left(A - BK_C - K_FC\right)\hat{\boldsymbol{x}} - K_F\boldsymbol{e} \end{cases}$$

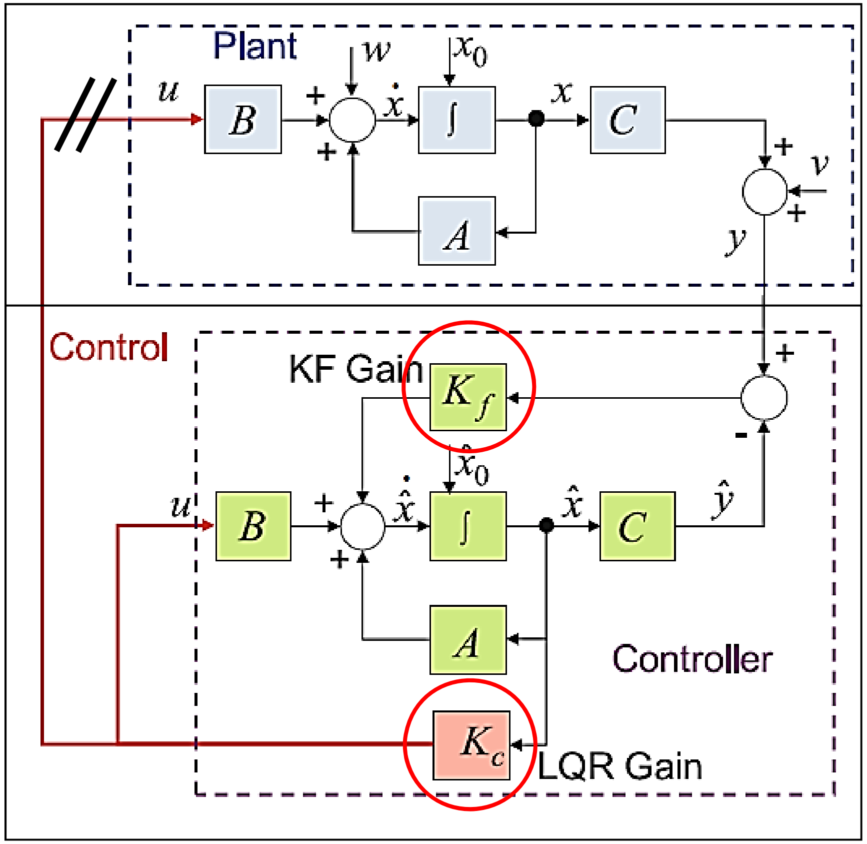
- Frequency Domain

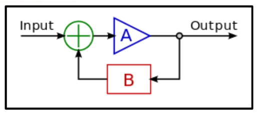
$$\Phi(s) = (sI - A)^{-1}$$

$$\begin{cases} G(s) = C\Phi(s)B \\ K(s) = K_C\left(sI - A + BK_C + K_FC\right)^{-1}K_F \end{cases}$$

$$K(s) = \left[\begin{array}{c|c} A - BK_C - K_FC & -K_F \\ \hline K_C & 0 \end{array}\right]$$

$$L(s) = K_C\left(sI - A + BK_C + K_FC\right)^{-1}K_FC(sI - A)^{-1}B$$





❑ **The LQG Interconnection is well-posed and internally stable provided the appropriate pairs are stabilizable and detectable:**

- **Assumption:** choose K_F to be a function of a design parameter q_0 such that:

$$\lim_{q_0 \to \infty} \left[\frac{K_F(q_0)}{q_0} \right] = BW$$

with W arbitrary, positive definite weighting matrix.
Then the RHS of **(1)** becomes:

$$K_F\left(I + C\Phi K_F\right)^{-1} = \frac{K_F(q_0)}{q_0} q_0 \left[I + C\Phi K_F(q_0)\right]^{-1} = \frac{K_F(q_0)}{q_0} \left[\frac{I}{q_0} + \frac{C\Phi K_F(q_0)}{q_0} \right]^{-1}$$

for $q_0 \to \infty$ $$BW\left[\frac{I}{q_0} + \frac{C\Phi K_F}{q_0}\right]^{-1} = BW\left(C\Phi BW\right)^{-1} = BWW^{-1}\left(C\Phi B\right)^{-1} = B\left(C\Phi B\right)^{-1}$$

- Now we must find $K_F(q_0)$

    - Consider the observer to be given by a Kalman Filter estimator (KBF), the filter gain matrix is a function of the Riccati equation solution:

$$A\Sigma + \Sigma A^T + Q_F - \Sigma C^T R_F^{-1} C\Sigma = 0$$
$$K_F = \Sigma C^T R_F^{-1}$$

- Rewrite the filter gain matrix

$$K_F(q_0) = \Sigma(q_0) C^T R_F^{-1}$$

- Define

$$\begin{cases} Q_F = Q_0 + q_0^2 BVB^T \\ R_F = R_0 \end{cases}$$

with Q_0 and R_0 original noise covariance matrices, and $V > 0$ arbitrary symmetric weighting matrix

- Using the above, the Riccati equation can be rewritten as:

$$A\left(\frac{\Sigma}{q_0^2}\right) + \left(\frac{\Sigma}{q_0^2}\right)A^T + \frac{Q_0}{q_0^2} + BVB^T - q_0^2\left(\frac{\Sigma}{q_0^2}\right)C^T R_0^{-1} C\left(\frac{\Sigma}{q_0^2}\right) = 0 \qquad \textbf{(2)}$$

❑ **Theorem:** **(Kwakernaak - Sivan, 'Linear Optimal Control Systems', Wiley 1972, Th. 3.14, pp. 306-307)**

- Given a stabilizable, detectable LTI stochastic system, assume B and C to be full rank:

$$\begin{cases} \dot{x} = Ax + Bu + v & v = (0, Q_0) \\ y = Cx + n & n = (0, R_0 = \rho N) \end{cases}$$

- Let Σ be the steady state solution of associate Riccati equation **(2)**, then the following hold:

1. The following limit exists: $\lim_{\rho \to 0} \Sigma = \Sigma_0$
2. If and only if the system is minimum phase(CΦB i invertible), then: $\dim(y) \leq \dim(u) \Rightarrow \Sigma_0 = 0$
3. Otherwise: $\dim(y) > \dim(u) \Rightarrow \Sigma_0 \neq 0$

- From the theorem, we can prove that, in Eq. **(2)** : $\lim_{q_0 \to \infty} \dfrac{\Sigma}{q_0^2} = 0$ note: $q_0 = \dfrac{1}{\rho_0}$

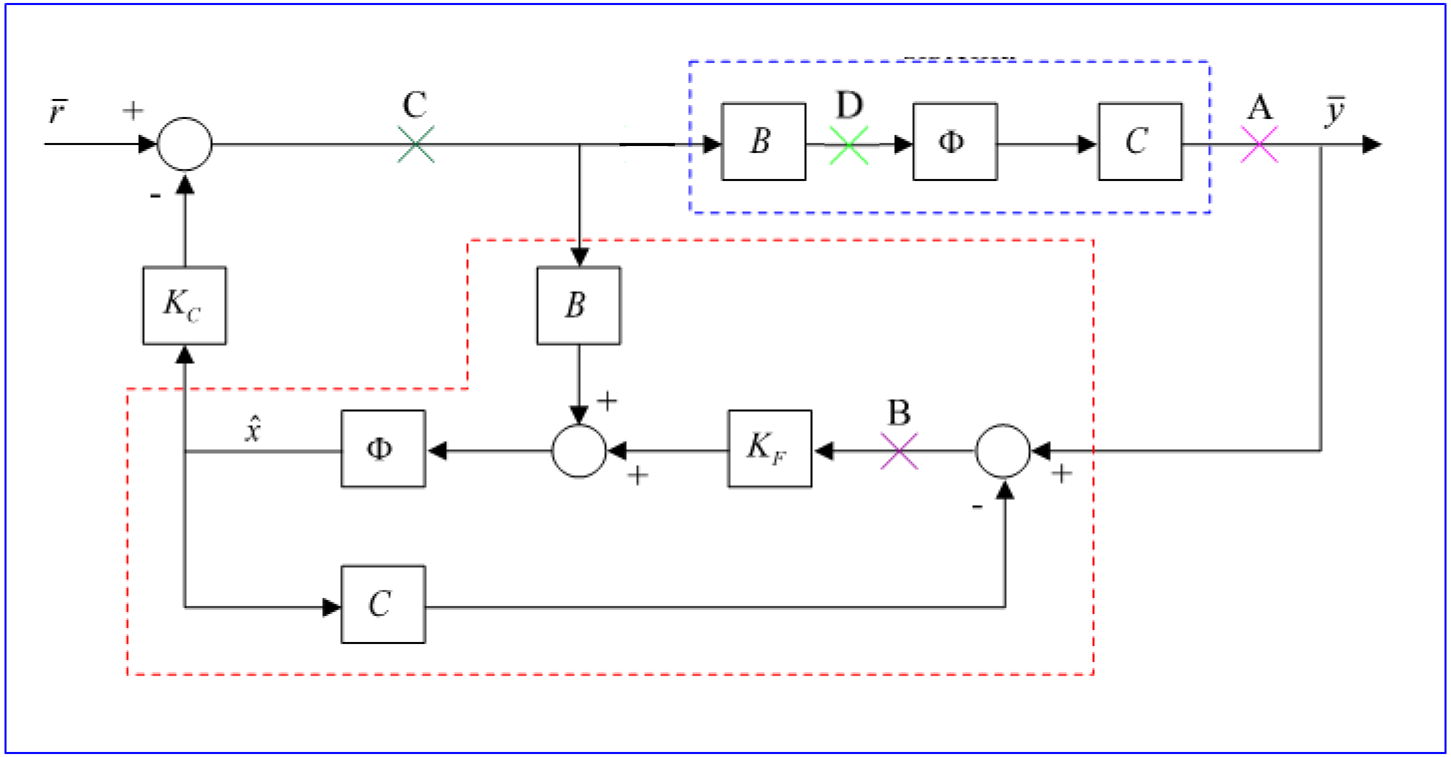
  - Therefore in **(2)** we have: $\lim_{q_0 \to \infty} q_0^2 \left[\dfrac{\Sigma}{q_0^2}\right] C^T R_0^{-1} C \left[\dfrac{\Sigma}{q_0^2}\right] = BVB^T$

- Recall: $$\begin{cases} K_F(q_0) = \Sigma(q_0) C^T R_F^{-1} \\ R_F = R_0 \end{cases}$$
- Therefore: $$\lim_{q_0 \to \infty} \frac{1}{q_0^2} K_F(q_0) R_0 K_F^T(q_0) = BVB^T$$
- or: $$\frac{K_F(q_0)}{q_0} \to B\sqrt{V}\sqrt{R_0}$$
- But from previous results: $$\frac{K_F(q_0)}{q_0} \to BW$$
- Therefore the non singular matrix W is given by: $$W = B[VR_0]^{\frac{1}{2}}$$

❑ **Summary:** the approach modifies the LQG (stochastic) compensator with a <u>deterministic observer – based compensator</u> in which the gain is a function of a parameter q_0 (or $1/\rho_0$) available to the designer, in order to match loop transfer function matrices.

- **The resulting compensator is NOT OPTIMAL anymore in the sense of the LQG**

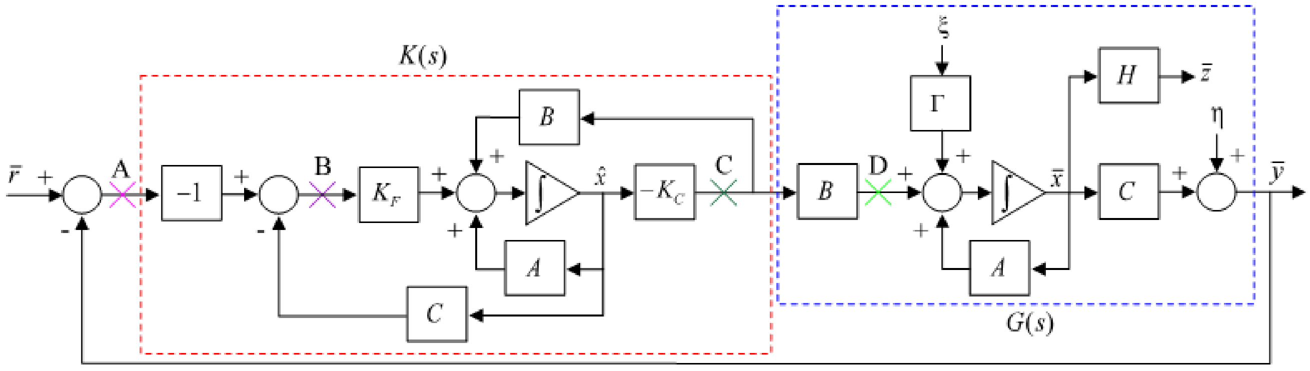
- Consider the general block diagram for an LQG-like compensator. Because of MIMO signals (TFM must be compatible with respect to multiplication)



- We can identify 2 sets of loop breaking points: **(A, B)** are the **plant output** breaking point, **(C, D)** are the **plant input** breaking points

- Modify the closed loop system into a unity feedback structure

Γ, μ



H, ρ

$$K(s) = K_c \left[sI - A + BK_c + K_f C\right]^{-1} K_f$$

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} + \Gamma \boldsymbol{w} \\ \boldsymbol{y} = C\boldsymbol{x} + \boldsymbol{n} \\ \boldsymbol{z} = H\boldsymbol{x} \end{cases} \qquad \begin{cases} Q = H^T H \\ R = \rho \ I \\ A^T P + PA + Q - \dfrac{1}{\rho} PBB^T P = 0 \end{cases}$$

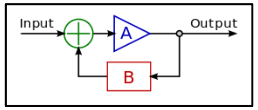
$$J = E\left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\boldsymbol{x}^T H^T H \boldsymbol{x} + \rho \boldsymbol{u}^T \boldsymbol{u} \right) dt \right\}$$

$$\begin{cases} \boldsymbol{u} = -K_c \hat{\boldsymbol{x}} \\ K_c = \dfrac{1}{\rho} B^T P \end{cases} \qquad \begin{cases} \boldsymbol{w} = (0, W \to I) \\ \boldsymbol{n} = (0, V \to \mu I) \end{cases}$$

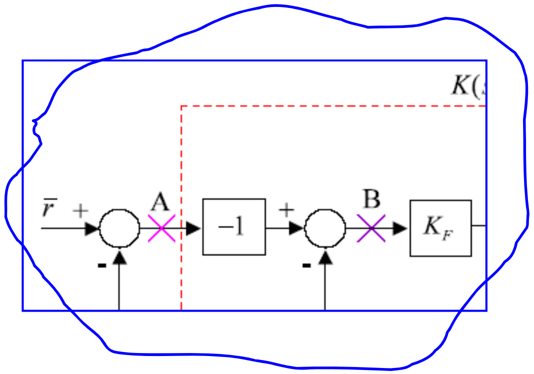
$$\dot{\hat{\boldsymbol{x}}} = A\hat{\boldsymbol{x}} + B\boldsymbol{u} + K_F(\boldsymbol{y} - C\hat{\boldsymbol{x}})$$

$$K_F = \frac{\Sigma C^T}{\mu}$$

$$A\Sigma + \Sigma A^T + \Gamma W \Gamma^T - \frac{1}{\mu} \Sigma C^T V^{-1} C \Sigma = 0$$



- Compute the transfer matrices at the opening points A, B, C, D.



Point A: signals $a = a'$

- The loop transfer matrix between a and a' can be computed as:

$$\boldsymbol{a} = C\Phi(s)B\boldsymbol{u}$$

$$\boldsymbol{u} = K_c\left(\Phi^{-1} + BK_c + K_FC\right)^{-1}K_F\boldsymbol{y}$$

- From which $\boldsymbol{a} = G(s)K(s)\boldsymbol{a}'$

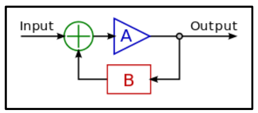
Point B: signals $b = b'$

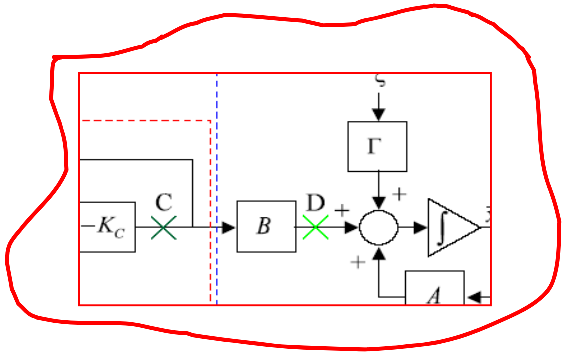
- The loop transfer matrix between b and b' can be computed as:

$$\boldsymbol{b} = \boldsymbol{y} - C\hat{\boldsymbol{x}} = \boldsymbol{y} - C\Phi\left[B\boldsymbol{u} + K_F\boldsymbol{b}'\right] =$$
$$= -C\Phi K_F\boldsymbol{b}'$$

- From which $\boldsymbol{b} = -C\Phi K_F\boldsymbol{b}'$

$$\boxed{\begin{cases} T_A(s) = G(s)K(s) \\ T_B(s) = C\Phi K_F \end{cases}}$$





Point C: signals $c = c'$

- The loop transfer matrix between c and c' can be computed:

$$\begin{aligned}\boldsymbol{c} &= -K_c\hat{\boldsymbol{x}} = -K_c\Phi\left[Bc' + K_F\left(\boldsymbol{y} - C\hat{\boldsymbol{x}}\right)\right] = \\ &= -K_c\left[\Phi B\boldsymbol{c}' + \Phi K_F y\right]\boldsymbol{y} + K_c\Phi C\hat{\boldsymbol{x}} = \\ &= -K_c\left(I + \Phi K_F C\right)^{-1}\left[\Phi B\boldsymbol{c}' + \Phi K_F\boldsymbol{y}\right]\end{aligned}$$

$$\boldsymbol{y} = C\Phi B\boldsymbol{c}'$$

$$\Rightarrow \boldsymbol{c} = -K_c\left(I + \Phi K_F C\right)^{-1}\left[\left(\Phi B + \Phi K_F C\Phi B\right)\boldsymbol{c}'\right]$$

$$\Rightarrow \boldsymbol{c} = -K_c\Phi B\boldsymbol{c}'$$

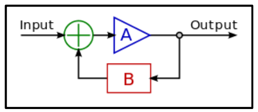
Point D: signals $d = d'$

- The loop transfer matrix between d and d' can be computed:

$$\boldsymbol{d} = \boldsymbol{u} = -K(s)\boldsymbol{y} = -K(s)G(s)\boldsymbol{d}'$$

$$\Rightarrow \boldsymbol{d} = -K(s)G(s)\boldsymbol{d}'$$

$$\boxed{\begin{cases} T_C(s) = K_c\Phi B \\ T_D(s) = K(s)G(s) \end{cases}}$$

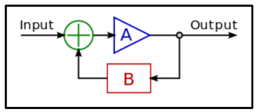


❑ Comments: depending on the location of the loop opening:

- **Assumptions:** square minimum phase plant.

- Performance requirements are defined by appropriate shaping of the frequency response (singular value behaviour)
- For the <u>loop opened at the input</u>, the ‘LQR’ component defines the performance requirement, and the ‘KBF’ component provided the loop transfer recovery.
- For the <u>loop opened at the output</u>, the ‘KBF’ component defines the performance requirement, and the ‘LQR’ component provided the loop transfer recovery.

1. Loop broken at the output:
   - Frequency shaping requirements (stability, performance, and robustness) are defined using the compensator dynamic behavior (“filter”)
   - The recovery is performed by selecting the LQR design parameters (performance index weights: Q, R) in order to match the desired frequency response **(H, ρ in the block diagram).**
2. Loop broken at the input:
   - Frequency shaping requirements (stability, performance, and robustness) are defined using the LQR feedback gain selection
   - The recovery is performed by selecting the compensator KBF design parameters (covariances: W, V) in order to match the desired frequency response **(Γ, μ in the block diagram).**



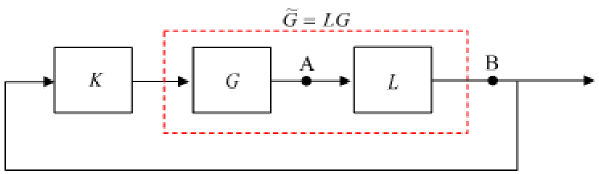
❑ LQG/LTR and Robustness with respect to unstructured Uncertainty

- Consider Input and Output uncertainties. The uncertainty is multiplicative, stable and bounded over frequency

$$\bar{\sigma}\left[L(j\omega)\right] \leq l_m(\omega); L = I + \Delta$$

- Frequency shaping requirements for stability and performance robustness

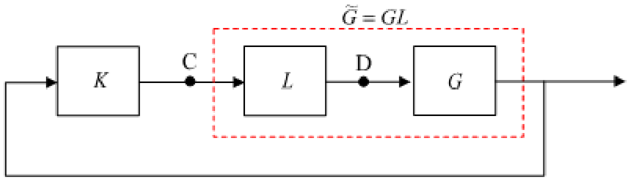
Uncertainty at the Output



- The loop is broken at the output of the plant (uncertainties in sensor dynamics, high frequency dynamics, etc.)

$$\begin{cases} \underline{\sigma}\left[GK\right] \geq \dfrac{p(\omega)}{1 - l_m(\omega)}, \omega < \omega_1 \ RP \\ \bar{\sigma}\left[GK\right] < \dfrac{1}{l_m(\omega)}, \omega > \omega_2 \ \boldsymbol{RS} \\ \underline{\sigma}\left[I + (GK)^{-1}\right] > l_m(\omega), \forall\omega \ RS \end{cases}$$

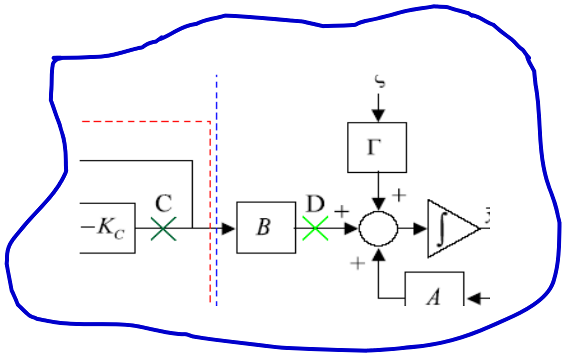
Uncertainty at the Input

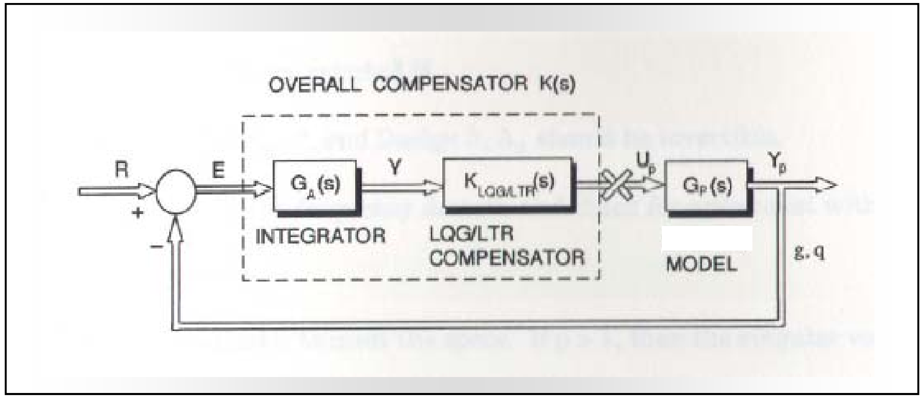


- The loop is broken at the input of the plant (uncertainties in actuator dynamics, controller dynamics, etc.)

$$\begin{cases} \underline{\sigma}\left[KG\right] \geq \dfrac{p(\omega)}{1 - l_m(\omega)}, \omega < \omega_1 \ RP \\ \bar{\sigma}\left[KG\right] < \dfrac{1}{l_m(\omega)}, \omega > \omega_2 \ \boldsymbol{RS} \\ \underline{\sigma}\left[I + (KG)^{-1}\right] > l_m(\omega), \forall\omega \ RS \end{cases}$$

❑ LQG/LTR **Design** with the loop open at the **Input**





Real Loop Transfer Matrix : $T_D(s) = K(s)G(s)$

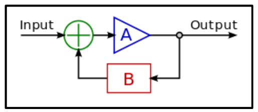
Desired Loop Transfer Matrix $T_{LQ}(s) = K_c \Phi B = T_C(s)$

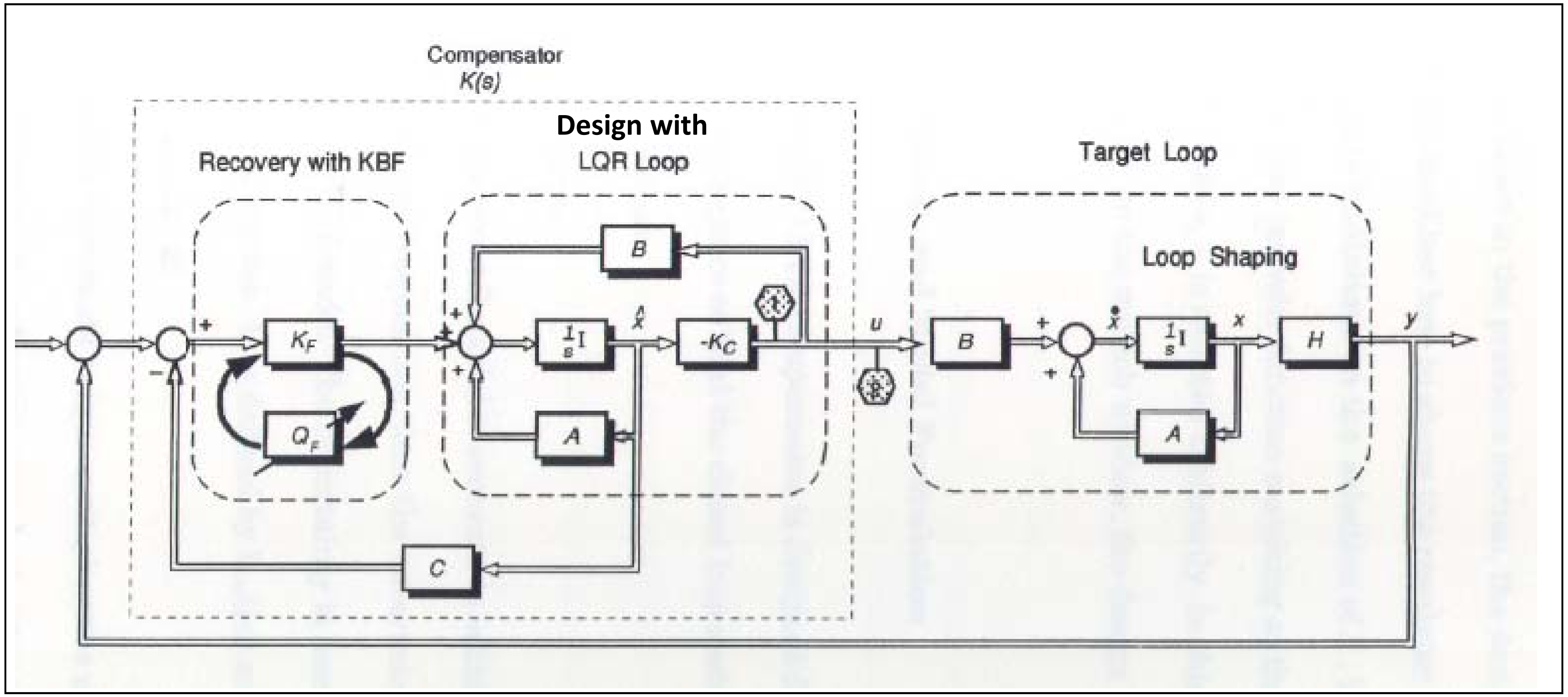
Let: $\begin{cases} Q = H^T H \\ R = \rho\ I \end{cases}$

- H and ρ are now design parameters for frequency shaping of $T_C(s)$

- Recall Kalman's Equality:

$$\rho I + B^T\left[-sI - A^T\right]^{-1} H^T H \left[sI - A\right]^{-1} B =$$
$$= \left[I + B^T\left(-sI - A^T\right)^{-1} K_c^T\right] \rho I \left[I + K_c\left(sI - A\right)^{-1} B\right]$$





- Kalman's Equality can be rewritten as:

$$\left[I+T_{LQ}(s)\right]^{*}\left[I+T_{LQ}(s)\right]=I+\frac{1}{\rho}[H\Phi B]^{*}\left[H\Phi B\right]$$

Thus: $\sigma_i\left[I+T_{LQ}(s)\right]=\sqrt{1+\frac{1}{\rho}\sigma_i^2\left[H\Phi B\right]}$ **(*)**

- **Low Frequency Performance**

At low frequency, we usually have $\sigma_{\min}\left[T_{LQ}\right] >> 1$ so we can use the following approximation:

$$\sigma_i\left[T_{LQ}\right] \approx \frac{1}{\sqrt{\rho}}\sigma_i\left[H\Phi B\right]$$

to verify the requirement it is sufficient to select (H, ρ) such that:

$$\boxed{\frac{1}{\sqrt{\rho}}\sigma_{\min}\left[H\Phi B\right] > \frac{p(\omega)}{1-l_m(\omega)}}$$

In addition, it is important to maintain "balance" keeping $\sigma_{\min}\left[T_{LQ}\right]$ and $\sigma_{\max}\left[T_{LQ}\right]$ close.

- **Robustness**

From **(*)** it is clear that $\sigma_i\left[I+T_{LQ}\right] > 1$. In addition Laub (1981) proved that:

$$\sigma_i\left[I+T_{LQ}\right] > 1 \Rightarrow \sigma_i\left[I+T_{LQ}^{-1}\right] > \frac{1}{2}$$

This implies that the LQR guarantees robust stability for all unstructured uncertainties such that

$$\left|l_m(\omega)\right| < 0.5$$

High Frequency Performance/Robustness

The previous conditions are inadequate when $l_m(\omega)$ grows (high frequency). The high frequency behavior requires a modification of $T_{LQ}(s)$ in that range.

- Assume $H\Phi B$ to be minimum phase.

- Compute $T_{LQ}(s)$ at high frequency $\rho \to 0$, thus we can set: $j\omega = s = j\dfrac{c}{\sqrt{\rho}}, c = const.$

this yields: $T\left(j\dfrac{c}{\sqrt{\rho}}\right) = \sqrt{\rho}\; K_c\left(jcI - \sqrt{\rho}A\right)^{-1} B \to \dfrac{WHB}{jc}$ **(**)**

- At crossover frequency $\sigma_{\max}\left[T_{LQ}(j\omega_{c\max})\right] = 1$ and we can write: $j\omega = \dfrac{j\omega_c}{\sqrt{\rho}} \to \omega = \dfrac{\omega_c}{\sqrt{\rho}}$

  - And the maximum crossover frequency is bound by: $\omega_{c_{MAX}} = \dfrac{\sigma_{\max}\left[HB\right]}{\sqrt{\rho}}$

- The determination of (H, ρ) must obviously take into account the constraint imposed by the crossover frequency.

$$\sigma_{\max}\left[\frac{HB}{\sqrt{\rho}}\right] = \omega_{c\max} \le \omega_c ; \omega_c := l_m(\omega_c) = 0\,dB$$

- Note that from **(**)** $T(j\omega) \to \dfrac{WHB}{j\omega\sqrt{\rho}}$ so the maximum attenuation is at the most–20 dB/dec

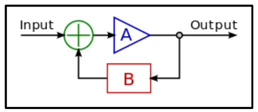
  If higher roll-off is required, the bandwidth must be reduced.

❑ LTR: Loop Transfer Recovery

- The first step in the recovery procedure, is to add fictitious columns and rows to the control gain and input matrices, in order to square $C\Phi B$ and $K_C\Phi B$

- The second step is to solve iteratively the following problem: $q_0 \to \infty$

$$\begin{cases} A\Sigma + \Sigma A^T + Q_F - \dfrac{1}{\mu}\Sigma C^T R_F^{-1} C\Sigma = 0 \\ Q_F = Q_0 + q_0^2 BVB^T \;, Q_0 = \Gamma\Gamma^T \,, \mathrm{V}>0 \\ R_F = \mu \; I \end{cases}$$



- Since the filter gain is given by: $K_F = \dfrac{\Sigma C^T}{\mu}$

- at each iteration compute:

$$A\Sigma + \Sigma A^T + \Gamma W \Gamma^T - \frac{1}{\mu}\Sigma C^T V^{-1} C \Sigma = 0$$

$$\begin{cases} K_{LQG}(s) = K_c \left[sI - A + BK_c + K_F C \right]^{-1} K_F \\ T_{INP}(s) = K_{LQG}(s) G(s) = T_D(s) \end{cases}$$

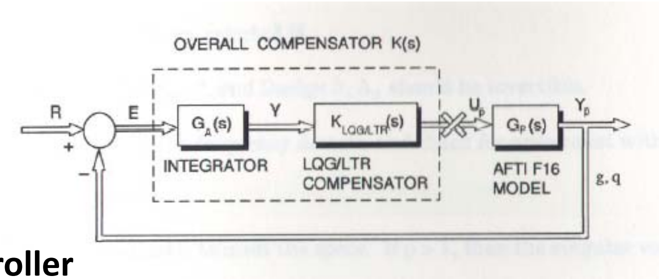
- The iterative procedure continues until:

$$\begin{cases} q_0 \to \infty \\ \sigma_i \left[T_{INP} \right] \approx \sigma_i \left[T_{LQ} \right] \end{cases}$$

- The procedure is complete when:

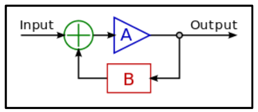
$$T_{INP}(s) = T_D(s) = K(s)G(s) \to K_C \Phi B \left(C \Phi B \right)^{-1} C \Phi B \approx K_C \Phi B = T_{LQ} = T_C$$

Input



Controller

1. Add integrators, and reduce the system to a minimal realization (if necessary);
2. Design the $K_c\Phi(s)B$;
3. From specifications find $p(\omega)$ and $l_m(\omega)$, and build the SV barriers;
4. Find (H, ρ) so that $\frac{1}{\sqrt{\rho}}\sigma_i\left[H\Phi B\right]$ satisfy low frequency barrier and $\bar{\sigma}\left[\frac{HB}{\sqrt{\rho}}\right] \leq \omega_c$ with ω_c such that $l_m(\omega_c) = 0dB$;
5. Once (H, ρ) are selected solve for K_C using Riccati. Verify that at low frequency $\sigma_i\left[K_c\Phi B\right] \approx \frac{1}{\sqrt{\rho}}\sigma_i\left[H\Phi B\right]$
6. Use $\sigma_i\left[I + T_{LQ}^{-1}\right] \geq \frac{1}{2}$ to verify robustness with $T_{LQ} = K_c\Phi B$;



7. Additional robustness test: $\frac{1}{l_m(\omega)} > \bar{\sigma}\left[T_{LQ}\left(I + T_{LQ}\right)^{-1}\right]$;
8. Compute transmission zeros of $C\Phi(s)B$ to verify that the system is minimum phase;
9. Perform recovery with $q_0 \to \infty$ using the system:

$$\begin{cases} A\Sigma + \Sigma A^T + \Gamma\Gamma^T - \frac{1}{\mu}\Sigma C^T C\Sigma = 0 \\ Q_F = Q_0 + q_0^2 BVB^T,\ Q_0 = \Gamma\Gamma^T,\ V > 0 \\ R_F = \mu I \end{cases}$$

since $K_F = \frac{\Sigma C^T}{\mu}$ at each iteration compute:

$$K_{LQG}(s) = K_c\left[sI - A + BK_c + K_F C\right]^{-1} K_F$$

and $T_{INP}(s) = K_{LQG}(s)G(s)$.

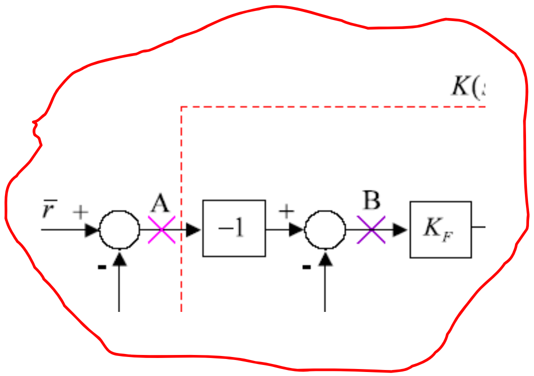
Iteration continues until:

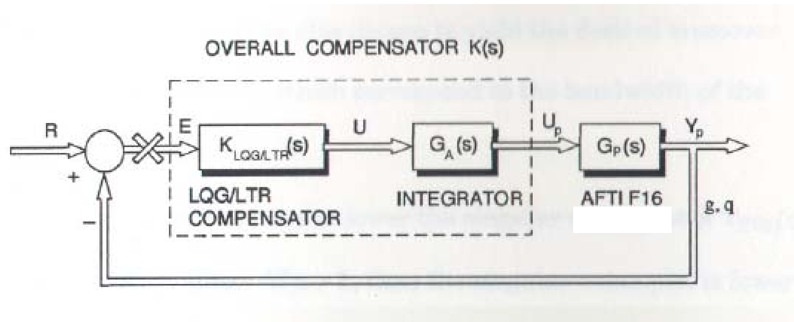
$$\sigma_i\left[T_{INP}\right] \approx \sigma_i\left[T_{LQ}\right]$$

10. Verify the design, checking $\underline{\sigma}\left[I + T_{INP}^{-1}\right] \geq \frac{1}{2}$

❑ LQG/LTR Design with the Loop open at the output

- The procedure is dual to the previous one. Performance and robustness are set by selecting the filter parameters, then the recovery is done using the LQR parameters.





Real Loop Transfer Matrix $\quad T_A(s) = G(s)K(s)$

Desired LoopTransfer Matrix $\quad T_B(s) = C\Phi K_F = T_{KF}(s)$

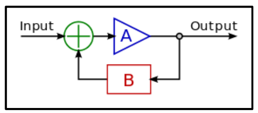
Let:
$$\begin{cases} \Gamma W \Gamma^T = \Gamma\Gamma^T \\ V = \mu \; I \end{cases}$$

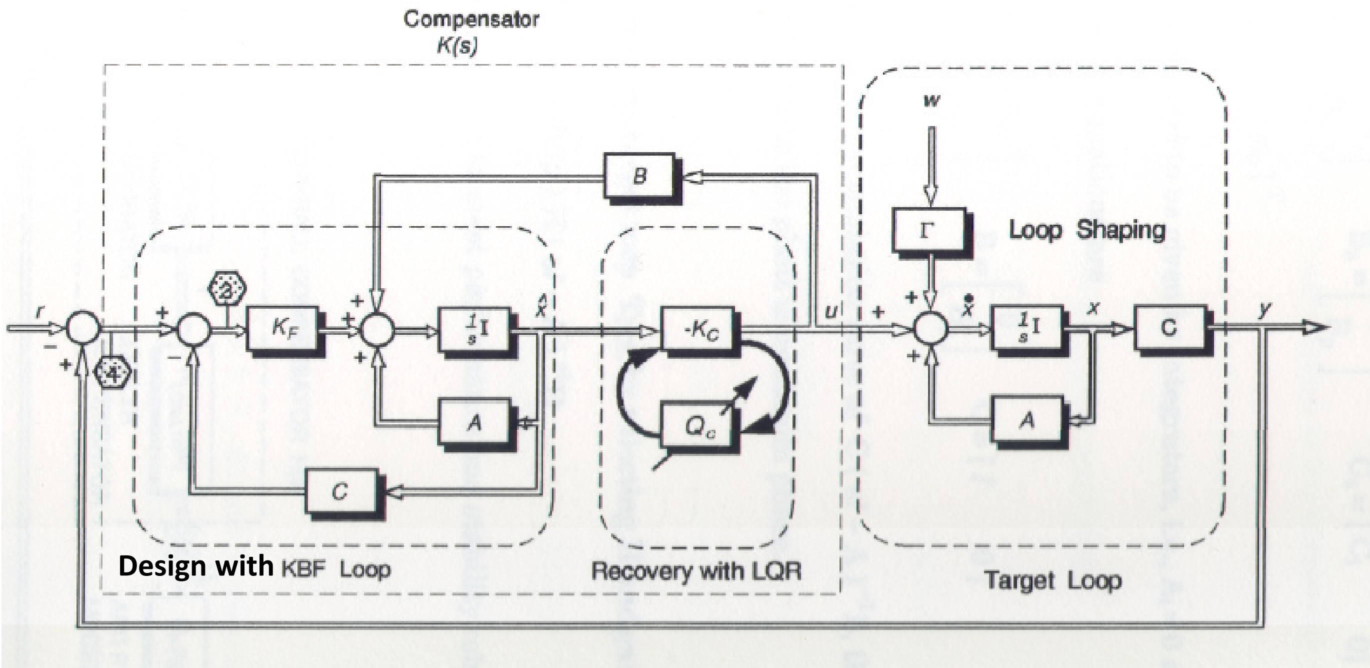
- The design parameters for shaping $T_B(s)$ are now Γ and μ.

- Kalman's Equality yields:

$$\sigma_i\left[I + T_{KF}(s)\right] = \sqrt{1 + \frac{1}{\mu}\sigma_i^2\left[C\Phi\Gamma\right]}$$

$$\sigma_i\left[T_{KF}\right] \approx \frac{1}{\sqrt{\mu}}\sigma_i\left[C\Phi\Gamma\right]$$





❑ Low Frequency Performance

$$\sigma_i\left[T_{KF}\right] \approx \frac{1}{\sqrt{\mu}}\sigma_i\left[C\Phi\Gamma\right]$$

- This is used to set the low frequency requirements on the design parameters (μ, Γ)

$$\boxed{\frac{1}{\sqrt{\mu}}\underline{\sigma}\left[C\Phi\Gamma\right] > \frac{p(\omega)}{1-l_m(\omega)}}$$

Robustness / Bandwidth

- The crossover frequency constraint must be also satisfied, relating the crossover frequency to the uncertainty upper bound, leading to:

$$\sigma_{\max}\left[\frac{C\Gamma}{\sqrt{\mu}}\right] = \omega_{c\max} \leq \omega_c ; \omega_c := l_m(\omega_c) = 0dB$$

Computation of $T_{KF}(s)$

- Once Γ and μ are selected we solve the “Filter” Riccati equation once.

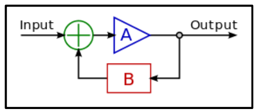
$$A\Sigma + \Sigma A^T + \Gamma\Gamma^T - \frac{1}{\mu}\Sigma C^T C\Sigma = 0$$

LTR: Loop Transfer Recovery

- The first step in the recovery procedure, is to add fictitious columns and rows to the control gain and input matrices, in order to square $C\Phi B$ and $C\Phi K_F$

- The second step is to solve iteratively the following problem:

$$\begin{cases} A^T P + PA + Q_C - PBR_c^{-1}B^T P = 0 \\ Q_C = H^T H + q_0^2 C^T V C \;\;, q_0 \to \infty \;\; V > 0 \\ R_c = \rho \;\; I \end{cases} \qquad q_0 \to \infty$$



Since: $K_C = \dfrac{B^T P}{\rho}$

- At each iteration compute:

$$A^T P + PA + Q - \frac{1}{\rho} PBB^T P = 0$$

$$\begin{cases} K_{LQG}(s) = K_c \left[sI - A + BK_c + K_F C\right]^{-1} K_F \\ T_{OUT}(s) = G(s) K_{LQG}(s) = T_B(s) \end{cases}$$

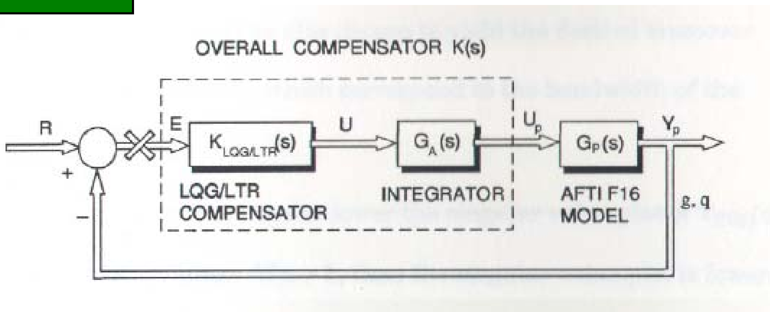
- The iterative procedure continues until:

$$\sigma_i \left[T_{OUT}\right] \approx \sigma_i \left[T_{KF}\right]$$

- In summary, as $q_0 \to \infty$

$$T_A(s) = G(s)K(s) \to C\Phi B \left(C\Phi B\right)^{-1} C\Phi K_F = C\Phi K_F = T_{KF} = T_B$$

Output



1. Add integrators, and reduce the system to a minimal realization (if necessary).
2. Design the filter computing $C\Phi(s)K_F$.
3. From specifications find $p(\omega)$ and $l_m(\omega)$, and build the SV barriers.
4. Find (Γ,μ) so that $\frac{1}{\sqrt{\mu}}\sigma_i[C\Phi\Gamma]$ satisfy the low frequency barrier, and $\bar{\sigma}\left[\frac{C\Gamma}{\sqrt{\mu}}\right] \le \omega_c$ with ω_c such that $l_m(\omega_c) = 0dB$.
5. Once (Γ,μ) are selected, solve for K_F using Riccati. Verify that at low frequency $\sigma_i[C\Phi K_F] \approx \frac{1}{\sqrt{\mu}}\sigma_i[C\Phi\Gamma]$
6. Verify robustness: $\sigma_i\left[I + T_{KF}^{-1}\right] \ge \frac{1}{2}$, with $T_{KF} = C\Phi K_F$
7. Additional robustness test: $\frac{1}{l_m(\omega)} > \bar{\sigma}\left[T_{LQ}\left(I + T_{LQ}\right)^{-1}\right]$;
8. Compute transmission zeros of $C\Phi(s)B$ to verify that the system is minimum phase;
9. Perform recovery with $q_0 \to \infty$ using the system:

$$\begin{cases} A^T P + PA + Q_C - PBR_c^{-1}B^T P = 0 \\ Q_C = H^T H + q_0^2 C^T V C \;,\; q_0 \to \infty \;\; V > 0 \\ R_c = \rho\, I \end{cases} \qquad q_0 \to \infty$$

since $K_C = \frac{B^T P}{\rho}$ at each iteration compute:

$$K_{LQG}(s) = K_c\left[sI - A + BK_c + K_F C\right]^{-1} K_F$$

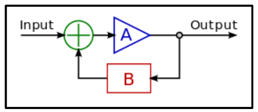
and $T_{OUT}(s) = G(s)K_{LQG}(s) = T_B(s)$

Iteration continues until:

$$\sigma_i\left[T_{OUT}\right] \approx \sigma_i\left[T_{KF}\right]$$

10. Verify the design, checking

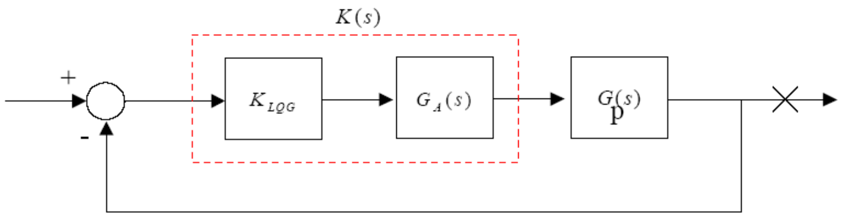
$$\underline{\sigma}\left[I + T_{INP}^{-1}\right] \ge \frac{1}{2}$$



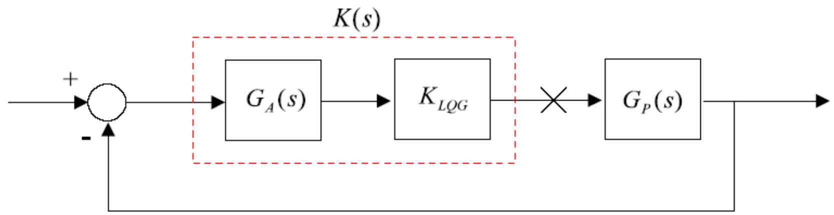
Loop Transfer Recovery for Tracking: Plant Augmentation

- Standard procedure when integrators are needed for steady state error requirements. In MIMO systems banks of integrators are added, in required number, in different points of the loop.

Output



Input



- The augmented system is:

$$G(s) = G_p(s)G_A(s) \qquad G_A(s) = \frac{1}{s}I$$

$$\begin{cases} \dot{x} = \begin{bmatrix} A_p & B_p C_A \\ 0 & A_A \end{bmatrix} \mathrm{x} + \begin{bmatrix} 0 \\ B_A \end{bmatrix} u + \Gamma\zeta \\ y_p = \begin{bmatrix} C_p & 0 \end{bmatrix} \mathrm{x} + n \end{cases}$$

- The augmented system is:

$$G(s) = G_A(s)G_p(s) \qquad G_A(s) = \frac{1}{s}I$$

$$\begin{cases} \dot{x} = \begin{bmatrix} A_A & B_A C_p \\ 0 & A_p \end{bmatrix} x + \begin{bmatrix} 0 \\ B_p \end{bmatrix} u + \Gamma\zeta \\ y_p = \begin{bmatrix} C_A & 0 \end{bmatrix} x + n \end{cases}$$

Parameter Selection for frequency response shaping

- Several approaches are present in the literature for helping the designer in the choice of the LQG/LTR design parameters (Γ and μ for filter shaping, H and ρ for regulator shaping), in order to achieve satisfactory frequency response for the loop transfer matrices.

  - **Note 1:** The scalars (μ, ρ) are gains, therefore tend to **translate** the singular values Bode plots. The matrices (Γ, H), on the other hand, modify the **shape** of the singular values.

  - **Note 2:** An important issue is the capability of making the frequency behavior of $\sigma_{\min}$ and $\sigma_{\max}$ similar, so that the system response closely resembles that one of the singular values. The available design parameters allow this at low and high frequency, while more uncertainty remains in the mid-range (crossover)

    - Following is an example of design parameters selection. For alternate techniques see Lavrestki, Wise, "Robust and Adaptive Control", Springer 2013, Chapters 3 and 6.

- **Loop open at the Output**

Assume integrators are added as needed for steady state error requirements (input to the plant). The system is (plant noises were neglected for simplicity):

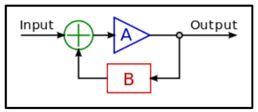
$$\begin{bmatrix}\dot{x}_a\\ \dot{x}_p\end{bmatrix}=\begin{bmatrix}0 & 0\\ B_pI & A_p\end{bmatrix}\begin{bmatrix}x_a\\ x_p\end{bmatrix}+\begin{bmatrix}I\\ 0\end{bmatrix}u=A\begin{bmatrix}x_a\\ x_p\end{bmatrix}+Bu \qquad A_a=0, B_a=C_a=I$$

$$y_p=\begin{bmatrix}0 & C_p\end{bmatrix}\begin{bmatrix}x_a\\ x_p\end{bmatrix}=C\begin{bmatrix}x_a\\ x_p\end{bmatrix}$$

- The transition matrix becomes

$$\Phi(s)=(sI-A)^{-1}=\begin{bmatrix}sI & 0\\ -B_p & sI-A_p\end{bmatrix}^{-1}=\begin{bmatrix}\frac{I}{s} & 0\\ (sI-A_p)^{-1}B_p & (sI-A_p)^{-1}\end{bmatrix}$$

$$T_B(s)=T_{KF}(s)=C\Phi(s)\Gamma=\begin{bmatrix}0 & C_p\end{bmatrix}\begin{bmatrix}\frac{I}{s} & 0\\ (sI-A_p)^{-1}\frac{B_p}{s} & (sI-A_p)^{-1}\end{bmatrix}\begin{bmatrix}\Gamma_1\\ \Gamma_2\end{bmatrix}=T_{FOL}(s)$$



$$\boxed{T_{FOL}(s)=C_p(sI-A_p)^{-1}\frac{B_p}{s}\Gamma_1+C_p(sI-A_p)^{-1}\Gamma_2}$$

- At low frequency we can assume:

$$(j\omega I - A_p)^{-1} \approx -A_p^{-1}; \omega = [0, \omega_{low}]$$

$$T_{FOL}(s) \approx -C_p A_p^{-1} \frac{B_p}{j\omega}\Gamma_1 - C_p A_p^{-1}\Gamma_2 \Rightarrow -C_p A_p^{-1} \frac{B_p}{j\omega}\Gamma_1$$

- Select: $\Gamma_1 = -(C_p A_p B_p)^{-1} = G_p^{-1}(0)$ $\qquad T_{FOL}(s) \approx -C_p A_p^{-1}\Gamma_2$

- For arbitrary Γ_2, The singular values of T_{FOL} have integral behavior and tend to coincide at low frequency

$$\Gamma = \begin{bmatrix} G_p^{-1}(0) \\ \Gamma_2 \end{bmatrix}$$

- At high frequency we can assume: $(j\omega I - A_p)^{-1} \approx \dfrac{I}{j\omega}; \omega \to \infty$

- In $T_{FOL}(s)$, the second term is dominant. Select: $\Gamma_2 = C_p^T (C_p C_p^T)^{-1}$

$$T_{FOL}(s) \approx C_p B_p \frac{\Gamma_1}{(j\omega)^2} + \frac{I}{j\omega}$$

- For arbitrary Γ_1, The singular values of T_{FOL} have integral behavior and tend to coincide at high frequency

$$\Gamma = \begin{bmatrix} \Gamma_1 \\ C_p^T (C_p C_p^T)^{-1} \end{bmatrix}$$

- **Loop open at the Input**

As before, assume the system is augmented with an appropriate number of integrators for steady state requirements

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} A_p & 0 \\ IC_p & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_a \end{bmatrix} + \begin{bmatrix} B_p \\ 0 \end{bmatrix} u = A \begin{bmatrix} x_p \\ x_p \end{bmatrix} + Bu_p \qquad A_a = 0, B_a = C_a = I$$

$$y_a = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} x_p \\ x_a \end{bmatrix} = C \begin{bmatrix} x_p \\ x_a \end{bmatrix}$$

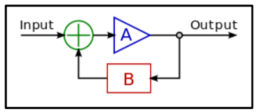
- **Results**:

Low frequency:

$$H = \begin{bmatrix} H_1 & G_p^{-1}(0) \end{bmatrix}$$

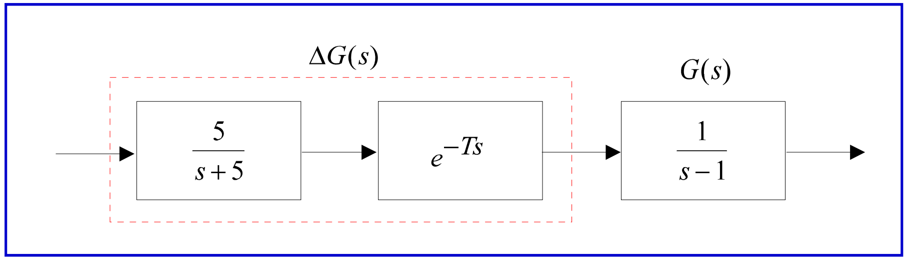
High frequency:

$$H = \begin{bmatrix} (B_p^T B_p)^{-1} B_p^T & H_2 \end{bmatrix}$$



LQG_LTR Procedure using MATLAB

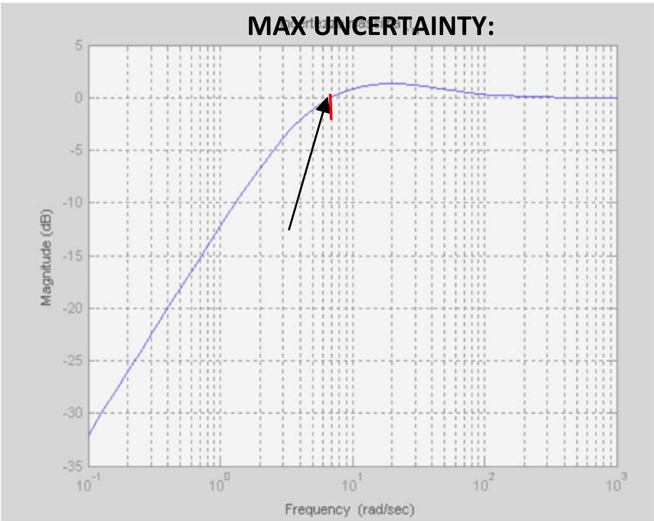
❑ Example



Given the system: $\tilde{G}(s) = \dfrac{5\ e^{-Ts}}{(s+5)(s-1)}$

The model is : $G(s) = \dfrac{1}{s-1}$ Delay time T=0.05 sec.

The frequency response of the maximum input uncertainty is given by:



$$
\begin{aligned}
l_m(s) &= \sigma_{\max}\left[L(s)\right] = \left|\Delta G(s) - 1\right| \\
&= \left|\frac{-0.125s + 4.9984 - 0.025s^2 - 1.1246s - 4.9984}{0.025s^2 + 1.1246s + 4.9984}\right| \\
&= \frac{s\ (s + 49.984)}{(s + 39.988)(s + 5)}
\end{aligned}
$$

with crossover frequency

$$\omega_{CR} \approx 7\ \mathrm{rad/sec}$$

We wish to design a controller such that:

- the closed loop system is stable
- the steady state error to a unit step is 0
- the closed loop system is robust to the assumed uncertainty.

- Phase 1: Plant Model Augmentation

An integrator is added for steady state requirements (the pole is set to -0.001 because of the recovery constraints).

$$G_{aug}(s) = \frac{1}{(s+0.001)(s-1)}$$

- Phase 2: Definition of Target TFM

The target TFM comes from the LQR design (input uncertainty)

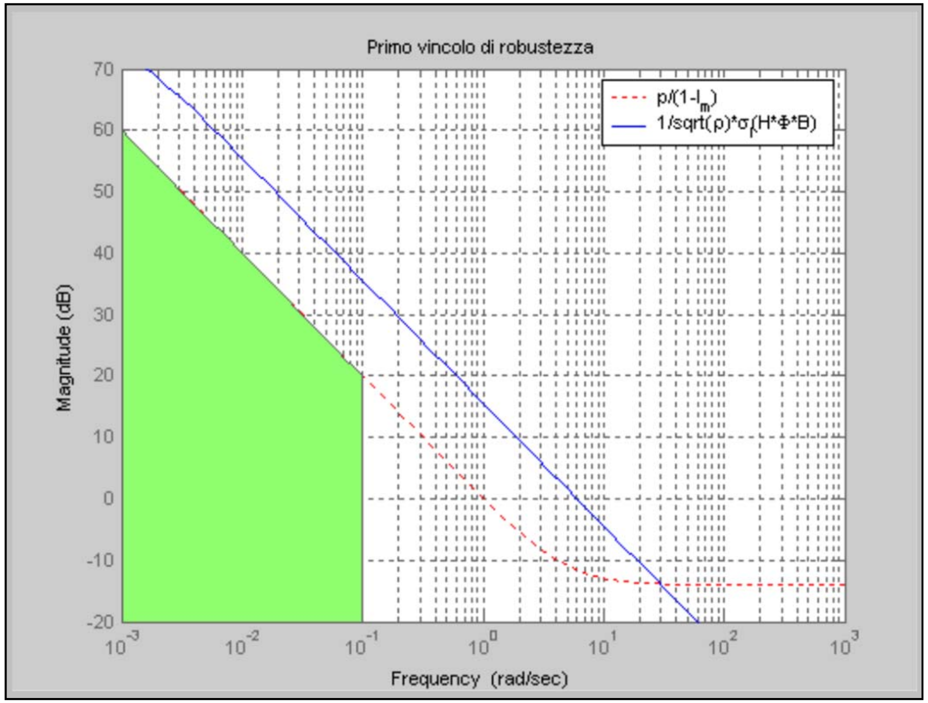
$$(H,\rho\ ) \rightarrow \quad \frac{1}{\sqrt{\rho}}\sigma_i\left[H\Phi B\right] \quad \rightarrow \quad K_c\Phi(s)B$$

The LQR design parameters are found based on low and high frequency requirements:

$$\frac{1}{\sqrt{\rho}}\sigma_{\min}\left[H\Phi B\right] \geq \frac{p}{1-l_m}, \omega \leq 0.1 \qquad \sigma_{\max}\left[\frac{HB}{\sqrt{\rho}}\right] \leq 7\, rad\ /\sec$$

- Phase 3: Performance and Stability Design with LQR

Select $H = \text{diag}\ (2,\ 2)$ and $r = 1$.



- Based on the above, we can compute K_c = (5.9930, 7.9990) using the LQR, and all requirements are satisfied.

The first plot shows the validity of the approximation

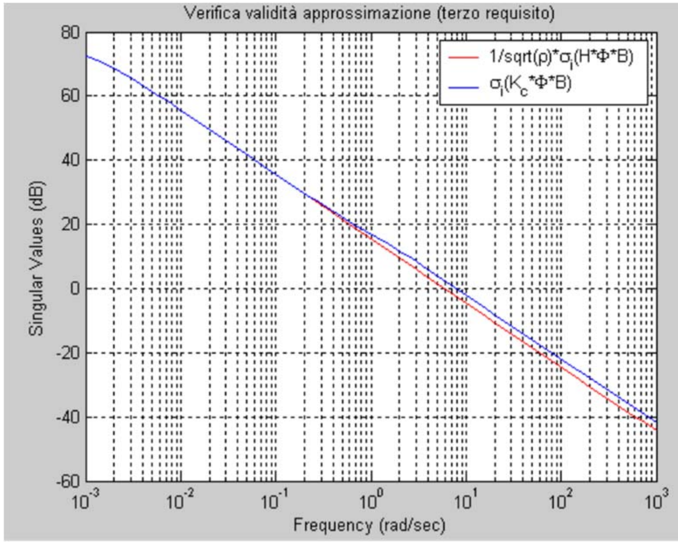
$$\sigma_i\left[T_{LQ}\right]=\sigma_i\left[K_c\Phi B\right]\approx\frac{1}{\sqrt{\rho}}\sigma_i\left[H\Phi B\right]$$

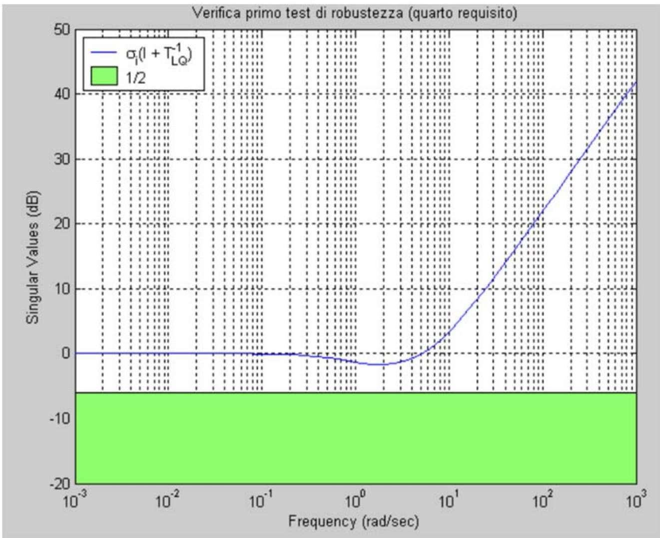
The second plot shows the LQR stability margins verification

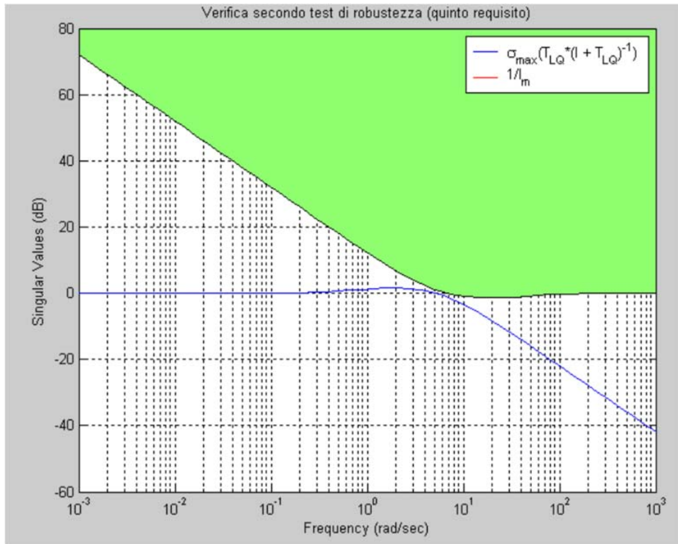
$$\sigma_i\left[I+T_{LQ}^{-1}\right]>\frac{1}{2}$$

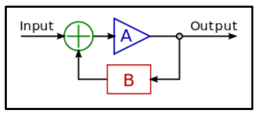
The third plot shows the stability robustness test

$$\frac{1}{l_m(\omega)}>\sigma_{\max}\left[T_{LQ}\left(I+T_{LQ}\right)^{-1}\right]$$









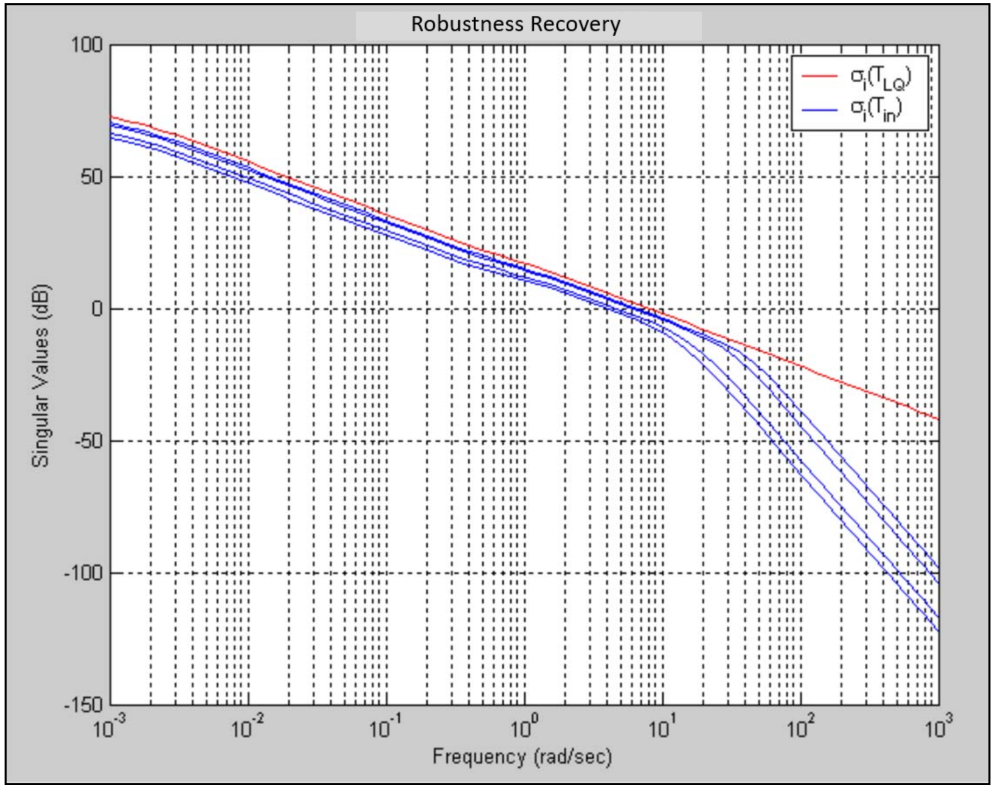
- Phase 4: Transfer Recovery using the Filter

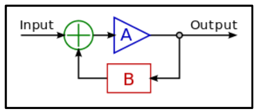
$$\begin{cases} A\Sigma + \Sigma A^T + \Gamma\Gamma^T - \frac{1}{\mu}\Sigma C^T C\Sigma = 0 \\ Q_F = Q_0 + q_0^2 BVB^T \;, Q_0 = \Gamma\Gamma^T = BB^T \;, V > 0 \\ R_F = \mu I = 0.01I \end{cases}$$

- The iteration using q_0 gives satisfactory values for $q_0=100$

- Loop transfer function open at the input

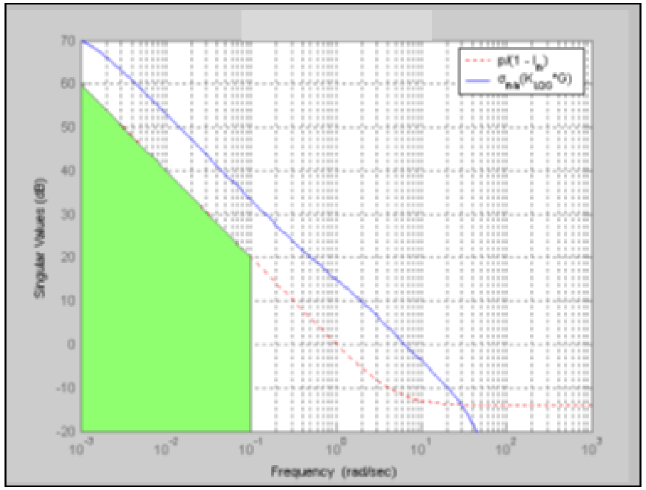
```
Zero/pole/gain:
        12071.133 (s+0.7031)
-------------------------------------
(s-1) (s+0.001) (s^2  + 61.19s + 1854)
```

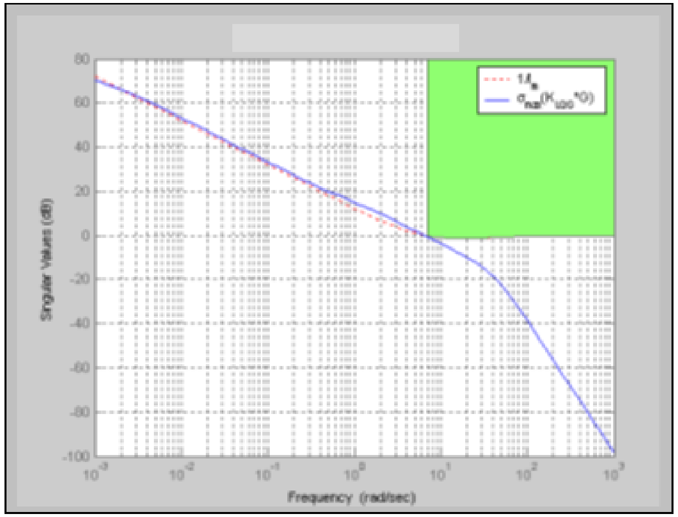


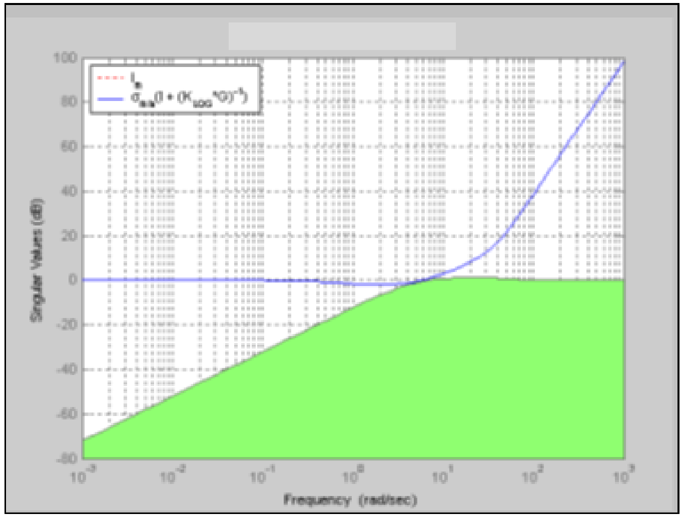


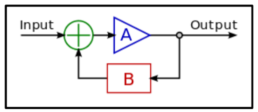
- Phase 5: Final Synthesis

$$
\begin{cases}
\sigma_{\min}\left[KG\right] \geq \dfrac{p(\omega)}{1-l_m\left(\omega\right)}, \omega < \omega_1 \ \Re^P \\
\sigma_{\max}\left[KG\right] < \dfrac{1}{l_m\left(\omega\right)}, \omega > \omega_2 \ \Re^P \\
\sigma_{\min}\left[I + \left(KG\right)^{-1}\right] > l_m\left(\omega\right), \forall\omega \ \Re^S
\end{cases}
$$





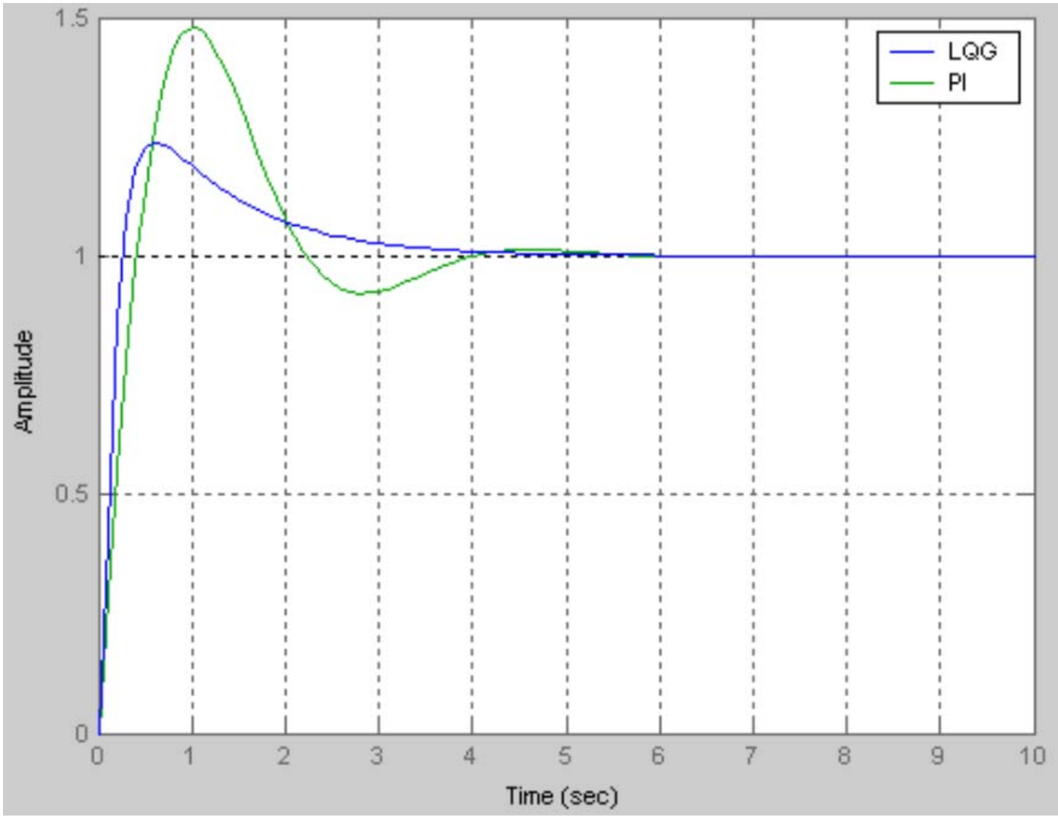




- Comparison with previous controller (PI)

Zero/pole/gain:

$$\frac{3\ (s+1.333)}{s}$$



❑ Example 2

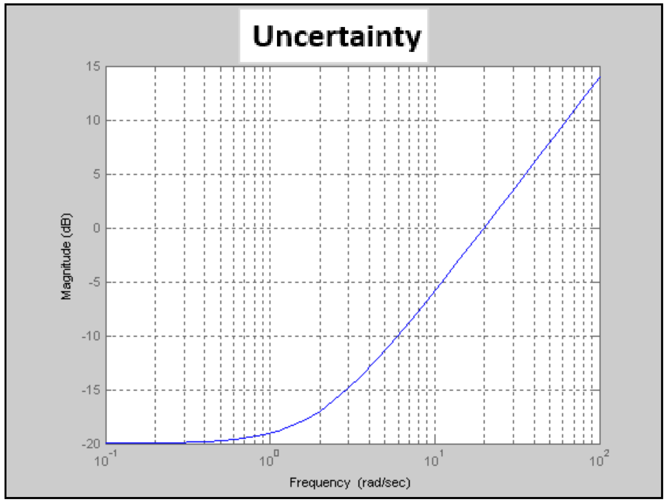
- **Design Specifications**

The design objectives are good steady state performance and robustness:

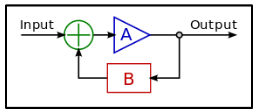
- -20 dB/dec slope at low frequency (<0.1)
- Bandwidth below 10 rad/sec (closed loop)
- Good output decoupling
- Good robustness with respect to an uncertainty due to actuator dynamics.

The uncertainty models a 10% inaccuracy in the model up to 2 rad/sec, and the fact that such inaccuracy grows at high frequency with slope of +20 dB/dec. Thus:

$$L(j\omega) = \frac{j\omega + 2}{20} \Rightarrow l_m(\omega) = \left|\frac{j\omega + 2}{20}\right|$$



- The crossover frequency is about 20 *rad/sec*. As shown in the uncertainty representation.



- System Model (Short Period Dynamics plus first order Actuators, landing flight condition)

$$x = \begin{bmatrix} \gamma & q & \alpha & \delta_E & \delta_F \end{bmatrix}^T$$

$$u = \begin{bmatrix} \delta_{EC} & \delta_{FF} \end{bmatrix}^T$$

$$y = x$$

$$z = \begin{bmatrix} \theta & \gamma \end{bmatrix}^T = C_p x$$

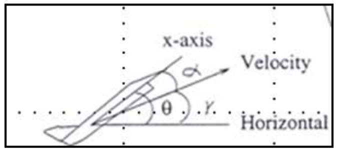
Ap =

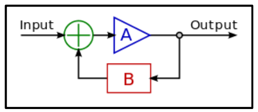
$$\begin{matrix} 0 & 0.0066 & 1.3411 & 0.1690 & 0.2518 \\ 0 & -0.8694 & 43.2230 & -17.2510 & -1.5766 \\ 0 & 0.9933 & -1.3411 & -0.1690 & -0.2518 \\ 0 & 0 & 0 & -20.0000 & 0 \\ 0 & 0 & 0 & 0 & -20.0000 \end{matrix}$$

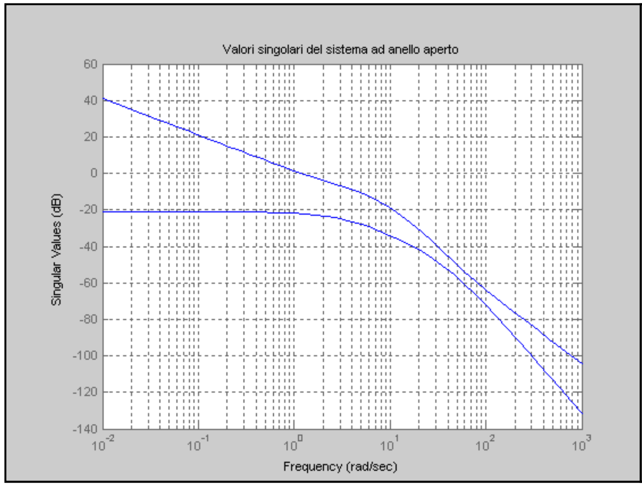
Bp =

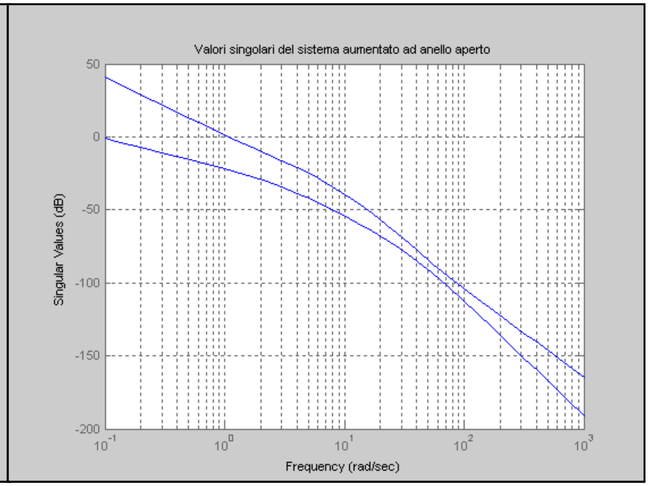
$$\begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{matrix}$$

- The singular values of the model indicate the necessity of adding integrators for steady state error requirements (2 integrators)









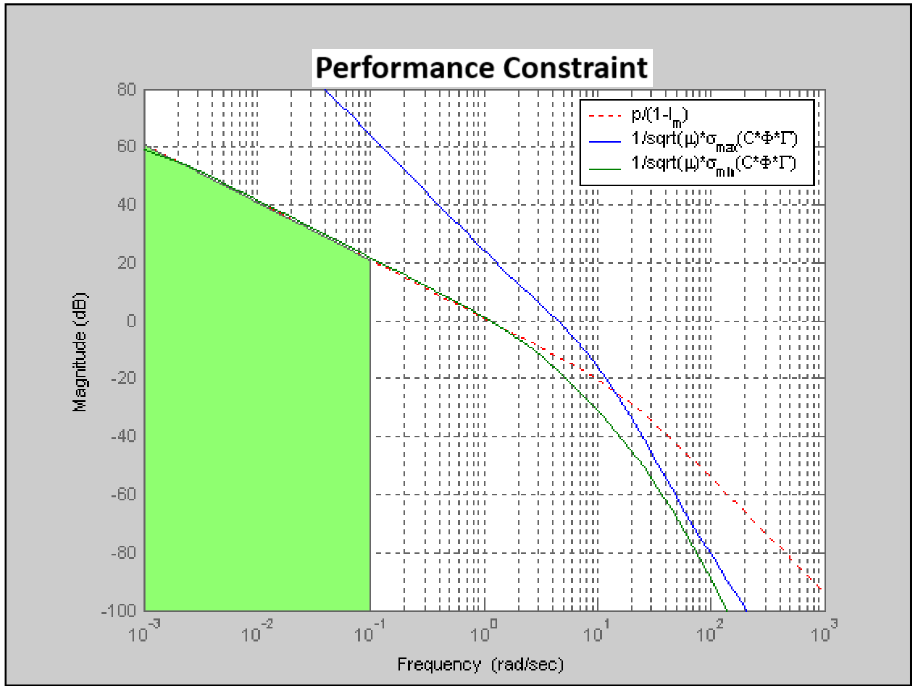
- Consider an output uncertainty, use of the filter for performance and robustness specifications

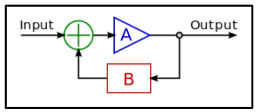
$$\sigma_{\max}\left[\frac{C\Gamma}{\sqrt{\mu}}\right] \le \omega_C;\ \frac{1}{\sqrt{\mu}}\sigma_i[C\Phi\Gamma] > \frac{p(\omega)}{1-l_m(\omega)}$$

$$\begin{cases} \omega_C = 20 \\ p(\omega) = \dfrac{1}{s} \\ l_m(\omega) = \left|\dfrac{j\omega+2}{20}\right| \end{cases}$$

- The first constraint is satisfied, $\sigma_{\max}\left[\frac{C\Gamma}{\sqrt{\mu}}\right] = 0; \omega_C \approx 5$ since:

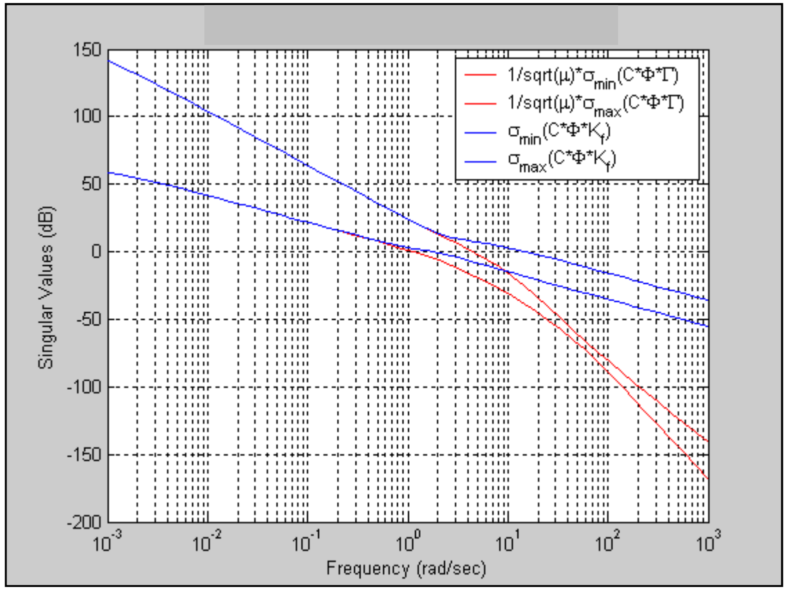
- The second constraint is also satisfied (barely)



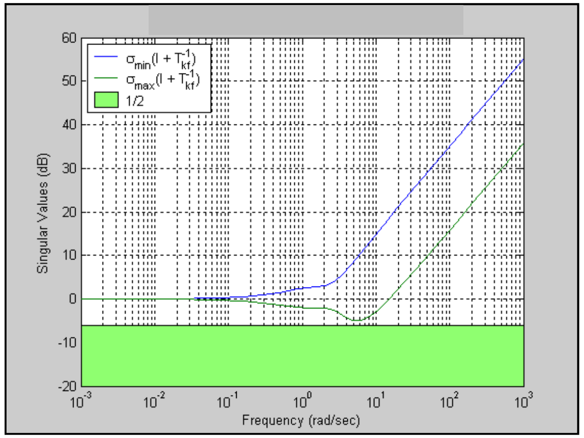


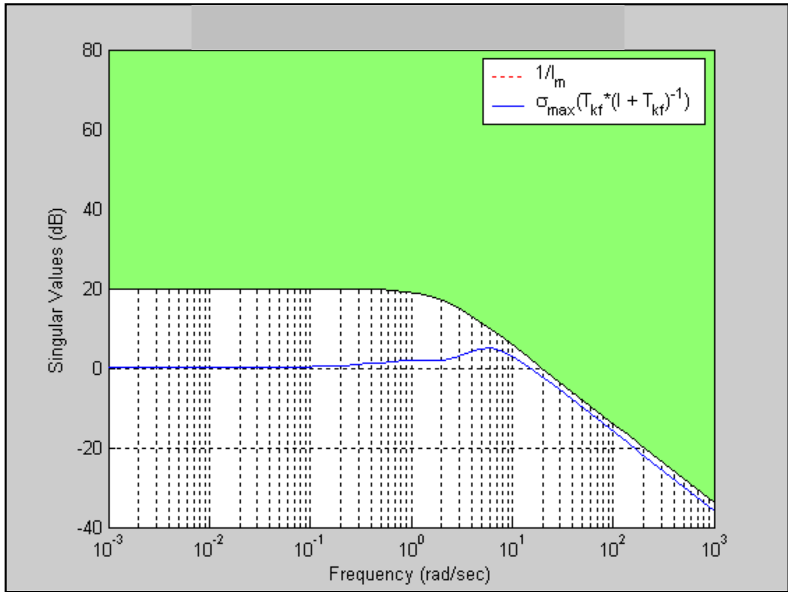
- Compute the Kalman filter gain, and evaluate the approximation

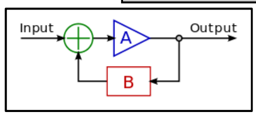
```
K_f =

    0.8850     1.8145
  130.3232    -2.7376
   15.2352    -0.9295
  -13.9719    -1.3389
   -1.2990    14.0128
  -14.0746    -1.3063
   -1.3118    14.0656
```



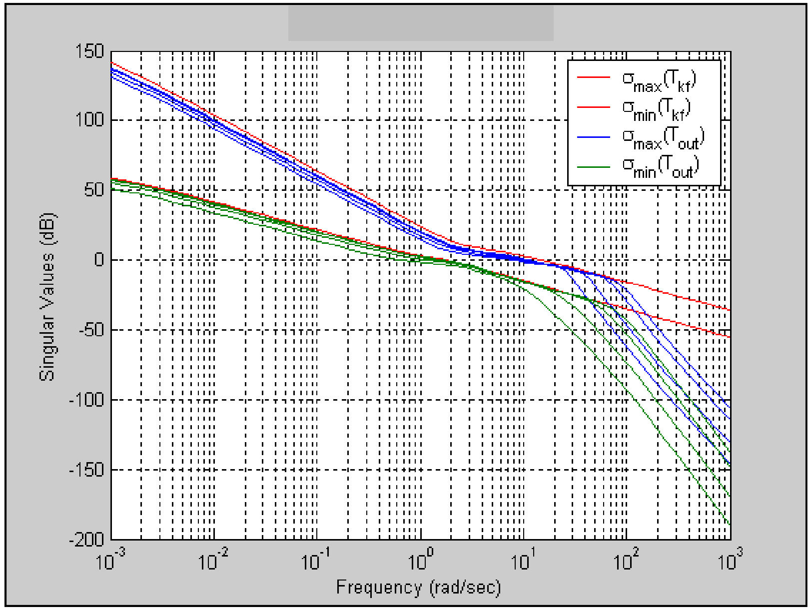
- Robustness Analysis

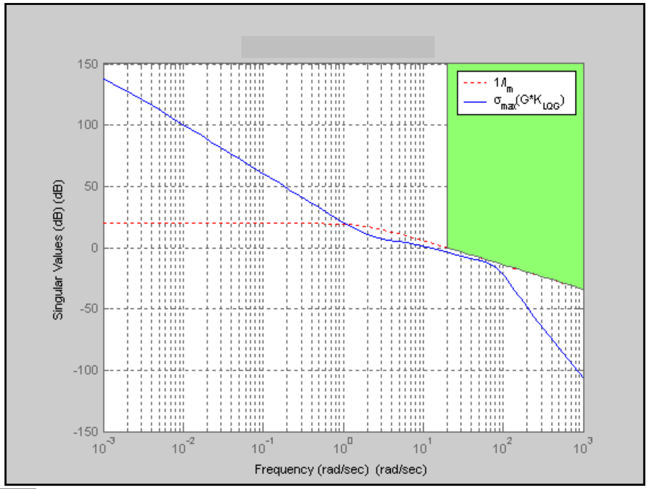


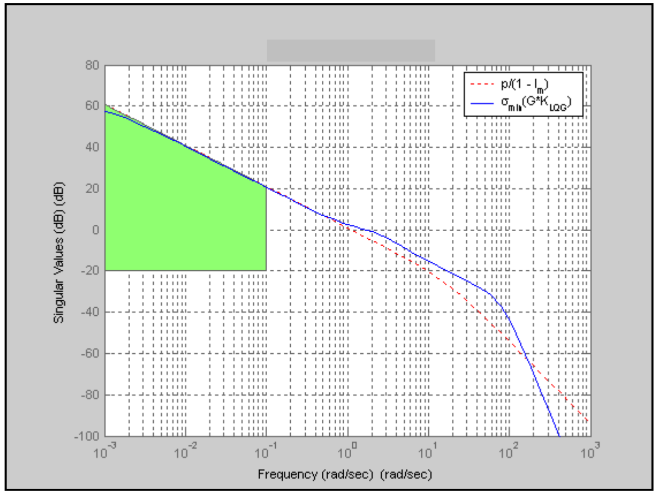


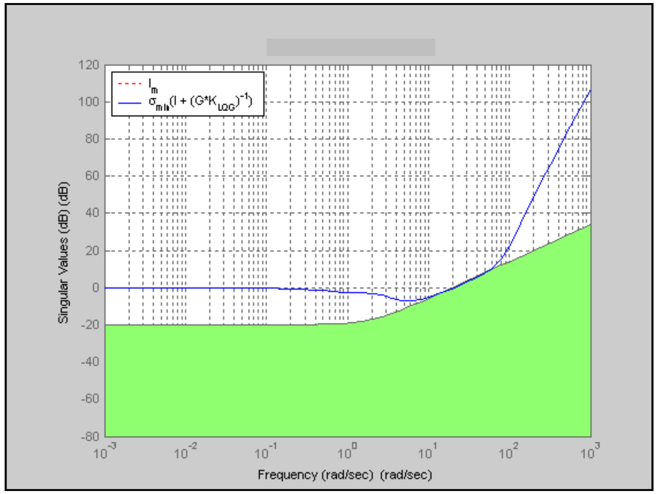


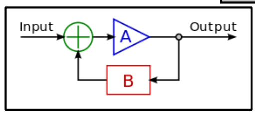
- **Recovery Process (q= 3162.23):**





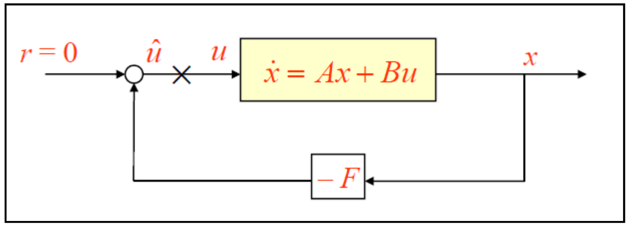






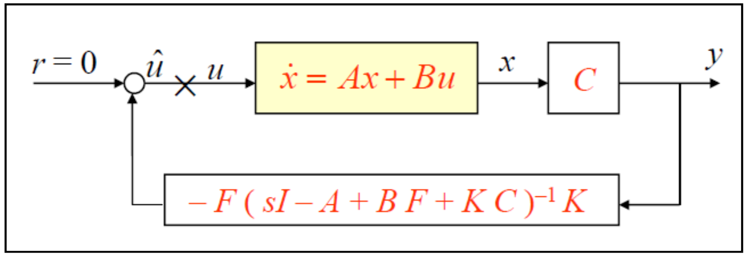
❑ The procedure illustrated previously has some limitations:

- Square plant (The recovery inverts the plant)
- Minimum phase system (Existence of inverse, well posedness)
- Full order observer (Compensator size)
- High Gain for full recovery (Highly sensitive to high frequency noise and disturbances)
- Perturbation Set limited to Unstructured uncertainties



$$L_t(s) = F(sI - A)^{-1}B$$

$$\begin{cases} u' = -H_1 H_2 \hat{x} \\ \hat{x} = \Phi(s)\left[B\left(C\Phi B\right)^{-1} - K_F\left(I + C\Phi K_F\right)^{-1}\right]C\Phi B u' \\ +\Phi\left[K_F\left(I + C\Phi K_F\right)^{-1}\right]C\Phi B u'' \end{cases}$$



$$L_o(s) = F(sI - A + BF + KC)^{-1}KC(sI - A)^{-1}B$$

- From previous results (Doyle – Stein Conditions): we wish to have $L_t = L_o$

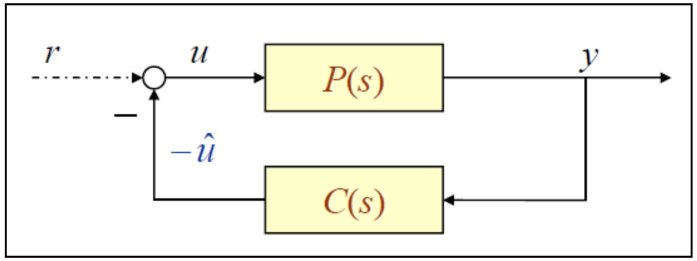
- **Procedure:** Consider a stabilizable and detectable system

$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x + D\,u \end{cases} \qquad P(s) = C\Phi B + D,\;\; \Phi = (sI - A)^{-1}$$

- **Synthesis Step 1:** Design a static compensator $u = -Fx$ (use **ANY** method) to meet:
  - Nominal closed loop stability and performance requirements
  - Resulting loop transfer matrix $L_t(s)$ has appropriate frequency shape for robustness

$$L_t(s) = F\Phi B$$

- **Synthesis Step 2:** Design a dynamic compensator $C(s)$ such that the loop transfer matrix $L_o(s)$ equals or approximates $L_t(s)$ (this becomes the recovery part of the design)

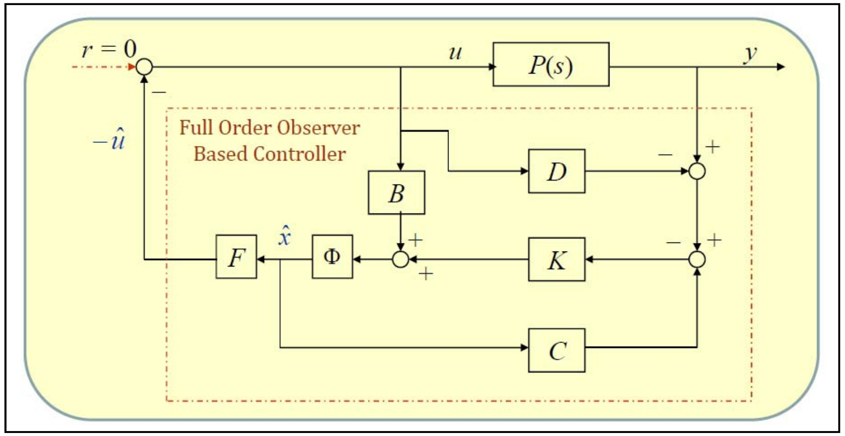


$$L_o(s) = C(s)P(s) \longrightarrow L_t(s)$$

- Make Error Mismatch go to zero using an optimization procedure (f.i. Newton – Rapson)

$$E(s) = L_t(s) - L_o(s) = F\Phi B - C(s)P(s) \;\longrightarrow\; 0$$

- Full observer structure



$$\dot{\hat{x}} = A\,\hat{x} + B\,u + K(y - C\hat{x} - D\,u), \quad \hat{u} = u = -F\,\hat{x}$$

$$C(s) = C_o(s) = F(\Phi^{-1} + BF + KC - KDF)^{-1}K$$

$$L_o(s) = C_o(s)P(s) = F(\Phi^{-1} + BF + KC - KDF)^{-1}K(C\Phi B + D)$$

Lemma: Recovery error, $E_o(s)$, i.e., the mismatch between the target loop and the resulting open-loop of the observer based controller is given by

$$E_o(s) = M(s)\left[I + M(s)\right]^{-1}(I + F\Phi B), \quad \underline{M(s) = F(\Phi^{-1} + KC)^{-1}(B - KD)}$$

LOOP TRANSFER RECOVERY FOR GENERAL NONMINIMUM PHASE NON-STRICTLY PROPER SYSTEMS, PART 1—ANALYSIS*

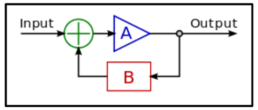
B. M. Chen,[1] A. Saberi[1] and P. Sannuti[2]

CONTROL-THEORY AND ADVANCED TECHNOLOGY
Vol. 8, No.1, pp.59-100, March, 1992

LOOP TRANSFER RECOVERY FOR GENERAL NONMINIMUM PHASE NON-STRICTLY PROPER SYSTEMS, PART 2—DESIGN*

B. M. Chen,[1] A. Saberi[1] and P. Sannuti[2]

CONTROL-THEORY AND ADVANCED TECHNOLOGY
Vol. 8, No.1, pp.101-144, March, 1992



- **Auxiliary 2 – Block Structure**: The previous Lemma is equivalent to finding a gain matrix K, for the system:

$$\Sigma_{\text{aux}}: \begin{cases} \dot{x} = A^{\mathrm{T}} x + C^{\mathrm{T}} u + F^{\mathrm{T}} w \\ y = x \\ z = B^{\mathrm{T}} x + D^{\mathrm{T}} u \end{cases} \quad + \quad u = -K^{\mathrm{T}} x$$

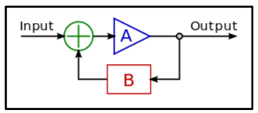
➡ Closed-loop transfer function from w to z is $(B^{\mathrm{T}} - D^{\mathrm{T}} K^{\mathrm{T}})(sI - A^{\mathrm{T}} + C^{\mathrm{T}} K^{\mathrm{T}})^{-1} F^{\mathrm{T}} = M^{\mathrm{T}}(s)$.

Thus, LTR design is equivalent to design a state feedback law for the above auxiliary system such that certain norm of the resulting closed-loop transfer function is made either identically zero or arbitrarily small. As such, the H_2 and H_∞ optimization techniques can be used to solve the LTR problem. There is no need to repeat all over again once this is formulated.

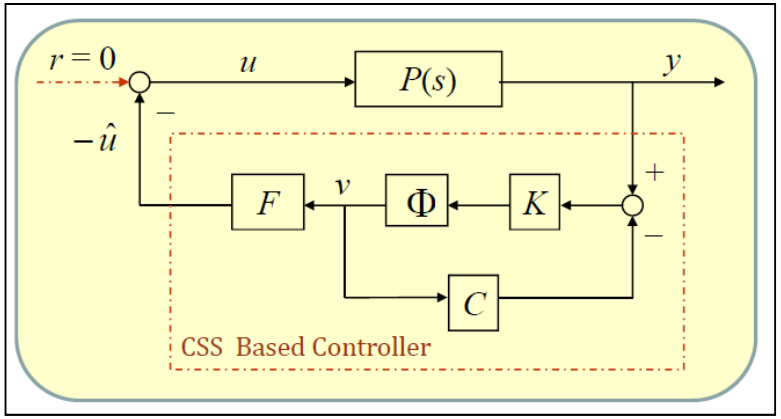
A New Stable Compensator Design for Exact and Approximate Loop Transfer Recovery*

B. M. CHEN,† A. SABERI† and P. SANNUTI‡

For minimum phase MIMO systems, a LTR design is presented via a new compensator which is stable and which allows much lower gain and thus lower controller band-width than the conventional observer based controller.



- **Reduced Order Observer**: The paper presents a recovery procedure using a reduced order observer:



Dynamic equations of $C(s)$: $\dot{v} = (A - KC)\, v + Ky, \quad \hat{u} = u = -F\, v$

Transfer function of $C(s) = C_c(s) = F(\Phi^{-1} + KC)^{-1} K$

Achieved open-loop: $L_c(s) = C_c(s)P(s) = F(\Phi^{-1} + KC)^{-1} K(C\Phi B + D)$

Lemma: Recovery error, $E_c(s)$, i.e., the mismatch between the target loop and the resulting open-loop of the CSS architecture based controller is given by

$$E_c(s) = M(s) = F(\Phi^{-1} + KC)^{-1}(B - KD)$$

Proof.

$$\begin{aligned} E_c(s) &= L_t(s) - L_c(s) \\ &= F\Phi B - F(\Phi^{-1} + KC)^{-1} K(C\Phi B + D) \\ &= F\underline{(\Phi^{-1} + KC)^{-1}}[\underline{(\Phi^{-1} + KC)}\Phi B - KC\Phi B - KD] \\ &= F(\Phi^{-1} + KC)^{-1}(B + KC\Phi B - KC\Phi B - KD) \\ &= M(s) \end{aligned}$$

It is clear that LTR via the CSS architecture based controller is achievable if and only if one can design a gain matrix K such that the resulting $M(s)$ can be made either identically zero or arbitrarily small. This is identical to the LTR design via the observer based controller. Thus, one can again using the H_2 and H_∞ techniques to carry out the design of such a gain matrix.

Example: Consider a given plant characterized by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 2 & 1 \end{bmatrix} x + 0 \cdot u$$

Let the target loop $L_t(s) = F\,\Phi B$ be characterized by a state feedback gain $F = \begin{bmatrix} 50 & 10 \end{bmatrix}$.