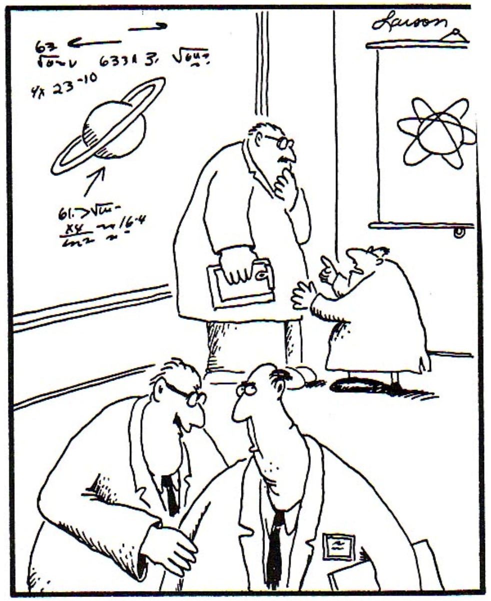
Orbital Mechanics

- **Newtonian Mechanics:**
  - Main assumptions/definitions
  - Reference Systems, Rigid Body Motion
- **Atmospheric Flight Dynamics:**
  - Reference systems
  - A/C Nonlinear Equations of Motion
  - Applied forces and moments
  - Geometry, Flight Envelope, Equilibrium Conditions
  - Linearization, Natural Modes, Stability Derivatives
- **A/C Control** Sensors, Actuators, Primary Control Loops
  - Stability Augmentation, Autopilots
  - Quadcopter D/C
- **Orbital Mechanics:**
  - Law of Gravitation, 2-Body Problem
  - Keplerian Orbits, Kepler Equation
  - Lambert Theorem, Orbit Transfer
  - Patched Conics, Restricted 3-Body Problem
- **S/C Control**
  - Stability, Open Loop/Closed Loop
  - Attitude Control
  - Sensors, Actuators, Power Limitations
- **Proximity Operations:**
  - Euler-Hill Equations
  - Rendez Vous and Docking
  - Formation Flight
- **Basic Human Operator in - the - Loop**
  - Operator in the Control Loop, Classical Models
  - MMI, Haptics

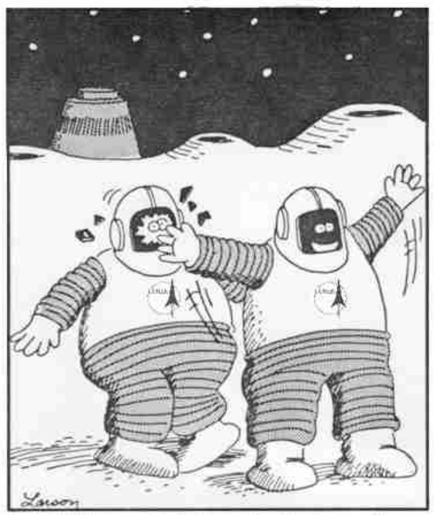
References:

- Richard H. Battin (MIT): An Introduction to the Mathematics and Methods of Astrodynamics, AIAA Education Series, American Institute of Aeronautics and Astronautics, Reston, VA, USA.
- Bong Wie (IASTATE): Space Vehicle Dynamics and Control, AIAA Education Series, American Institute of Aeronautics and Astronautics, Reston, VA, USA.
- Curtis Howard (Embry-Riddle): Orbital Mechanics for Engineering Students, Elsevier, 2014.
- Giovanni Mengali (UNIPI): Meccanica del Volo Spaziale, Edizioni Plus, Pisa.
- Bate Roger (USAF Academy): Fundamentals of Astrodynamics, Dover 1971.

- Kinematics of particles: Geometry of motion with negligible mass in different coordinate systems
- Kinetics: Branch of Dynamics that describes the motion of particles and bodies
- Dynamics: Branch of Mechanics that studies the forces, which produce motion
- Euclidean Geometry: relative to 2D flat space
- Trigonometry
- Newtonian Mechanics (Law of gravity)



"There goes Williams again…trying to win support for his Little Bang theory."

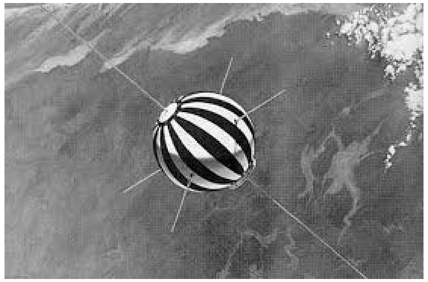


"We've made it, Warren! ... The moon!"

Space Flight Mechanics - Indian Institute of Technology

David Butler - How Far Away is it Series

Italy and Space



Launch Date	Vehicle	Payload	NSSDC ID	Comments
15-12-1964	Scout-X4	San Marco 1	1964-084A	
26-04-1967	Scout B	San Marco 2	1967-038A	
24-04-1971	Scout B	San Marco 3	1971-036A	
18-02-1974	Scout D-1	San Marco 4	1974-009A	
01-07-1983	Scout G-1	San Marco D/M	SANM-DM	Cancelled
25-03-1988	Scout G-1	San Marco D/L	1988-026A	

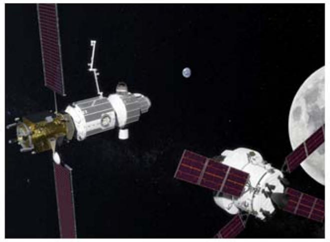
- Tethered Satellite System-1 (TSS-1) was proposed by NASA and the Italian Space Agency (ASI) in the early 1970s. It was a joint NASA-Italian Space Agency project, was flown in 1992, during STS-46 aboard the Space Shuttle Atlantis from 31 July to 8 August.
- It stuck at 78 meters; after that snag was resolved its deployment continued to a length of 256 meters (840 ft) before sticking again, where the effort finally ended (the total proposed length was 20,000 meters (66,000 ft)). A protruding bolt due to a late-stage modification of the deployment reel system, jammed the deployment mechanism and prevented deployment to full extension. Despite this issue, the results showed that the basic concept of long gravity-gradient stabilized tethers was sound.
- Four years later, as a follow-up mission to TSS-1, the TSS-1R satellite was released in latter February 1996 from the Space Shuttle Columbia on the STS-75 mission. The TSS-1R mission objective was to deploy the tether 20.7 km (12.9 mi) above the orbiter and remain there collecting data. The TSS-1R mission was to conduct exploratory experiments in space plasma physics. Projections indicated that the motion of the long conducting tether through the Earth's magnetic field would produce an EMF that would drive a current through the tether system. TSS-1R was deployed (over a period of five hours) to 19.7 km (12.2 mi) when the tether broke. The break was attributed to an electrical discharge through a broken place in the insulation

https://www.youtube.com/watch?v=tsEKGxoMGdA

Italy and Space

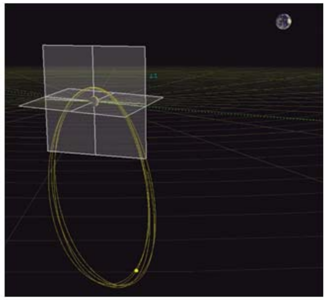
Member state/Source	ESA convention	National programme	Contr. (mill. €)	Contr. (%)	Contr. per capita (€)
Austria[note 1]	30 December 1986	FFG	47.6	1.3%	5.48
Belgium[note 2]	3 October 1978	BELSPO	188.9	5.0%	16.70
Czech Republic[note 3]	12 November 2008	Ministry of Transport	15.6	0.4%	1.48
Denmark[note 2]	15 September 1977	DTU Space	29.5	0.8%	5.17
Estonia[note 3]	4 February 2015	ESO	0.9	0.0%	0.68
Finland[note 3]	1 January 1995	TEKES	21.6	0.6%	3.94
France[note 2]	30 October 1980	CNES	844.5	22.6%	12.65 **2**
Germany[note 2]	26 July 1977	DLR	872.6	23.3%	10.62 **1**
Greece[note 3]	9 March 2005	ISARS	11.9	0.3%	1.10
Hungary[note 3]	24 February 2015	HSO	5.0	0.1%	0.51
Ireland[note 1]	10 December 1980	EI	23.3	0.6%	4.93
Italy[note 2]	20 February 1978	ASI	512.0	13.7%	8.44 **3**
Luxembourg[note 3]	30 June 2005	Luxinnovation	22.0	0.6%	38.18
Netherlands[note 2]	6 February 1979	NSO	102.6	2.7%	6.04
Norway[note 1]	30 December 1986	NSC	59.6	1.6%	11.43
Poland[note 3]	19 November 2012	POLSA	29.9	0.8%	0.79
Portugal[note 3]	14 November 2000	FCT	16.0	0.4%	1.55
Romania[note 3]	22 December 2011	ROSA	26.1	0.7%	1.32
Spain[note 2]	7 February 1979	INTA	152.0	4.1%	3.27
Sweden[note 2]	6 April 1976	SNSB	73.9	2.0%	7.50
Switzerland[note 2]	19 November 1976	SSO	146.4	3.9%	17.58
United Kingdom[note 2]	28 March 1978	UKSA	324.8	8.7%	4.97
Other	—	—	204.4	5.5%	

- Thales – Alenia Space
- Leonardo
- Telespazio
- Selex (Galileo Avionica)
- ELV
- MBDA
- ASI
- ESA
- Toscana Spazio (Alta, Kaiser, FlyBy, Intecs,…)



European Space Agency

Rendezvous Operations Simulation Software on Near-Rectilinear-Orbit – ROSSONERO

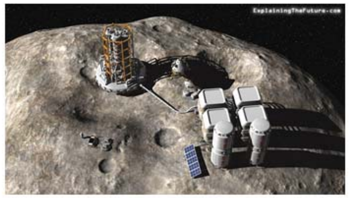


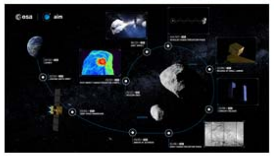
Study of the relative dynamics in three-body scenarios, and development of a simulation tool for preliminary guidance design.

Reference operational scenario: rendezvous and docking in a lunar near-rectilinear halo orbit with the *Deep Space Gateway* (NASA, Roscosmos).

European Space Agency

COSMO – COntrol and Stabilization of a spacecraft during asteroid Mining Operations





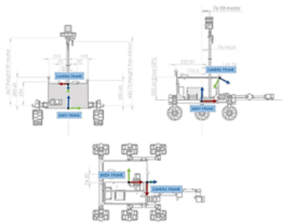
- The research activity "COntrol and Stabilization of a spacecraft during asteroid Mining Operations (COSMO)" aims at the definition of control algorithms and of a preliminary control technology evaluation needed for successful and safe mining operations on the surface of one or more asteroids)
- Development of a robust stabilization and precision control system that would allow the vehicle to maintain stability of operation in the presence of poorly known environmental parameters (soil characteristics at the site, variable mass and center of mass distribution, property of gravitational field, among others) and uncertainties in the force distribution (vibrations induced by drilling, relative motion between vehicle and body, stresses on the anchoring devices with potential actuator failures, etc.).

European Space Agency

ThalesAlenia Space

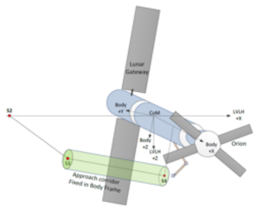
European Space Agency

Visual Odometry for Mars Lander dynamic Motion



- The ESA **Planetary Robotics Laboratory** is an engineering research laboratory that specialises in addressing the challenges robot probes face in the exploration of the surface of Moon and Mars, with particular attention to the motion aspect. As a probe moves on a far planetary surface it must have physical locomotion ability, navigation ability and logical autonomy. The lab is equipped to study and support research, development, validation and verification of all three aspects.

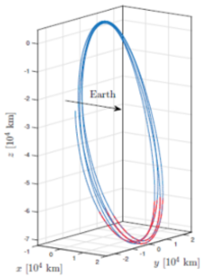
- The objective of the work is to evaluate the use of Visual Odometry (VO) in providing precision location data for ground navigation, and to test VO parameters in the ESTEC laboratory using the EXOTER rover.



European Space Agency

spin.works

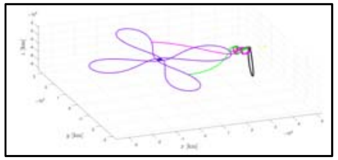
ANGLE Angles-only NRO rendezvous Guidance for Lunar Exploration



- **Bearings-Only rendezvous** consists in approaching a target in the far range having only angular measurements to this target. This technique has the potential to significantly reduce the mass of the navigations sensors required for rendezvous, since angular measurements can be obtained with an optical camera, which has significantly lower mass and power requirements. Nevertheless, since the **navigation problem is unobservable** without range measurements, smart guidance must be implemented to ensure the manoeuvres performed allow range estimation and ensure navigation filter convergence, which is compulsory for autonomous rendezvous. This technique has the potential to reduce mission cost by increasing autonomy and reducing requirements for the rendezvous in exploration missions such as the Cis-Lunar Transport Habitat (CTH) or sample return missions.

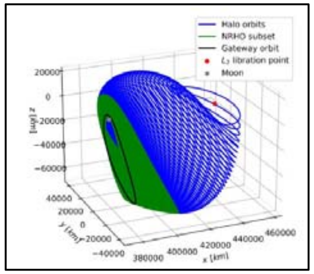
C-SLAM for heterogeneous mobile robots in lunar environment

- A challenge in current space robotics research is to maximise the potential for reuse and identification of extensions to existing building blocks, while minimising the risks associated with the integration activities. Using robotics hardware and software building blocks developed within the Strategic Research Cluster on Space Robotics Technologies, the EU-funded CoRob-X project introduces a multi-agent robotic team for the exploration of planetary surfaces.



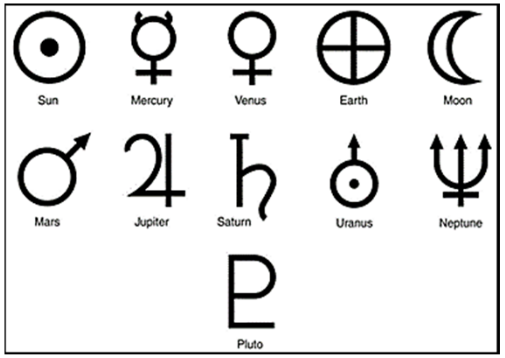
Cycler trajectory design for Earth – Moon transfer

- Gli attuali piani di ritorno sulla Luna prevedono la collocazione di una stazione spaziale che faccia da centro di comunicazione sull'orbita NRHO, una particolare traiettoria molto stabile facente parte della famiglia Southern Halo intorno il punto lagrangiano L2.
- L'obiettivo è quello di individuare possibili traiettorie che permettono una continua comunicazione tra la Terra e la NRHO. Questo può essere realizzato con un cycler, una tipologia di orbita che incontra periodicamente due corpi; in questo caso, individuare un cycler potrebbe garantire un continuo rifornimento di materiale e di necessità per i passeggeri della Lunar Gateway.



Design of Butterfly orbits for Earth - Moon phasing

- The increasing interests in Moon exploration has led in recent years to international collaborations between space agencies aimed to assemble and operate the Gateway, an orbiting spaceport located on a Near Rectilinear Halo Orbit (NRHO) about the second libration point of the Earth-Moon system (EML2). In this context, transfer strategies between the Earth and the spaceport cover a key role for both assembly and resupply missions.
- Under CR3BP, Poincare maps are employed to identify unstable/stable manifolds intersections in search of low-energy phasing trajectories that leave the reference orbit along the unstable branch before re-entering it via the stable one.

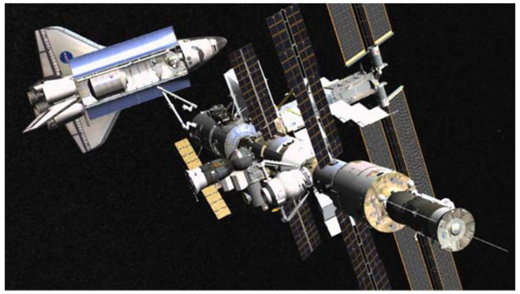


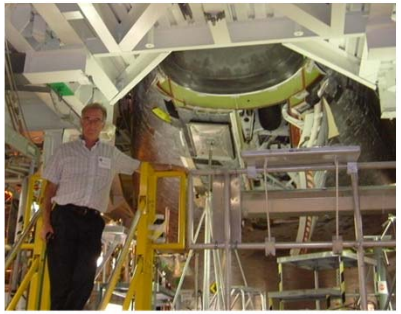
Orbital Mechanics

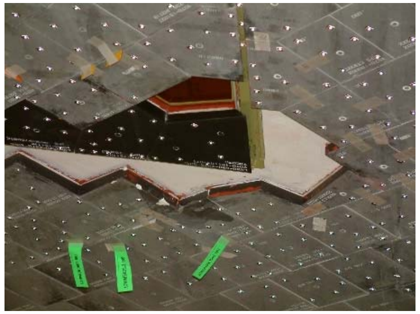
Salyut USSR 1975 - 1986

MIR USSR/RUS 1986 - 2001

ISS 1998 – 202?





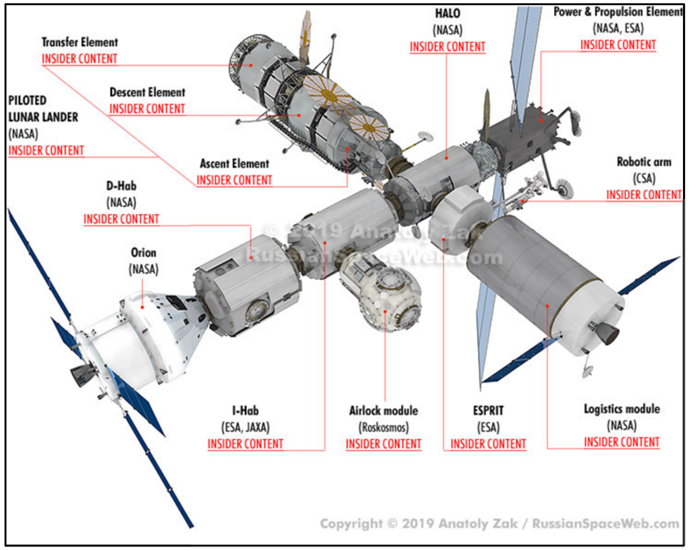


- The ISS is maintained in a nearly circular orbit with a minimum mean altitude of 330 km (205 mi) and a maximum of 410 km (255 mi), at an inclination of 51.6 degrees to Earth's equator. It completes 15.54 orbits per day (93 minutes per orbit).

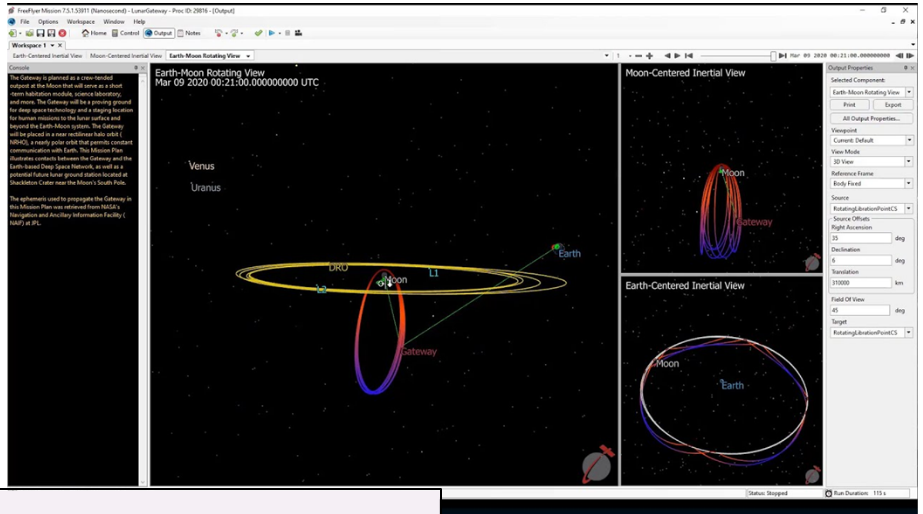
- Tiangong is a space station that the Chinese Manned Space Agency (CMSA) is building in low Earth orbit. In May 2021, China launched Tianhe, the first of the orbiting space station's three modules, and the country aims to finish building the station by the end of 2022.

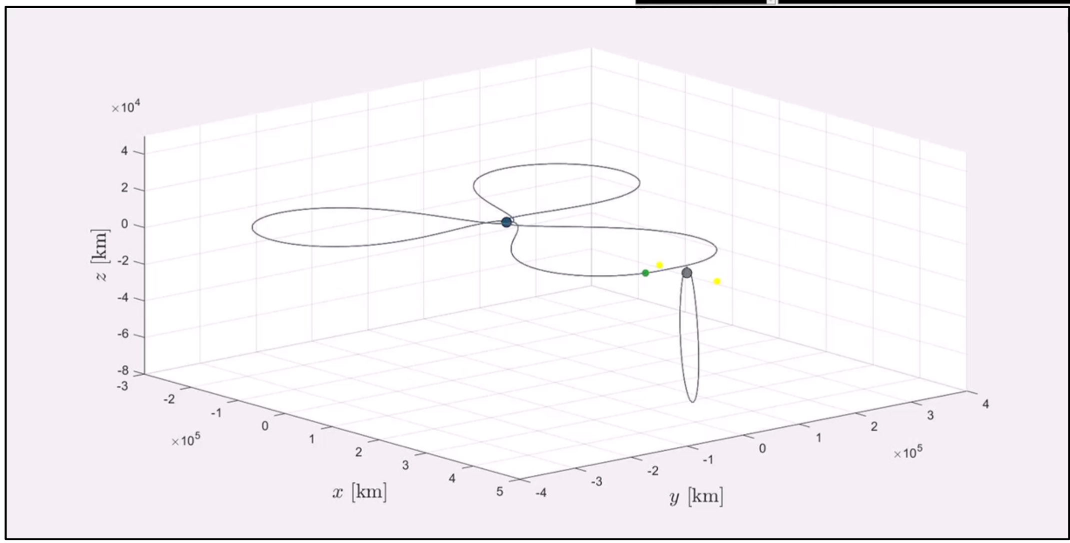
https://www.youtube.com/watch?v=iLEm5Nva2HI

Orbital Mechanics



- The Gateway, a vital component of **NASA's Artemis** program, will serve as a multi-purpose outpost orbiting the Moon that provides essential support for long-term human return to the lunar surface and serves as a staging points for deep space exploration. NASA is working with commercial and international partners to establish the Gateway.



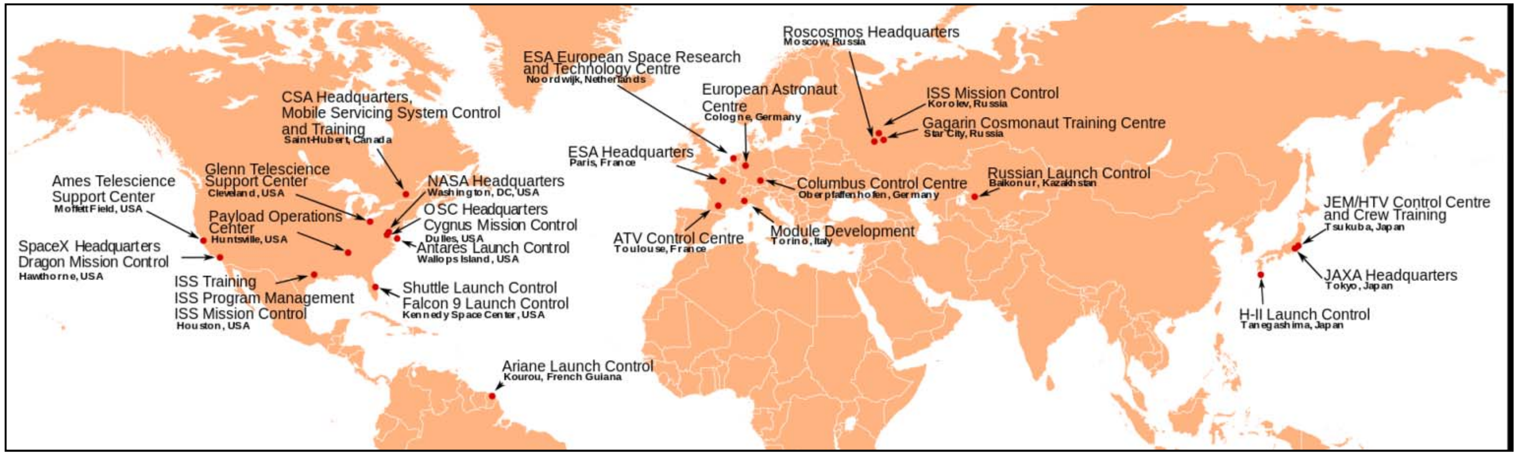


https://www.nasa.gov/specials/artemis/

http://lunarexploration.esa.int/intro

Orbital Mechanics

[SPACE X and Falcon 9 Back on Earth]



https://www.nasa.gov/mission_pages/juno/main/index.html

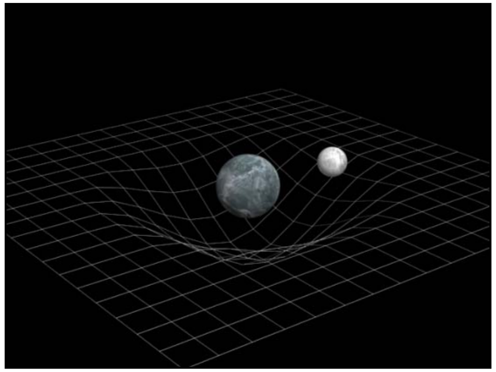
https://www.nasa.gov/mission_pages/newhorizons/main/index.html

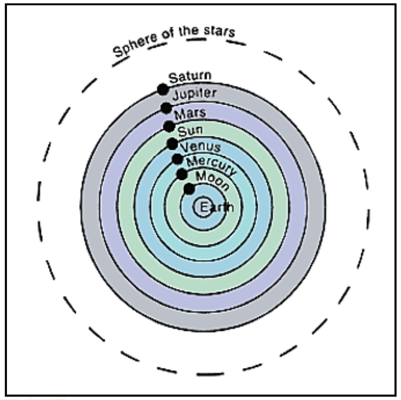
https://sci.esa.int/web/bepicolombo

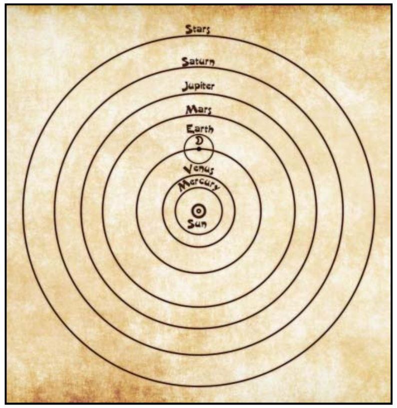
Definitions:

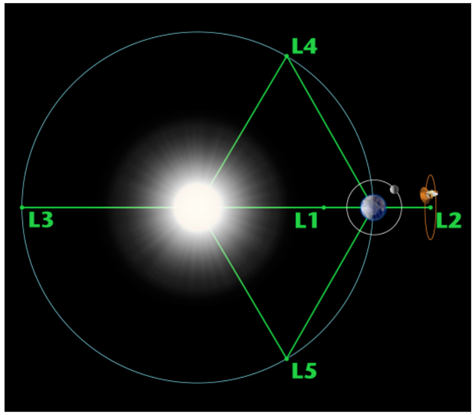
- **Celestial Mechanics**: motion of bodies in space, including relativity
- **Space Dynamics** = Orbital Mechanics + Attitude Dynamics
  - **Orbital Mechanics:** motion of space vehicles under Newtonian Forces
  - **Attitude Dynamics**: orientation and control of space vehicles
- **Astrodynamics**: motion of celestial objects under law of gravity (Newton)
- **Astronomy**: is the study of celestial objects and processes, the physics, chemistry, and evolution of such objects and processes. Astronomy is the oldest of the natural sciences
- **Physical Cosmology**: is concerned with studying the Universe as a whole.

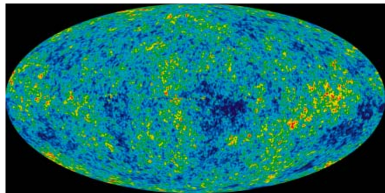
1. Eratosthenes
2. Ptolemy (accurate predictions)
3. Copernicus, Tyho Brahe, Galileo, Kepler
4. Newton, Halley
5. Einstein
6. Hubble
7. Von Braun



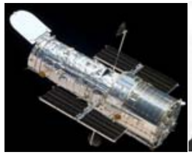








Edwin Hubble




- **Until circa 1920 the only known universe was the Milky Way (as predicted by Galileo)**

- **Cepheid Variable Star**: a type of star that pulsates radially, varying in both diameter and temperature and producing changes in brightness with a well-defined stable period and amplitude **Cepheid V1 - M31**

- Henrietta Swan Leavitt, 1908

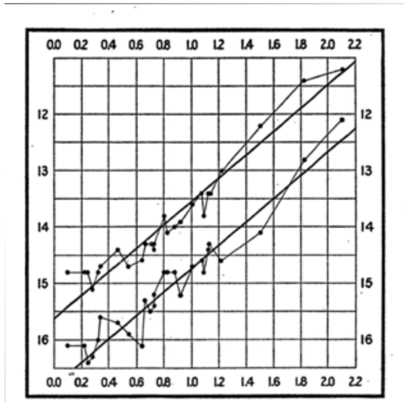


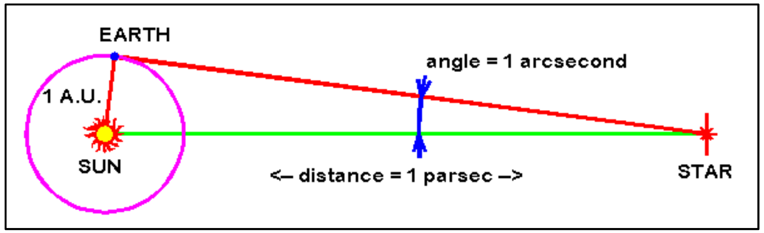
FIG. 2.

$$\begin{cases} M_v = -2.87\log P - 1.40 \\ M_v = m - 5\log d + 5 \end{cases}$$

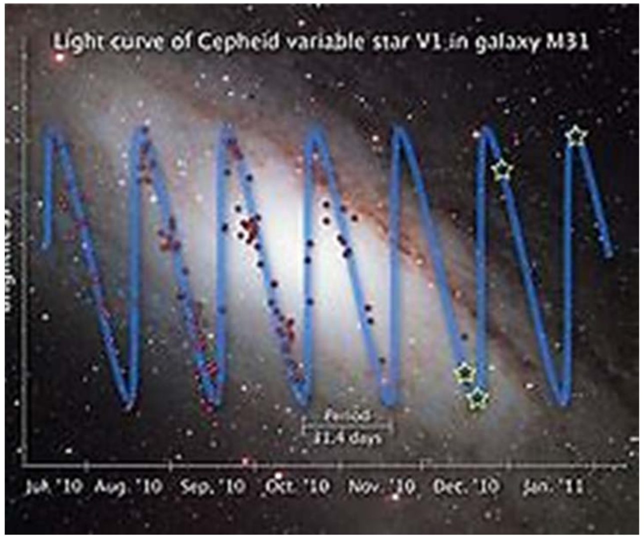
- Edwin Hubble
- Discovery of Andromeda Galaxy, 1924
- Discovery of redshift 1929 (Accelerating Universe)

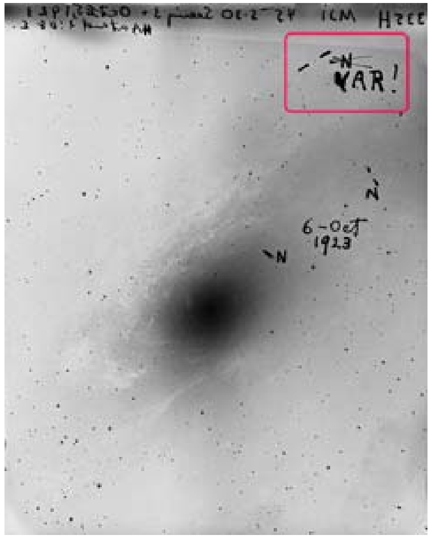
$$z = \frac{H_0 D}{c}$$

$$d = 1\,par\sec \approx 3.26LY$$



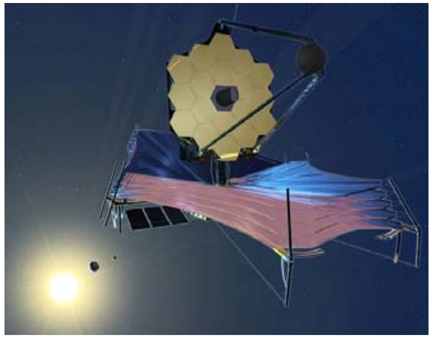
Orbital Mechanics





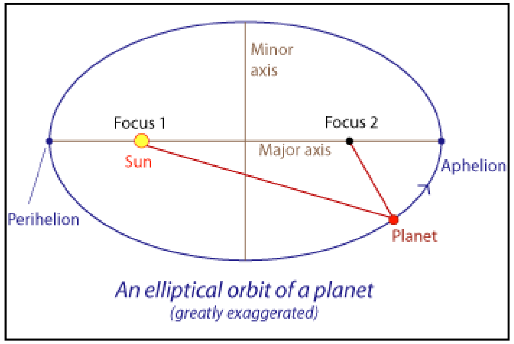
[Hubble's Legacy](#)

https://jwst.nasa.gov/content/webbLaunch/index.html



Keplerian Mechanics

- **Basic Comment:** unless otherwise stated, we will study the motion of celestial bodies and artificial satellites under mutual gravitational attraction limited to TWO bodies, according to Newtonian Mechanics



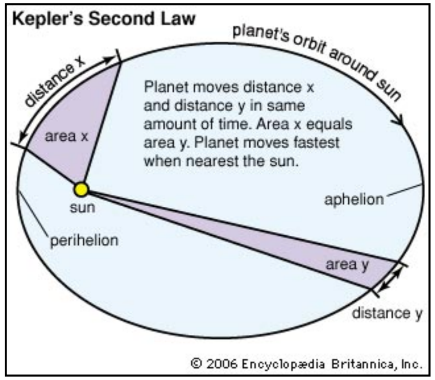
An elliptical orbit of a planet
(greatly exaggerated)

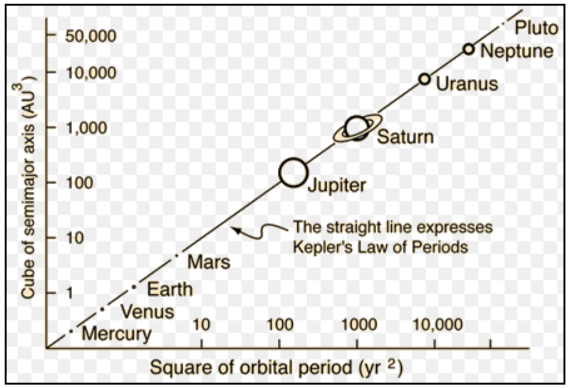
- **Kelper's Observations (for celestial bodies):**

Law 1: Planets move in elliptic orbits with one focus at the planet they orbit.

Law 2: Planets sweep out equal areas of the ellipse in equal time.

Law 3: The square of the period of the orbit is proportional to the cube of the semi-major axis.





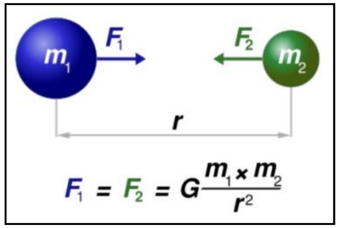
$$\prod^2 \left(\boldsymbol{years}\right) = a^3 \left(\boldsymbol{astr.\,units}\right)$$

1 AU ~ 1.5x10^6 Km mean distance Earth - Sun

FEYNMAN LOST LECTURE

- **Newton's Law of universal Gravitation**

N-Body problem



Cavendish Experiment 1797

"Big G" $G = 6.67 \times 10^{-11} \text{N m}^2 / \text{kg}^2$

- **Statement:** a particle attracts every other particle in the universe using a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.
- **Gravitational two-body problem:** is the motion of two mass point particles that interact only with each other, due to gravity. This means that influences from any third body are neglected.
- **Central-force problem:** to determine the motion of a particle under the influence of a single central force. A central force is a force that points from the particle directly towards (or directly away from) a fixed point in space, the center, and whose magnitude only depends on the distance of the object to the center. In many important cases, the problem can be solved analytically, i.e., in terms of well-studied functions such as trigonometric functions. Examples include gravity and electromagnetism as described by Newton's law of universal gravitation and Coulomb's law, respectively.

https://www.youtube.com/watch?v=et7XvBenEo8 NBODY

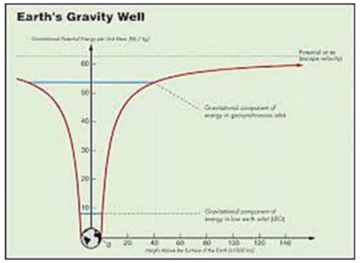
Newton Mechanics

- A conservative system is a system in which work done by a force is
  - 1. Independent of path.
  - 2. Equal to the difference between the final and initial values of an energy function.
  - 3. Completely reversible.

- The two most notable conservative systems are gravitational and electric fields. With gravity for example, the gravitational potential energy acquired or lost by a mass depends only on the difference between heights (or between distances from the origin of the force), and not on the path taken to get from one state to the other.

❑ Main Properties of a Gravitational Field

$$\boldsymbol{F} = -\frac{Gm_1 m_2}{r^3}\boldsymbol{r};\ \left|\boldsymbol{F}\right| = \frac{Gm_1 m_2}{r^2}$$



- The total mechanical (specific) energy is constant:

$$\begin{cases} m_1 \gg m_2 \\ E_{SPEC} = \left(\dfrac{v^2}{2} - \dfrac{\mu}{r}\right), \mu = Gm_1 \Rightarrow \dfrac{dE_{SPEC}}{dt} = 0 \end{cases}$$

- The total (specific) angular momentum is constant:

$$\begin{cases} \boldsymbol{h}_{SPEC} = \boldsymbol{r} \times \dot{\boldsymbol{r}} \\ \dfrac{d\boldsymbol{h}_{SPEC}}{dt} = 0 \end{cases}$$

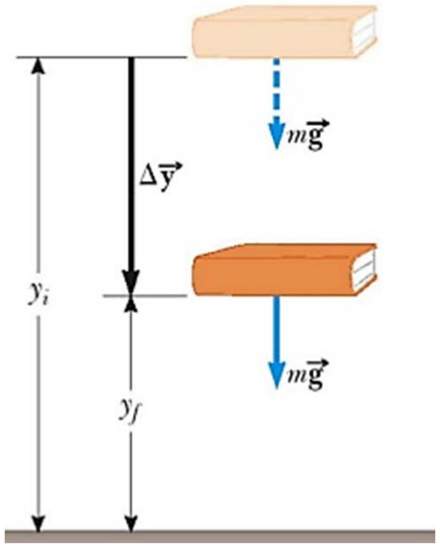
- The total linear momentum is constant:

$$(m_1 + m_2)\dot{\boldsymbol{r}}_{cm} = \mathbb{C}$$

- The gravitational potential $\boldsymbol{U}$, generates a potential energy $\boldsymbol{V}$

$$V = -U = -\frac{Gm_1m_2}{r}$$
$$U = \int \boldsymbol{F}(\boldsymbol{r})d\boldsymbol{r}$$

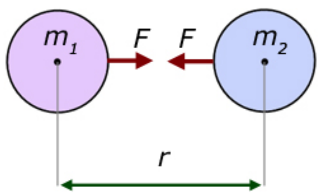
- Potential energy is a state of the system, a way of storing energy as of virtue of its configuration or motion, while work done in most cases is a way of chaining this energy from one body to another. When a body is dropped from a height, its gravitational energy a virtue of its configuration with respect to the Earth is converted into kinetic energy a virtue of its motion, due to the work done by gravity in bringing it down.



- Variation of potential energy due to height for a mass m (relative measure w.r.t. Earth).

$$\Delta V = mg\Delta y$$

- Seek a more general expression using Newton's law of gravitation



$$F = G\frac{mM}{r^2}$$

Newton Mechanics

- The gravitational potential energy of mass m in the gravitational field of M is therefore

$$V = mG\frac{M}{r^2}r = G\frac{mM}{r}$$

- **Note:** there is a paradox in that, according to the above, the potential energy decreases the higher the mass m is (!). So the convention is a sign change that makes the potential energy always negative, but with a maximum value of zero at the surface.

$$V = -G\frac{mM}{r}$$

❑ **Example:** How much potential energy is needed to lift a spacecraft of mass m = 9000 Kg, to an altitude of 400 Km?

$$\begin{cases} M_E = 5.96 \cdot 10^{24}\, Kg \\ R_E = 6.37 \cdot 10^{3}\, m \\ G = 6.674 \cdot 10^{-11}\, m^3 Kg^{-1} s^{-2} \end{cases}$$

$$\Delta V = V_{400} - V_0 = -G\frac{mM_E}{R_E + 400} - \left[-G\frac{mM_E}{R_E}\right] = 3.32 \cdot 10^{10}\, J$$

- Avg. Italian family consumption: 225 Kwh/month = $0.81 \cdot 10^9$ J = 41 months

- **Example:** Compute escape velocity (velocità di fuga) from the surface of M. The escape velocity v_{ESC} is the minimum velocity necessary to escape the gravity of a mass, in general it refers to planets and stars.

$$\begin{cases} t = 0 & v_1 = v_{ESC} \\ t \to \infty & v_2 = 0 \end{cases}$$

- Apply conservation of mechanical energy

$$\frac{1}{2} m v_{ESC}^2 - G\frac{mM}{r_1} = 0 - G\frac{mM}{\infty} = 0$$

1. Escape velocity from Earth: v_{ESC} = 11.29 Km/sec = 40,000 Km/h
2. Escape velocity from the Sun: v_{ESC} = 42 Km/sec = $1.68 \cdot 10^6$ Km/h

    - the velocity of rotation of the Earth at the equator is 1.7 Km/sec. The velocity of revolution of the Earth about the Sun is 30 Km/sec

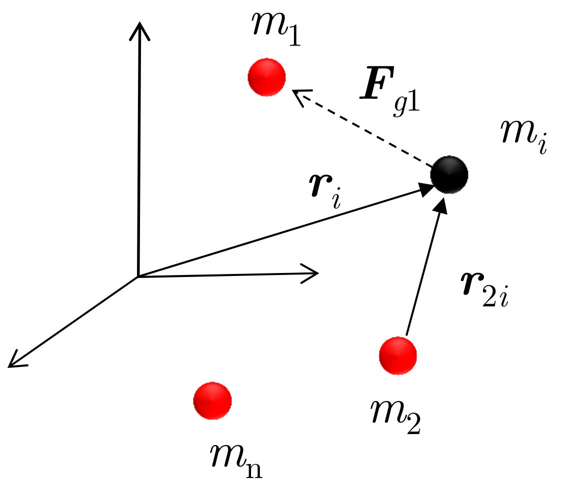
Two – Body Problem

- **Derivation of relative motion of mass particles under mutual gravitational attraction due to Newton's Laws** (this holds true even for spherical bodies with uniform density)

  - **Trajectory Equations**
  - 2 – Body Problem
  - Restricted 3 – Body Problem
  - 3 – Body Problem
  - N – Body Problem

  - **Invariants of Motion (2BP, R3BP)**
  - Conservation of Mechanical Energy
  - Conservation of Linear Momentum
  - Conservation of Angular Momentum

- **Problem Statement:** Derive the equations necessary to describe the motion of N mass particles under **<u>only</u>** mutual gravitational attraction with respect to some inertial frame



$\boldsymbol{F}_{gj}$ = Gravitational force on mass i due to mass j

$\boldsymbol{r}_{ji}$ = Distance vector from mass j to mass i

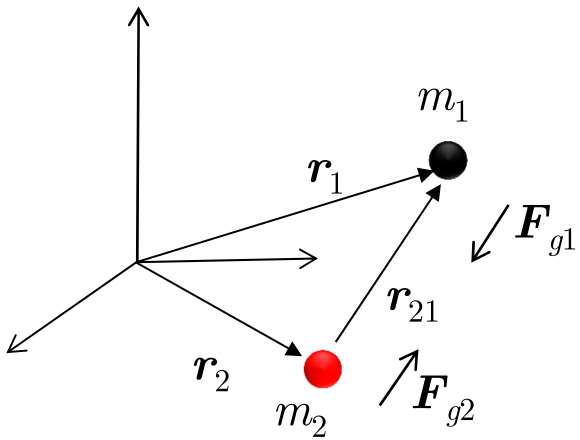
$\boldsymbol{r}_i$ = Inertial position of mass i

$$m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_{gj} = -\frac{G m_i m_j}{r_{ij}^3}\boldsymbol{r}_{ji} \Rightarrow -G\sum_{j=2}^{N}\frac{m_i m_j}{r_{ij}^3}\boldsymbol{r}_{ji}$$

NOTE: The acceleration of the center of mass of an isolated system is equal to zero, thus it is at rest or moves in a rectilinear motion and can be used as origin of an inertial coordinate system

Two – Body Problem

- **Reduction to a 2 – Body problem** (relative motion of two particles)



$$\frac{Gm_1m_2}{r^2}\frac{(\mathbf{r}_2-\mathbf{r}_1)}{r}=m_1\frac{d^2\mathbf{r}_1}{dt^2}$$

$$\frac{Gm_2m_1}{r^2}\frac{(\mathbf{r}_1-\mathbf{r}_2)}{r}=m_2\frac{d^2\mathbf{r}_2}{dt^2}$$

$$|\boldsymbol{r}|=|\boldsymbol{r}_2-\boldsymbol{r}_1|=|\boldsymbol{r}_1-\boldsymbol{r}_2|$$

- Consider the relative motion between the two constant masses (easier to compute than absolute motion of each single mass)

$$\frac{d^2}{dt^2}(m_1\mathbf{r}_1+m_2\mathbf{r}_2)=\mathbf{0}$$

$$-\frac{G(m_1+m_2)}{r^2}\frac{(\mathbf{r}_2-\mathbf{r}_1)}{r}=\frac{d^2}{dt^2}(\mathbf{r}_2-\mathbf{r}_1)$$

- From the definition of center of mass:

$$\mathbf{r}_{cm}\stackrel{\text{def}}{=}\frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2}{m_1+m_2}$$

- The Linear Momentum of the system is an **invariant** of motion (constant):

$$(m_1+m_2)\dot{\boldsymbol{r}}_{cm}=\mathbb{C}$$

$$\frac{d}{dt}[(m_1+m_2)\dot{\boldsymbol{v}}_{cm}]=m_1\frac{d\boldsymbol{r}_1^2}{dt}+m_2\frac{d\boldsymbol{r}_2^2}{dt}=0$$

Two – Body Problem

- From previous page, the relative motion dynamics yield:

(*) $$\boxed{\frac{d^2\mathbf{r}}{dt^2} + \frac{\mu}{r^3}\mathbf{r} = 0} \quad \text{or} \quad \frac{d\mathbf{v}}{dt} = -\frac{\mu}{r^3}\mathbf{r} \quad \text{where} \quad \begin{aligned} \mathbf{r} &= \mathbf{r}_2 - \mathbf{r}_1 \\ r &= |\mathbf{r}| = |\mathbf{r}_2 - \mathbf{r}_1| \\ \mu &= G(m_1 + m_2) \end{aligned}$$

- If one mass is much larger than the other (Earth vs Satellite), for example, then the motion of the center of mass approximates the motion of the smaller mass (Central Force Motion)

$$m_1 = M \gg m_2 \Rightarrow \mu = G(m_1 + m_2) \approx GM$$

- Consider (*), post multiply by velocity (inner product) and rewrite terms:

$$\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} + \frac{\mu}{r^3}\boldsymbol{r} \cdot \dot{\boldsymbol{r}} = 0 \Rightarrow \frac{1}{2}\frac{d}{dt}(\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}) + \frac{\mu}{r^3}\boldsymbol{r}\dot{\boldsymbol{r}} = 0 \Rightarrow \frac{d}{dt}\left(\frac{v^2}{2} - \frac{\mu}{r}\right) = 0 \Rightarrow E_{SPEC} = \mathbb{C}$$

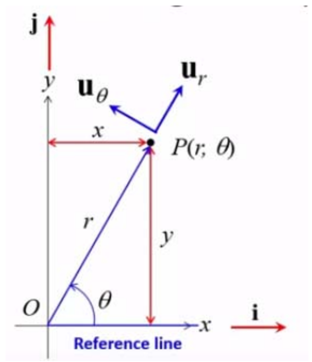
- The total specific mechanical energy of the mass m is an **invariant** of motion (Note that μ/r is the Gravitational Potential)
- Take the cross product of (*) with the position vector:

$$\boldsymbol{r} \times \ddot{\boldsymbol{r}} + \boldsymbol{r} \times \frac{\mu}{r^3}\boldsymbol{r} = 0 \Rightarrow \frac{d}{dt}(\boldsymbol{r} \times \dot{\boldsymbol{r}}) = 0 \Rightarrow \frac{d}{dt}(\boldsymbol{h}_{SPEC}) = 0 \Rightarrow \boldsymbol{h}_{SPEC} = \mathbb{C}$$

- The total specific angular momentum is an **invariant** of motion (The trajectory of motion is therefore a planar path)

- **Note:** the previous equations can be also written in polar coordinates recalling:

$$\boldsymbol{r} = r\boldsymbol{u}_r \qquad \begin{cases} \boldsymbol{u}_r = \cos\theta\,\boldsymbol{u}_x + \sin\theta\,\boldsymbol{u}_y \\ \boldsymbol{u}_\theta = -\sin\theta\,\boldsymbol{u}_x + \cos\theta\,\boldsymbol{u}_y = \dfrac{d\boldsymbol{u}_r}{d\theta} \end{cases}$$



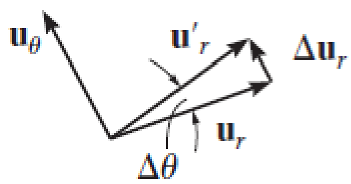
$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{d(r\boldsymbol{u}_r)}{dt} = \dot{r}\boldsymbol{u}_r + r\frac{d\boldsymbol{u}_r}{dt} = \dot{r}\boldsymbol{u}_r + r\dot{\theta}\boldsymbol{u}_\theta = v_r\boldsymbol{u}_r + v_\theta\boldsymbol{u}_\theta$$

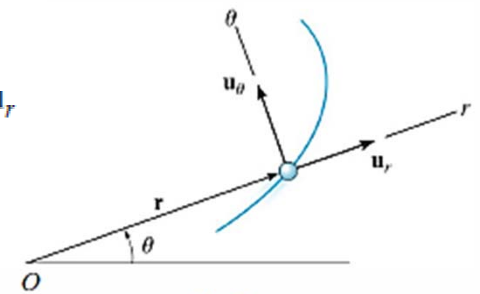
$$\boldsymbol{a} = \frac{d(v_r\boldsymbol{u}_r + v_\theta\boldsymbol{u}_\theta)}{dt} = \ddot{r}\boldsymbol{u}_r + \dot{r}\dot{\mathbf{u}}_r + \dot{r}\dot{\theta}\boldsymbol{u}_\theta + r\ddot{\theta}\boldsymbol{u}_\theta + r\dot{\theta}\dot{\mathbf{u}}_\theta$$

$$\boldsymbol{a} = a_r\boldsymbol{u}_r + a_\theta\boldsymbol{u}_\theta = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{u}_\theta$$

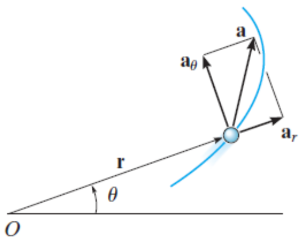
$$a = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2}$$

$$\boldsymbol{h}_{SPEC} = r^2\frac{d\theta}{dt}\boldsymbol{u}_z$$

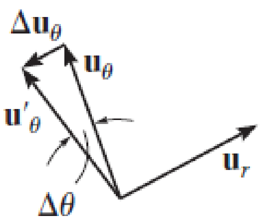


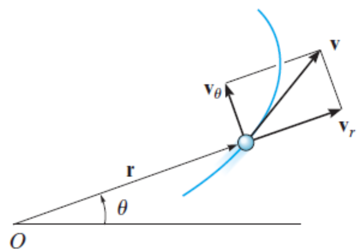


Position



Acceleration





Velocity

- The 2 – Body problem solution can be used to prove analytically Kepler's laws (f.i. *Wie, Sect. 3.1, Mengali, Sect. 1.4., Curtis,…*)

Kepler's first Law (Orbit equation):

- Right cross (*) with the specific angular momentum vector $\boldsymbol{h}$:

$$\ddot{\boldsymbol{r}} \times \boldsymbol{h} + \frac{\mu}{r^3}\boldsymbol{r} \times \boldsymbol{h} = 0$$

- Rewrite as:

$$\frac{d}{dt}[\boldsymbol{r} \times \boldsymbol{h} - \frac{\mu}{r}] = 0$$

- Integrate and define the constant eccentricity vector $e,$ and f as the angle between $\boldsymbol{r}$ and $\boldsymbol{e}$:

$$\mu\boldsymbol{e} = \dot{\boldsymbol{r}} \times \boldsymbol{h} - \frac{\mu}{r}\boldsymbol{r}$$

- The vector $\mu\boldsymbol{e}$ can be written as:

$$\mu\vec{e} = \vec{v} \times \vec{h} - \frac{\mu}{r}\vec{r} = \vec{v} \times (\vec{r} \times \vec{v}) - \frac{\mu}{r}\vec{r}$$
$$= [v^2 - (\mu/r)]\vec{r} - (\vec{r} \cdot \vec{v})\vec{v}$$

- Take the dot product yields:

$$\dot{\boldsymbol{r}} \cdot \mu\boldsymbol{e} = \dot{\boldsymbol{r}} \cdot [\dot{\boldsymbol{r}} \times \boldsymbol{h} - \frac{\mu}{r}\boldsymbol{r}]$$

- But: $\vec{r} \cdot \dot{\vec{r}} \times \vec{h} = (\vec{r} \times \dot{\vec{r}}) \cdot \vec{h} = h^2,$

$$\ddot{\vec{r}} \times \vec{h} + \mu \frac{\vec{r}}{r^3} \times \vec{h} = 0 \tag{3.17}$$

which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} [\dot{\vec{r}} \times \vec{h} - (\mu/r)\vec{r}\,] = 0 \tag{3.18}$$

because

$$\vec{r} \times \dot{\vec{h}} = \vec{r} \times (\vec{r} \times \ddot{\vec{r}}\,) = (\vec{r} \cdot \dot{\vec{r}}\,)\dot{\vec{r}} - (\vec{r} \cdot \vec{r}\,)\dot{\vec{r}} \tag{3.19}$$

Integrating Eq. (3.18) gives

$$\dot{\vec{r}} \times \vec{h} - (\mu/r)\vec{r} = \text{constant vector} = \mu\vec{e} \tag{3.20}$$

where a constant vector $\mu\vec{e}$ is introduced and $\vec{e}$ is called the *eccentricity vector*. Note that the constant vector $\mu\vec{e}$ can also be written as

$$\begin{aligned} \mu\vec{e} &= \vec{v} \times \vec{h} - \frac{\mu}{r}\vec{r} = \vec{v} \times (\vec{r} \times \vec{v}\,) - \frac{\mu}{r}\vec{r} \\ &= [v^2 - (\mu/r)]\vec{r} - (\vec{r} \cdot \vec{v}\,)\vec{v} \end{aligned}$$

Taking the dot product of Eq. (3.20) with $\vec{r}$ gives

$$\vec{r} \cdot \dot{\vec{r}} \times \vec{h} - \vec{r} \cdot (\mu/r)\vec{r} = \vec{r} \cdot \mu\vec{e} \tag{3.21}$$

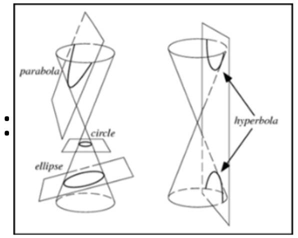
Because $\vec{r} \cdot \dot{\vec{r}} \times \vec{h} = (\vec{r} \times \dot{\vec{r}}\,) \cdot \vec{h} = h^2$, Eq. (3.21) becomes

$$h^2 - \mu r = \mu r e \cos\theta \tag{3.22}$$

Two – Body Problem

- Obtain: $h^2 - \mu r = \mu r e \cos f \Rightarrow r = \dfrac{h^2 / \mu}{1 + e\cos f}$

- Which is the equation of a conic section in polar coordinates with origin at one focus:



$$\boxed{r = \frac{p}{1 + e\cos f}} \qquad \textbf{(**)}$$

  - p is a geometric constant known as **semilatus rectum** (dimensions of length)
  - e is the **eccentricity** magnitude of the trajectory (orbit). The orbit is bounded if $e < 1$, otherwise is unbounded
  - f is called **true anomaly**, and it is the current angle between the position vector and the eccentricity direction (also indicated with θ or ν in several texts)

- **Energy Integral – Vis Viva equation**: Recall the invariance of the specific mechanical energy

$$\frac{d}{dt}\left(\frac{v^2}{2} - \frac{\mu}{r}\right) = 0 \Rightarrow E_{SPEC} = \mathbb{C}$$

  - Define the constant parameter (dimensions of length): $a = \left(\dfrac{2}{r} - \dfrac{v^2}{\mu}\right)^{-1}$

$$\left(\frac{v^2}{2} - \frac{\mu}{r}\right) = -\frac{\mu}{2a} = const. \quad \Longrightarrow \quad v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

Energy Integral

Two – Body Problem

- From algebra the following relationship holds: $p = a(1 - e^2)$

- We can rewrite (**) in Cartesian coordinates with the origin at one focus and the x axis along the eccentricity vector direction:

$$r = \frac{p}{1 + e\cos f} \Rightarrow p = r + \boldsymbol{e} \cdot \boldsymbol{r} \Rightarrow \begin{cases} x = r\cos f \\ y = p - ex \end{cases}$$

- For e ≠ 1, equation (**) becomes:

$$\frac{(x + ea)^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

(Circle, Ellipse, Hyperbola)

- For e = 1, equation (**) becomes:

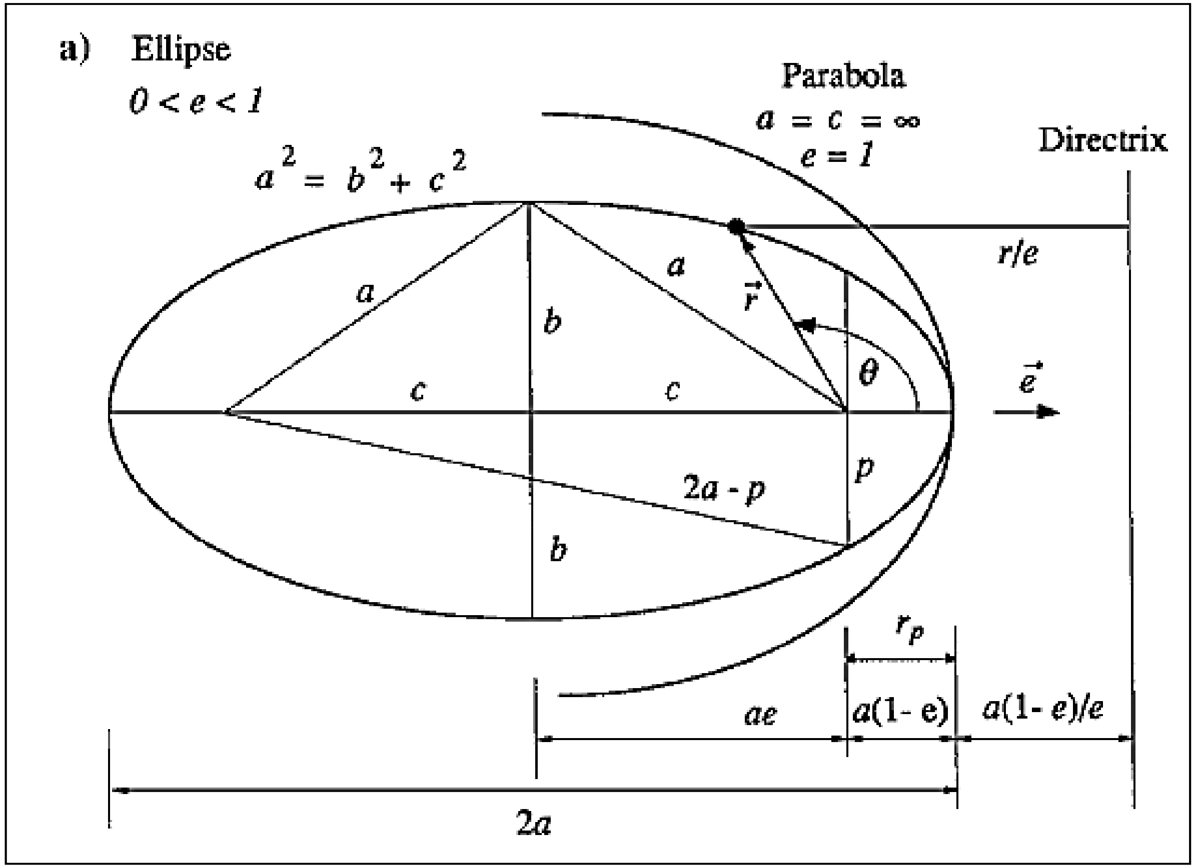
$$y^2 = 2p(\tfrac{1}{2}p - x)$$

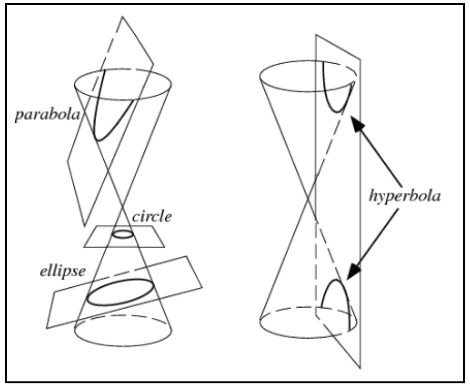
(Parabola)

- a is the ellipse semi major axis
- c is distance from the center of the ellipse
- $e = c/a$ is the eccentricity
- r_p is the periapsis (closest distance to the focus)
- r_a is the apoapsis (largest distance to the focus)

$$\begin{cases} r_p = a(1 - e) & \text{ellipse, parabola, hyperbola} \\ r_a = (1 + e) & \text{ellipse} \end{cases}$$

- Graphical representations (from Wie, with permission)





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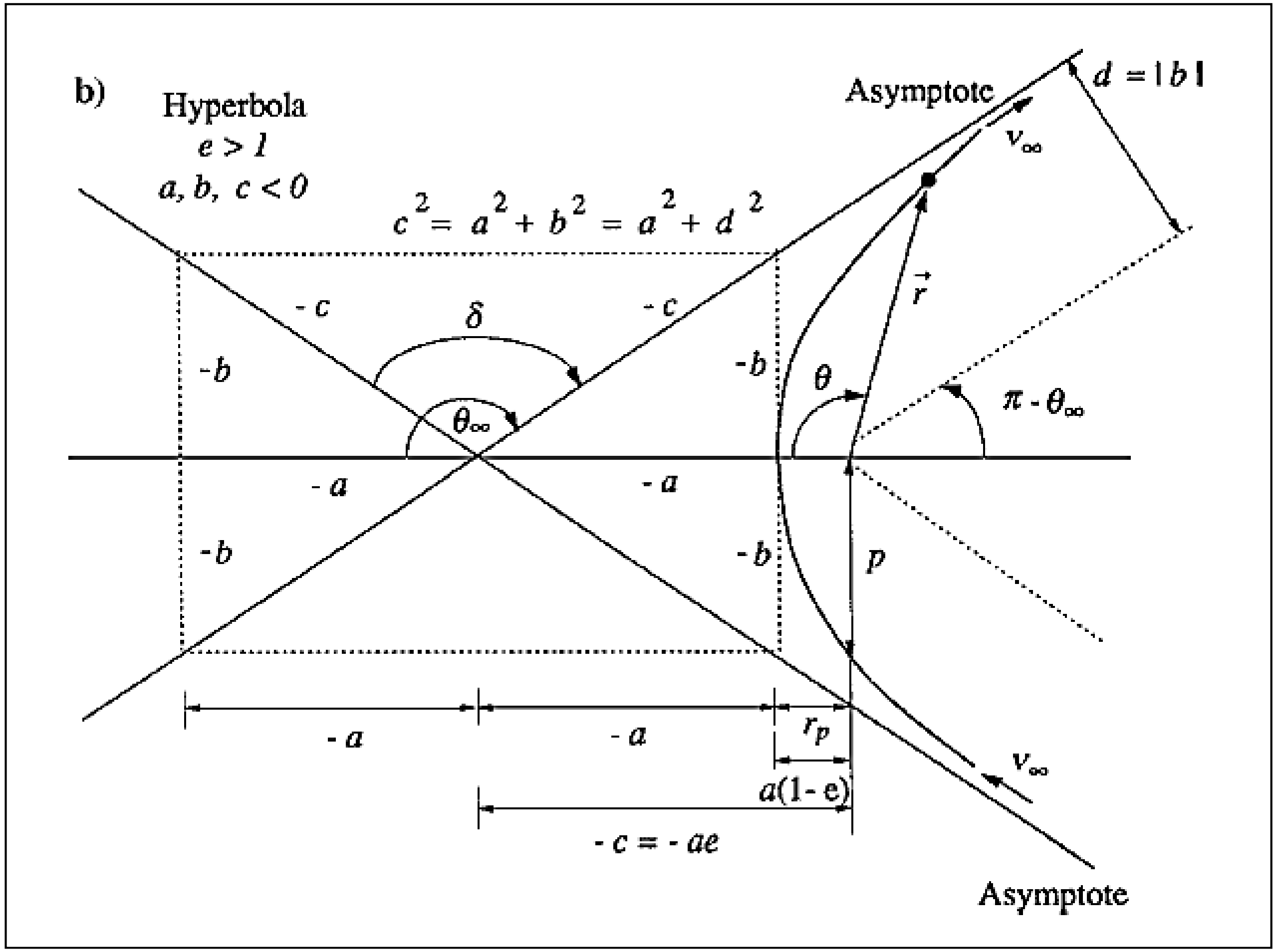
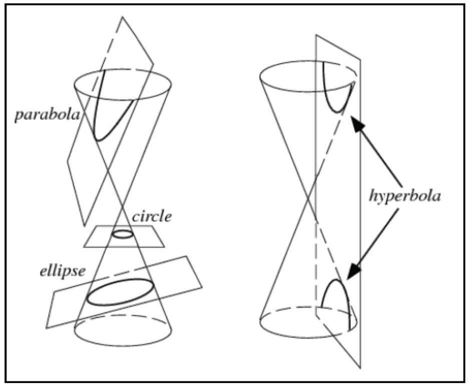




Two – Body Problem

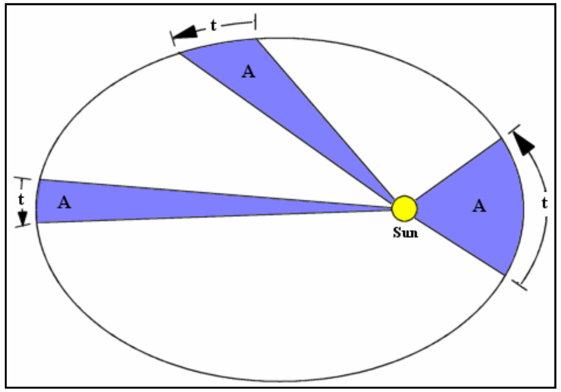
- Kepler's second Law (Areas Law)

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} h = \text{const}$$

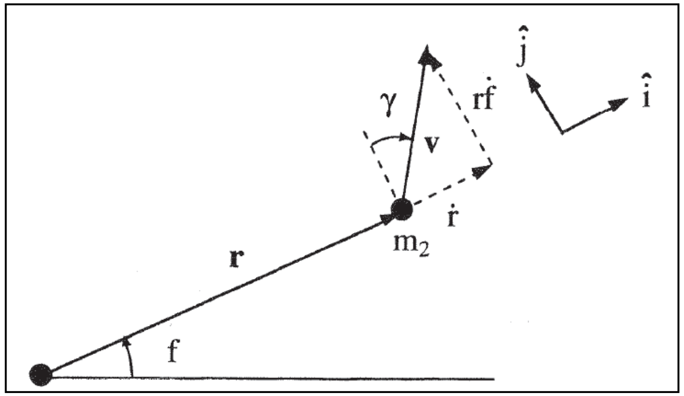
- Kepler's third Law (Relationship between period and area of the orbit)

  - Consider an infinitesimal area ΔA swept by the radius vector $\boldsymbol{r}$, as it moves of a Δf in a time Δt:

$$\Delta A = \frac{1}{2} r (r \Delta f)$$



$$\begin{cases} \dfrac{dA}{dt} = \lim\limits_{\Delta t \to 0} \left(\dfrac{\Delta A}{\Delta t} \right) = \lim\limits_{\Delta t \to 0} \left(\dfrac{1}{2} r^2 \dfrac{\Delta f}{\Delta t} \right) = \dfrac{1}{2} r^2 \dot{f} = \dfrac{1}{2} h = const \\ \boldsymbol{h} = \boldsymbol{r} \times \dot{\boldsymbol{r}} \Rightarrow h = \left| \boldsymbol{h} \right| = r(r\dot{f}) \end{cases}$$



  - The period P of an elliptical orbit can be found from the above:

$$\boxed{P = \frac{A}{dA/dt} = \frac{\pi ab}{h/2} = \frac{\pi a^2 \sqrt{1-e^2}}{\sqrt{\mu a(1-e^2)}/2} = 2\pi \sqrt{\frac{a^3}{\mu}}}$$

Two – Body Problem

- Recall the expression of the semi minor axis b (for a bounded orbit): $b^2 = a(1 - e^2)$

$$\boxed{P^2 = (4\frac{\pi^2}{\mu})a^3}$$

- From energy equation, we can find velocities at particular points of the orbit. Recall the vis – viva equation:

$$(\frac{v^2}{2} - \frac{\mu}{r}) = -\frac{\mu}{2a}$$

- For a circular orbit, a = r, therefore: $v_C = \sqrt{\frac{\mu}{r}}$

- The **Escape velocity** is the minimum velocity necessary to leave a closed orbit (enter a parabolic orbit, with specific energy equal to 0 and $v_\infty = 0$):

$$(\frac{v_E^2}{2} - \frac{\mu}{r_E}) = -\frac{\mu}{2a} = 0 \Rightarrow v_E = \sqrt{\frac{2\mu}{r_E}} = \sqrt{2v_C}$$

- If at some point r_0 the velocity $v_0 > v_E$, the body is on a hyperbolic orbit, yielding a positive velocity at infinity (Hyperbolic excess velocity v_∞), with specific energy greater than zero:

$$(\frac{v_0^2}{2} - \frac{\mu}{r_0}) = (\frac{v_\infty^2}{2} - \frac{\mu}{r_\infty}) = -\frac{\mu}{2a} > 0 \Rightarrow v_\infty = \sqrt{-\frac{\mu}{a}}$$

Summary of important relationships

$$\frac{d^2\boldsymbol{r}}{dt^2} + \frac{\mu}{r^3}\boldsymbol{r} = 0 \qquad r = \frac{p}{1 + e\cos f}$$

$$r(t) = \frac{h^2}{\mu\,(1 + e\cos f(t))}$$
$$r_p = a(1-e)$$
$$r_a = a(1+e)$$
$$p = a(1-e^2) = \frac{h^2}{\mu}$$
$$b = a\sqrt{1-e^2}$$
$$E = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$$
$$h = r_p v_p = r_a v_a \cong r^2 \dot{f}$$

$$\dot{\vec{e}} = 0$$
$$\dot{\vec{h}} = 0$$
$$\vec{h} = \vec{r} \times \dot{\vec{r}}$$
$$\vec{e} = \frac{1}{\mu}\left(\dot{\vec{r}} \times \vec{h} - \mu\frac{\vec{r}}{\|\vec{r}\|}\right)$$
$$\vec{r}(t) \times \vec{h} = 0$$
$$\vec{e} \times \vec{h} = 0$$
$$\dot{\vec{r}} = \dot{r}\hat{i} + r\dot{f}\hat{j}$$

The Sun

Mass $= 1.989 \cdot 10^{30}$ kg

Radius $= 6.9599 \cdot 10^{5}$ km

$\mu_{sun} = Gm_{sun} = 1.327 \cdot 10^{11}\ \text{km}^3/\text{s}^2$

The Earth

Mass $= 5.974 \cdot 10^{24}$ kg

Radius $= 6.37812 \cdot 10^{3}$ km

$\mu_{earth} = Gm_{earth} = 3.986 \cdot 10^{5}\ \text{km}^3/\text{s}^2$

Mean distance from sun $= 1$ au $= 1.495978 \cdot 10^{8}$ km

The Moon

Mass $= 7.3483 \cdot 10^{22}$ kg

Radius $= 1.738 \cdot 10^{3}$ km

$\mu_{moon} = Gm_{moon} = 4.903 \cdot 10^{3}\ \text{km}^3/\text{s}^2$

Mean distance from earth $= 3.844 \cdot 10^{5}$ km

Orbit eccentricity $= 0.0549$

Orbit inclination (to ecliptic) $= 5^\circ\ 09'$

Example

Problem: Suppose we observe at perigee, $r_p = 15000km$ and at apogee, $r_a = 25000km$. At time t_0, we observe $r(t_0) = 20000km$. Determine $v(t_0)$.
Solution: We first find the total energy, E from

$$E = -\frac{\mu}{2a}$$

where a can be found from

$$a = \frac{r_p + r_a}{2} = 20000km$$

Therefore

$$E = -\frac{\mu}{2a} = -9.96$$

Now we use

$$E = v^2/2 - \mu/r = -9.96$$

to find

$$v = \sqrt{2\left(\left(E + \frac{\mu}{r}\right)\right.} = \sqrt{2\left(-9.96 + 19.93\right)} = 4.464km/s$$

Example

Question: Suppose a satellite of earth is initially tracked at radius of $r_1 = 20,000km$ at a velocity of $1000m/s$. The satellite is later spotted at a radius of $10,000km$. Determine the velocity of the satellite.

Solution: Find the Energy at the initial time and use it to find the kinetic energy at the final time.

$$E = T_1 + V_1 = \frac{1}{2}\|\vec{v_1}\|^2 - \frac{\mu}{\|\vec{r_1}\|}$$

$$= .5 - \frac{398601}{20,000} = -19.43$$

$$V_2 = -\frac{\mu}{\|\vec{r_2}\|} = -\frac{398601}{10,000} = -39.86.$$

So $T_1 + V_1 = T_2 + V_2$ implies

$$T_2 = E - V_2 = -19.43 + 39.86 = 20.43$$

Example

Escape velocity is the kinetic energy needed to leave the sphere of influence of a planet. To achieve escape, net energy, E must be positive, so that as $r \to \infty$, we still have forward motion.

At $r \to \infty$, $V_\infty = \lim_{r\to\infty} \frac{\mu}{\|\vec{r}\|} = 0$, so

$$E = T_\infty + V_\infty = T_\infty$$

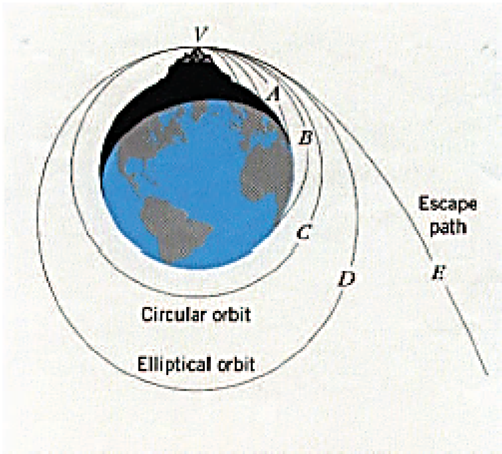
Question: Find the escape velocity at $r = 20,000 km$.

Solution: As we know, at $r = 20,000$, $V = -19.93$. So in order for $E > 0$, we need

$$E = T + V = T - 19.93 > 0$$

So we need $T > 19.93$. This yields a velocity of

$$\|v\| = \sqrt{2T} > 6.313$$



Question: Compute the escape velocity from the surface of the Earth

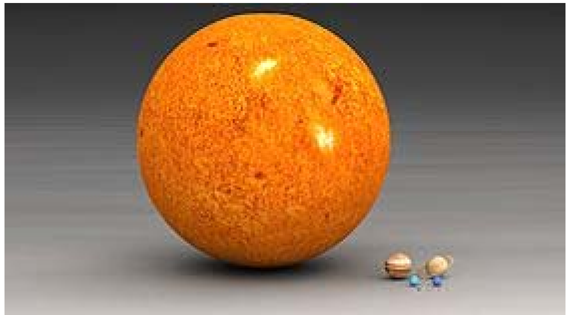
$$\boxed{v_E^{EARTH} = 11{,}186\,\frac{km}{\text{sec}} = 40{,}270\,\frac{km}{hr}}$$

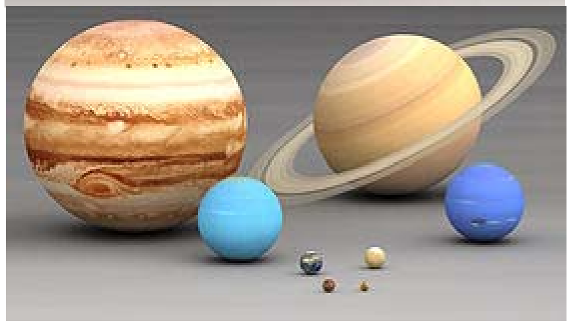
Planet	Distance from Sun (in millions of kilometers)
Mercury	58
Venus	108
Earth	150
Mars	228
Jupiter	778
Saturn	1,427
Uranus	2,871
Neptune	4,497
Pluto	5,914

TABLE 2.1 Some Solar System Dimensions

Planet	Orbital Semimajor Axis, a (AU)	Orbital Period, P (years)	Orbital Eccentricity, e	P^2/a^3
Mercury	0.387	0.241	0.206	1.002
Venus	0.723	0.615	0.007	1.001
Earth	1.000	1.000	0.017	1.000
Mars	1.524	1.881	0.093	1.000
Jupiter	5.203	11.86	0.048	0.999
Saturn	9.537	29.42	0.054	0.998
Uranus	19.19	83.75	0.047	0.993
Neptune	30.07	163.7	0.009	0.986

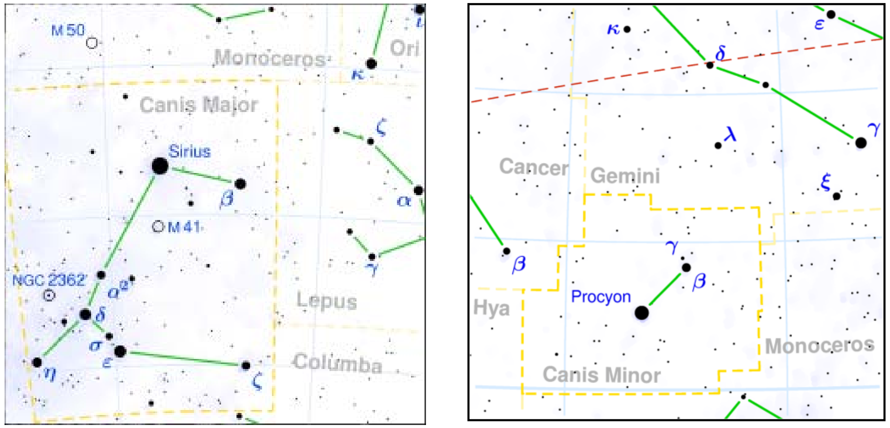
Body	μ ($km^3\ s^{-2}$)
Sun	132 712 440 018.(9)[1]
Mercury	22032(9)
Venus	324859(9)
Earth	398 600.4418(9)
Moon	4904.8695(9)
Mars	42828(9)
Ceres	63.1(9)[2][3]
Jupiter	126 686 534(9)
Saturn	37 931 187(9)
Uranus	5793939(9)[4]
Neptune	6 836 529(9)
Pluto	871(9)[5]
Eris	1108(9)[6]

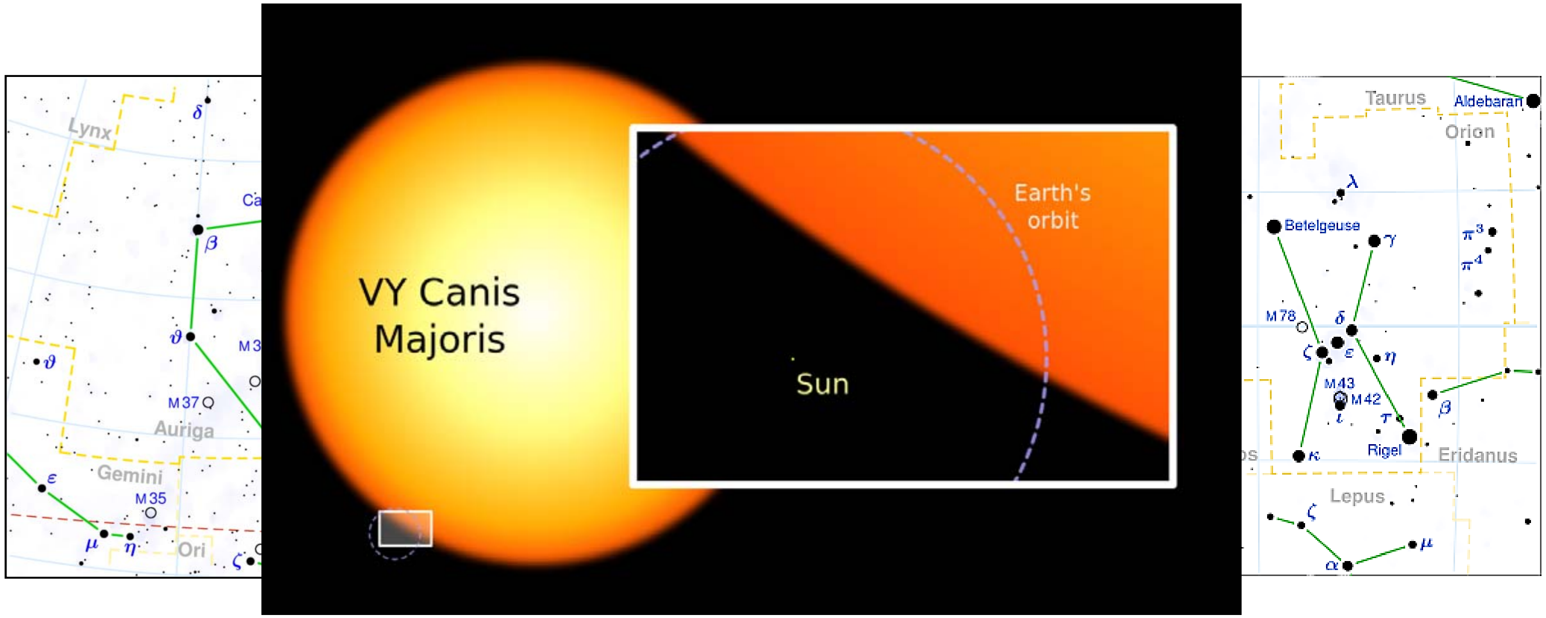




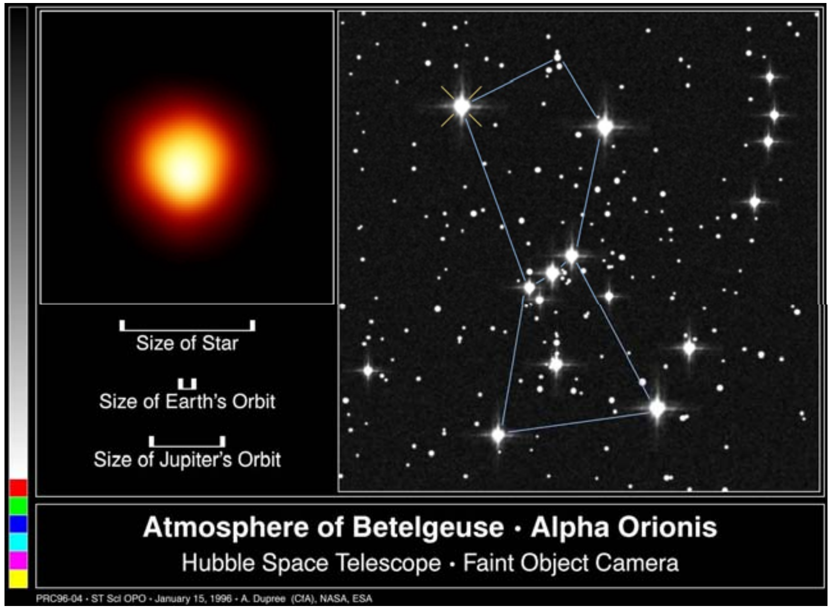
- Jupiter (69,911 km / 43,441 miles) – 1,120% the size of Earth
- Saturn (58,232 km / 36,184 miles) – 945% the size of Earth
- Uranus (25,362 km / 15,759 miles) – 400% the size of Earth
- Neptune (24,622 km / 15,299 miles) – 388% the size of Earth
- Earth (6,371 km / 3,959 miles)
- Venus (6,052 km / 3,761 miles) – 95% the size of Earth
- Mars (3,390 km / 2,460 miles) – 53% the size of Earth
- Mercury (2,440 km / 1,516 miles) – 38% the size of Earth

Sirius (8.6 LY 2 W+WD) α - Canis Major
Procyon (12LY 2 W+WD) α - Canis Minor
Capella (43LY 2+2 (Y+W) + 2 RD) α – Aurigae
Aldebaran (65LY RG) α – Tauri
Rigel (860 LY BG) β - Orionis
Betelgeuse (642.5 LY) α – Orionis
VY Canis Majoris (..3900 LY Red SG)

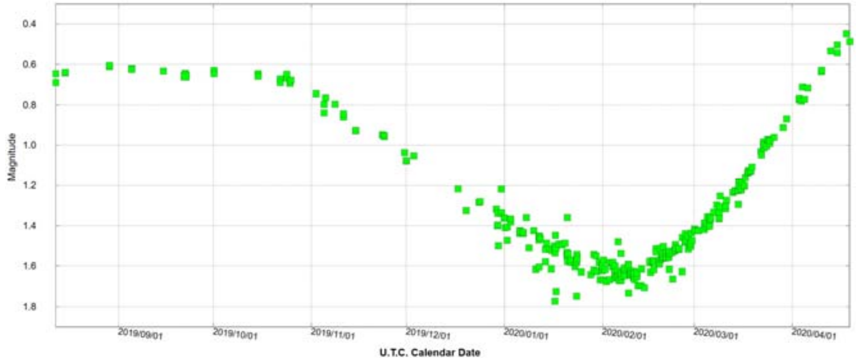


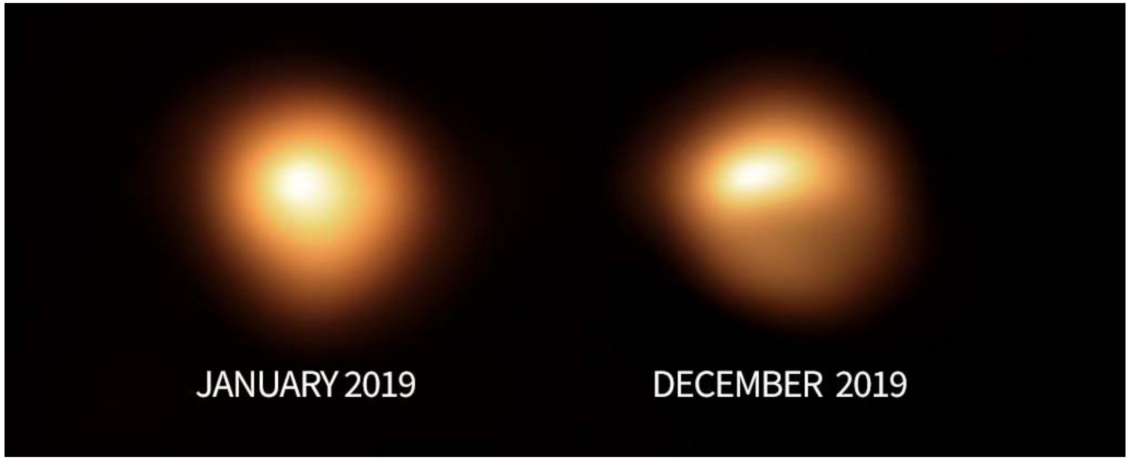


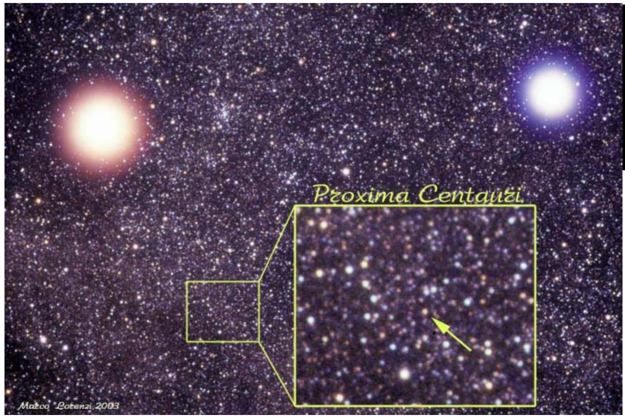
Brightest stars

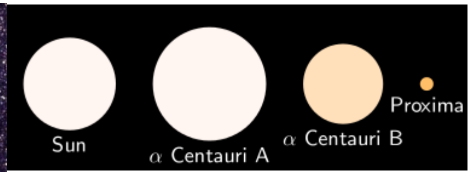


- ❑ Change in luminosity Late 2019 early 2020





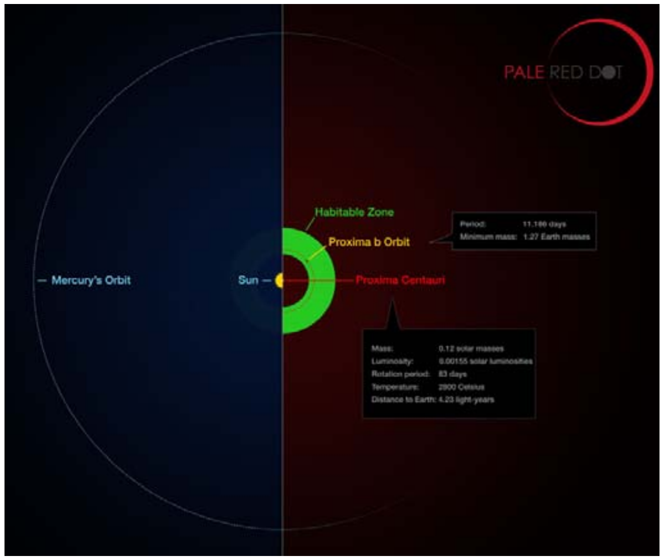




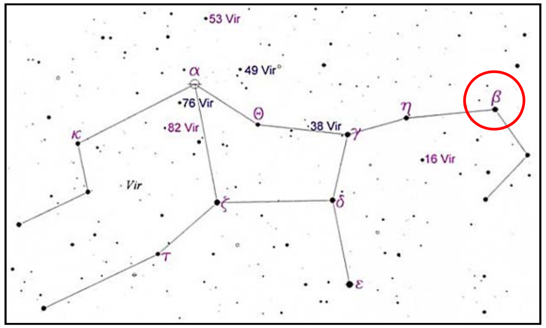
How to find an Exoplanet

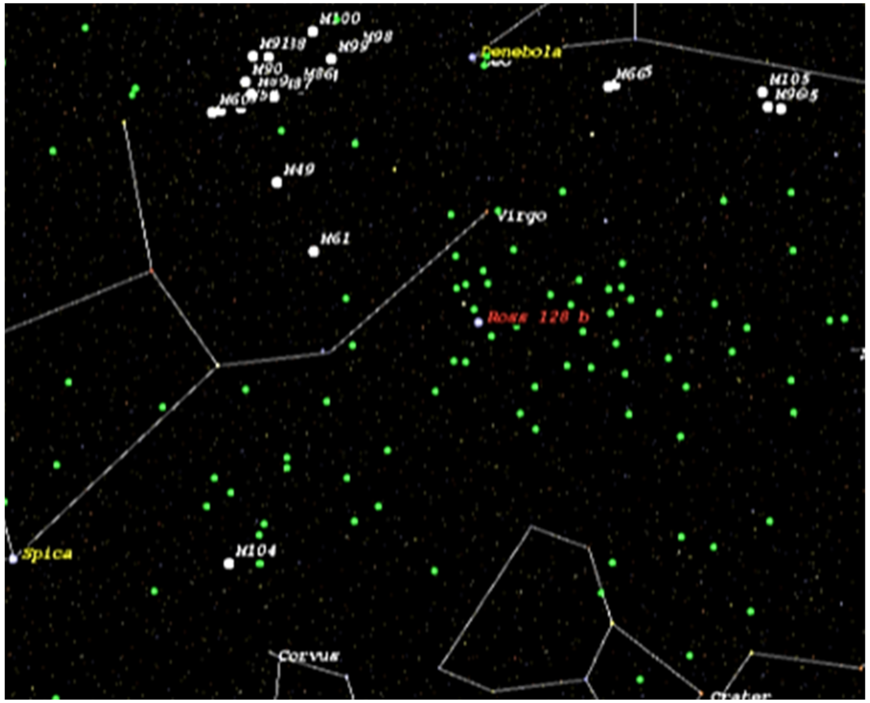
http://spaceengine.org/download/spaceengine/

On August 24, 2016, the European Southern Observatory announced the discovery of Proxima b, a planet orbiting the star at a distance of roughly 0.05 AU (7,500,000 km) and an orbital period of approximately 11.2 Earth days. Its estimated mass is at least 1.3 times that of the Earth. Moreover, the equilibrium temperature of Proxima b is estimated to be within the range where water could exist as liquid on its surface, thus placing it within the habitable zone of Proxima Centauri.



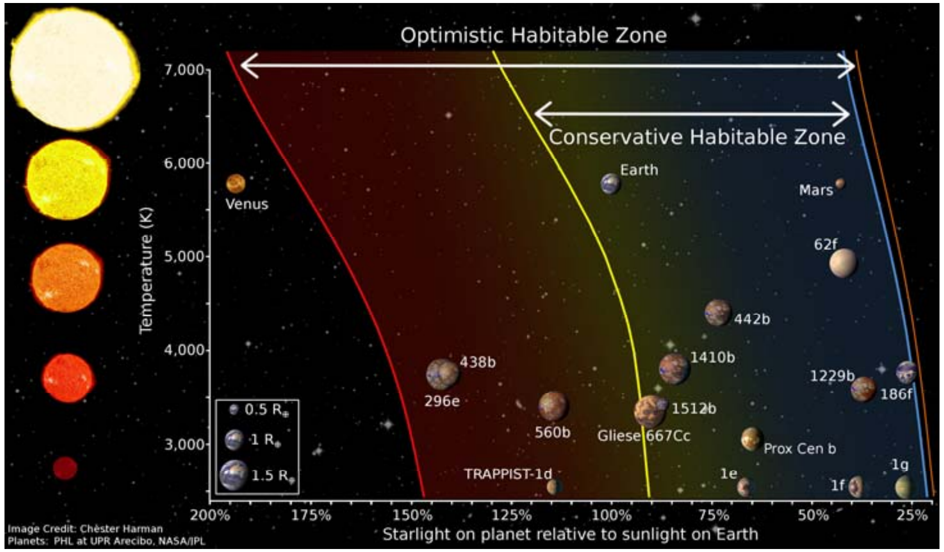
Alpha Centauri 2W 4.37 LY
Proxima Centauri RD 4.27 LY

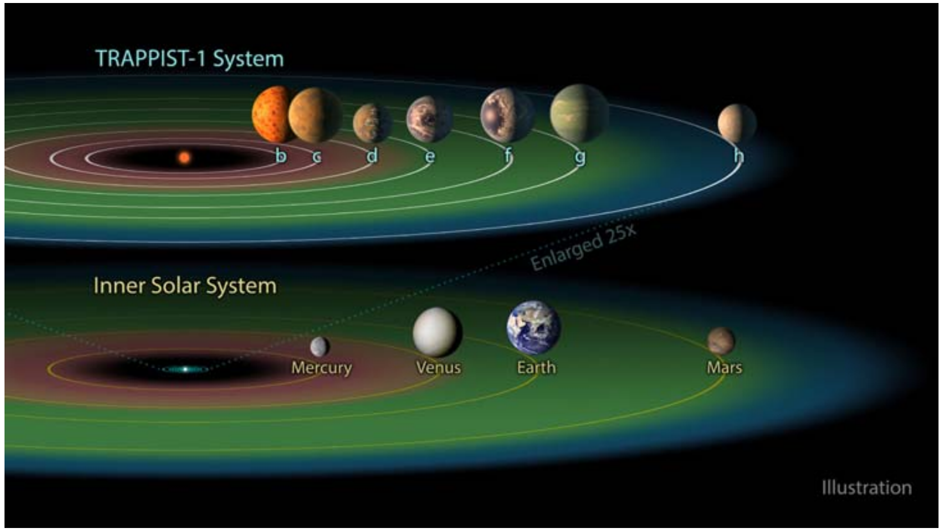




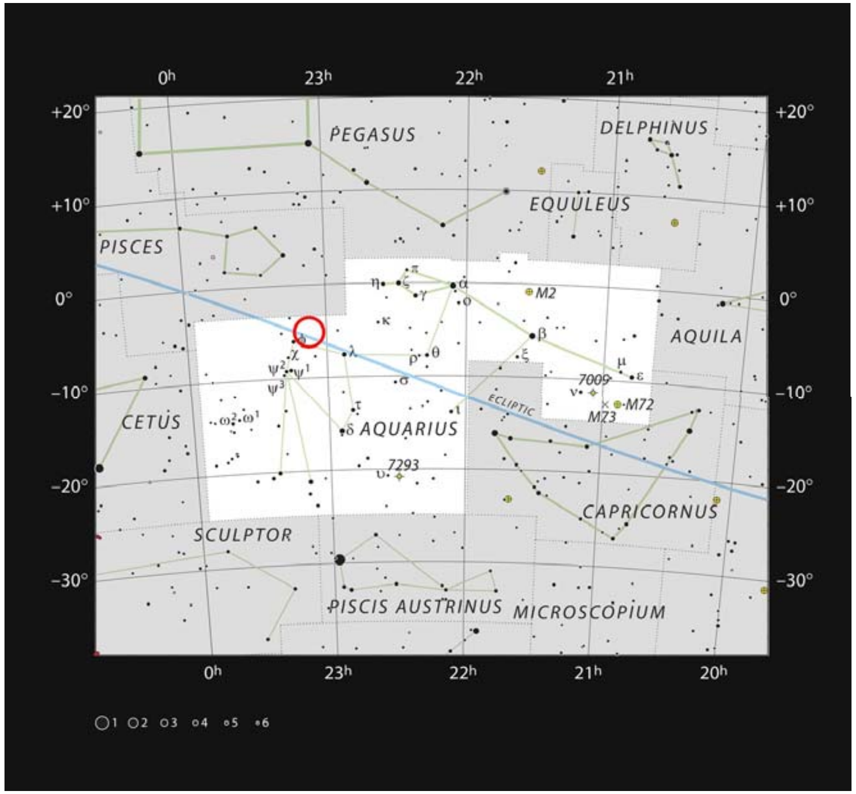
Ross 128 is a red dwarf in the equatorial zodiac constellation of Virgo, near β Virginis. The apparent magnitude of Ross 128 is 11.13, which is too faint to be seen with the unaided eye. The distance of this star from Earth is 10.89 light-years , making it the twelfth closest stellar system to the Solar System.

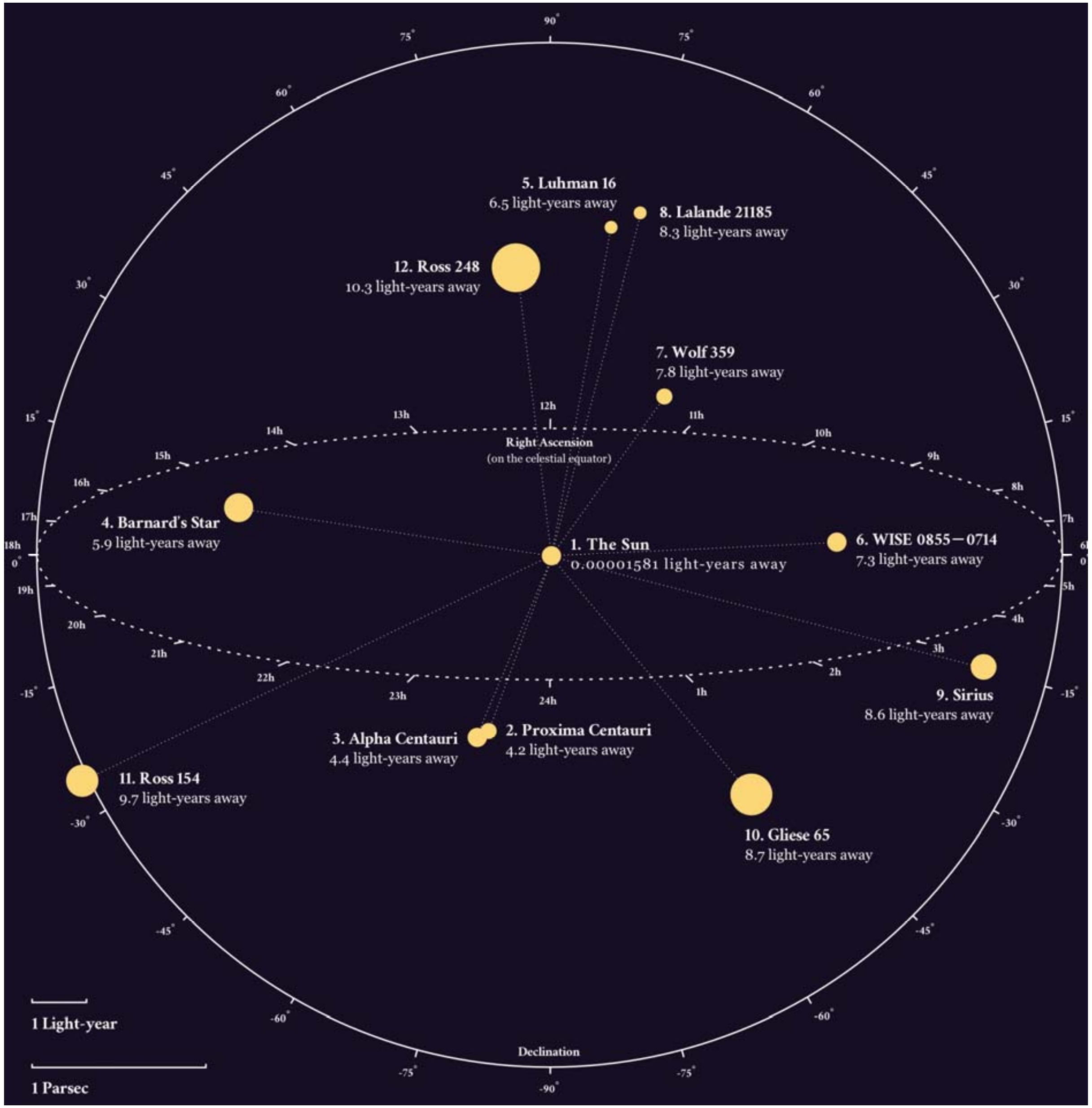
Ross 128 b is a confirmed Earth-sized exoplanet, likely rocky, orbiting within the inner habitable zone. The exoplanet was found using a decade's worth of radial velocity data with the HARPS spectrograph (High Accuracy Radial velocity Planet Searcher) at the La Silla Observatory in Chile. The planet is only 35% more massive than Earth, receives only 38% more sunlight, and is expected to be a temperature suitable for liquid water to exist on the surface, if it has an atmosphere.





Trappist 1, RD 40 LY





1. The Sun
0.00001581 light-years away
2. Proxima Centauri
4.2 light-years away
3. Alpha Centauri
4.4 light-years away
4. Barnard's Star
5.9 light-years away
5. Luhman 16
6.5 light-years away
6. WISE 0855−0714
7.3 light-years away
7. Wolf 359
7.8 light-years away
8. Lalande 21185
8.3 light-years away
9. Sirius
8.6 light-years away
10. Gliese 65
8.7 light-years away
11. Ross 154
9.7 light-years away
12. Ross 248
10.3 light-years away
Right Ascension
(on the celestial equator)
Declination
1 Light-year
1 Parsec

- Kepler's Equation tutorials

https://www.youtube.com/watch?v=xqFrLvBnBGU

Space Flight Mechanics - Indian Institute of Technology (Lec. 11 and 12)

- Other Units (canonical units) based on Earth – Sun orbit
  - 1 AU = 1.4959787x10^{11} m ≈ 150000000 km
  - 1 YR = 3.15x10^{7} sec ≈ 365.25 days

$$\begin{cases} 1DU = R_{PRIM} \\ \mu_{PRIM} = 1\dfrac{DU^3}{TU^2} \\ TU_{PRIM} = \sqrt{\dfrac{R_{PRIM}^3}{\mu_{PRIM}}} \\ 1MU = m_{SAT/PLAN} \end{cases}$$

$$\begin{cases} DU_{\odot} = 1AU \\ TU_{\odot} = \sqrt{\dfrac{AU^3}{\mu_{\odot}}} = 58.13\, days \end{cases}$$

$$\begin{cases} DU_{\oplus} = 1R_{\oplus} = 6378\, km \\ TU_{\oplus} = \sqrt{\dfrac{R_{\oplus}^3}{\mu_{\oplus}}} = 806.78\, \mathrm{sec} \end{cases}$$

Kepler's Equation

- The problem is to determine the relationship between the position of a body in orbit (elliptical) and time (Kepler's Equation).

  - The vectors $\boldsymbol{h}$ and $\boldsymbol{e}$ together determine the size, shape, and orientation of the orbit with respect to the frame of reference. Their components provide six scalar constants of integration of the two-body equation of motion, but these constants are not independent since they are perpendicular, which means $\boldsymbol{h}\cdot\boldsymbol{e} = 0$.

  - An **additional integration constant** will be required to complete the solution. What is missing, of course, is the location of the body in orbit at some particular instant of time.

  - Recall Kepler's second law and the two – body problem solution (**Newton unanswered question**)

$$\begin{cases} h = |\boldsymbol{h}| = r(r\dot{f}) \\ r = \dfrac{p = h^2 / \mu}{1 + e\cos f} \end{cases} \Longrightarrow \dot{f} = \frac{df}{dt} = \frac{h}{r^2} \Longrightarrow dt\sqrt{\frac{\mu}{p^3}} = \frac{df}{(1+e\cos f)^2}$$

$$E_{SPEC} = -\frac{\mu}{2a}$$

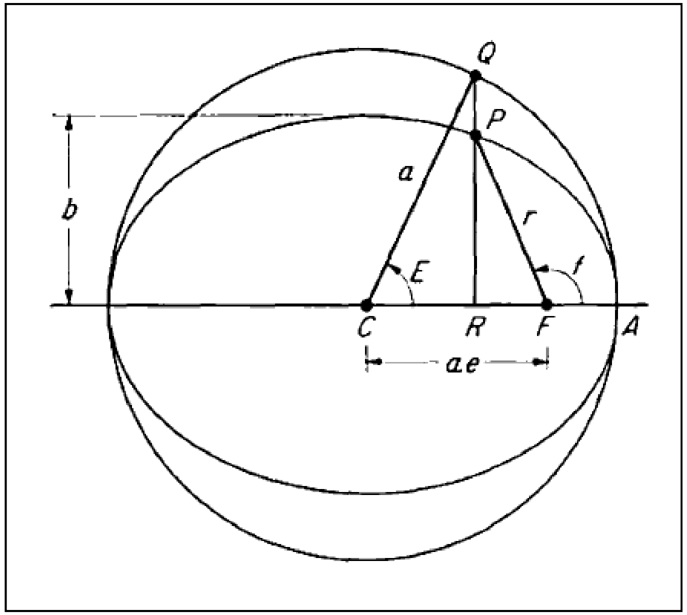
$$P = 2\pi\sqrt{\frac{a^3}{\mu = GM}}$$

  - If we take as initial point the periapsis ($f_0 = t_0 = 0$), integration of above gives the time of periapsis passage t_p for a given true anomaly value.

$$(t_p - t_0) = \int_{f_0}^{f_p} \frac{\sqrt{\dfrac{p^3}{\mu}}}{(1+e\cos f)^2}\, df$$

Kepler's Equation

- Kepler's derivation using the **eccentric anomaly *E*** (not a physical angle)



- Draw the auxiliary circle, from which we obtain the ellipse scaling law:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad y_e=\frac{b}{a}\sqrt{a^2-x^2}$$

$$y_c(x)=\sqrt{a^2-x^2} \qquad \boxed{y_e=\frac{b}{a}y_c}$$

- Choose the periapsis as initial time = 0, then the time at some point P can be found from the Law of Areas:

$$\frac{t}{P}=\frac{A_{FAP}}{A_{ellipse}}=\frac{A_{FAP}}{\pi ab} \qquad A_{FAP}=A_{PRA}-A_{PRF}=A_{PRA}-\frac{1}{2}(ae-a\cos E)\cdot\frac{b}{a}(a\sin E)$$

- Use the ellipse scaling law to get:

$$A_{PRA}=\frac{b}{a}A_{QRA} \qquad A_{QRA}=A_{QCA}-A_{QCR}$$

Kepler's Equation

$$A_{FAP} = \frac{1}{2}ab(E - \cos E \sin E) - \frac{1}{2}ab(e - \cos E)\sin E = \frac{1}{2}ab(E - e\sin E)$$

- Note that E is in radians. Combining:

$$\begin{cases} \dfrac{t}{P} = \dfrac{A_{FAP}}{A_{ellipse}} = \dfrac{(E - e\sin E)}{2\pi} \\ P = \sqrt{\dfrac{4\pi^2 a^3}{\mu}} \end{cases}$$

Third Law

$$\frac{(E - e\sin E)}{2\pi} = \frac{1}{2\pi} t \sqrt{\frac{\mu}{a^3}}$$

- **Definition #1:** The mean motion n is defined as $n = \dfrac{2\pi}{P} = \sqrt{\dfrac{\mu}{a^3}}$

- **Definition #2:** The mean Anomaly $M(t)$ is defined as $M(t) = nt$

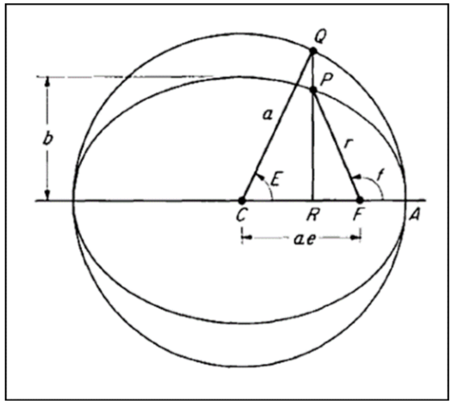
$$\boxed{M(t) = E(t) - e\sin E(t)}$$

The time required to travel between any two points in an elliptical orbit can be computed simply by first determining the eccentric anomaly E corresponding to a given true anomaly f, and then using Kepler's time equation. <u>However, Kepler's time equation does not provide time values (t – tp) greater than one-half of the orbit period, but it gives the elapsed time since periapsis passage in the shortest direction</u>.

Kepler's Equation

$$\boxed{M(t) = E(t) - e \sin E(t)}$$

- If we know the eccentric anomaly, we can find the true anomaly and then the position (and viceversa)



$$CF = ae = a\cos E + r\cos(\pi - f)$$

$$CF = ae = a\cos E - a\frac{(1-e^2)}{1+e\cos f}\cos f$$

$$\cos E = \frac{e + \cos f}{1 + e\cos f}$$

- From the ellipse scaling law:

$$r\sin f = \frac{b}{a}(a\sin E) \Rightarrow \sin E = \frac{\sqrt{1-e^2}\,\sin f}{1+e\cos f}$$

$$\boxed{\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2}}$$

Kepler's Equation

- **Comments:**

$$\boxed{nt = M(t) = E(t) - e\sin E(t)}$$

- Computation of true anomaly knowing eccentric anomaly is straightforward
- Computation of mean motion from eccentric anomaly is straightforward
- Computation of eccentric anomaly from mean motion is a very difficult problem and it requires numerical methods
- Kepler's equation can be derived for parabolic and hyperbolic orbits as well

Kepler's Equation for parabolic orbits

$$\begin{cases} t - t_p = \dfrac{1}{2\sqrt{\mu}}\left(pD + \dfrac{1}{3}D^3\right) \\ D = \sqrt{p}\tan\dfrac{f}{2} \end{cases}$$

Kepler's Equation for hyperbolic orbits

$$\begin{cases} t - t_p = \sqrt{\dfrac{(-a^3)}{\mu}}(e\sinh F - F) \\ F = \ln(\cosh F + \sqrt{(\cosh F)^2 - 1} \end{cases}$$

$$\cosh F = \frac{e + \cos f}{1 + e\cos f}$$

Problem: Given an orbit with $a = 10,000km$ and $e = .5$, determine the times at which $r = 14,147km$.

Solution: First solve for the true anomaly, f. we have

$$r(t) = \frac{a(1-e^2)}{1 + e\cos f(t)}$$

which yields

$$\cos f(t) = \frac{a(1-e^2) - r(t)}{er(t)} = -.9397$$

Solving for f yields two solutions $f = 160\,\mathrm{deg}, 200\,\mathrm{deg}$.

Now we want to find $E(t)$.

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2} = \pm 3.27$$

This yields

$$E = \pm 146.0337\,\mathrm{deg},$$

Kepler's Equation

Solving for mean anomaly (*in radians!!*),

$$M(t) = E(t) - e \sin E(t) = 2.2694 rad, 4.0138 rad$$

Now the mean motion is

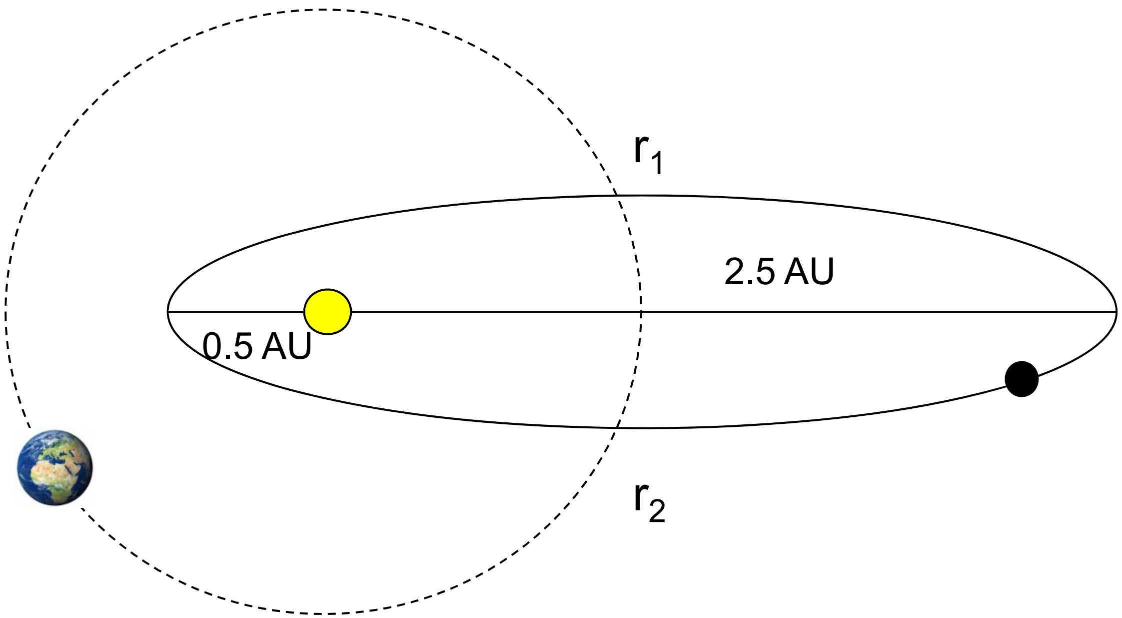
$$n = \sqrt{\frac{\mu}{a^3}} = 6.3135E - 4$$

So finally, the times of arrival are

$$t = \frac{M(t)}{n} = 3594s, 6357s$$

Note: In this way, it is easy to find the time between any 2 points in the orbit. e.g. from $f = 160 \deg$ to $f = 200 \deg$ takes time $\Delta t = 6357 - 3594 = 2763s$.

Problem: A space probe is following an elliptical orbit around the Sun with $r_p = 0.5\ \mathrm{AU}$ and $r_a = 2.5\ \mathrm{AU}$. How many days during each orbit is the probe closer than 1 AU to the Sun?



- We must find the time of flight between t1 corresponding to r1 = 1 AU and t2 corresponding to r2 = 1 AU
- Due to symmetry we can compute the time from perihelion to point 2 and then double it

Kepler's Equation

Orbit parameters:

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{2}{3} \qquad a = \frac{r_p + r_a}{2} = 1.5\,AU = 225 \times 10^6\ Km \qquad P = \frac{2\pi}{\sqrt{\mu_{SUN}}} a^{3/2} \approx 1.845\,Years$$

From orbit equation:

$$r = \frac{a(1-e^2)}{1+e\cos f} \Rightarrow \cos f_2 = \frac{a(1-e^2)}{er_2} r_2 = -\frac{1}{4}$$

Eccentric anomaly:

$$\cos E_2 = \frac{e+\cos f_2}{1+e\cos f_2} = \frac{1}{2} \Rightarrow E_2 = 60^o = 1.048\,rad \Rightarrow \sin E_2 = 0.866$$

From Kepler's equation:

$$\sqrt{\frac{\mu_{sun}}{a^3}}(t-t_p) = 1.048 - \frac{2}{3} \cdot 0.866 \Rightarrow TOF = 100\,days$$

Kepler's Equation

Problem: consider a spacecraft following a parabolic orbit around the Earth. If at the initial time **t_0 = 0** the spacecraft is at the **Perigee**, at a distance from the center of the Earth of **r_P = 2DU**, find the **Δt** needed to reach a distance of **r_1 = 60DU** (about the Earth-Moon Distance)

- Parabolic orbit eccentricity e = 1 compute semilatus rectum

$$r_P = \frac{p}{1+\cos(0)} = \frac{p}{2} \Rightarrow p = 4DU$$

- Compute true anomaly at final time

$$f_1 = \cos^{-1}\left(\frac{p}{r_1} - 1\right) = 2.7744\, rad = 159 \deg$$

- Compute eccentric anomaly for the orbit (Recall tha at initial time D_0 = 0)

$$\begin{cases} t - t_p = \dfrac{1}{2\sqrt{\mu}}\left(pD + \dfrac{1}{3}D^3\right) \\ D = \sqrt{p}\tan\dfrac{f}{2} = 5.3852 \end{cases}$$

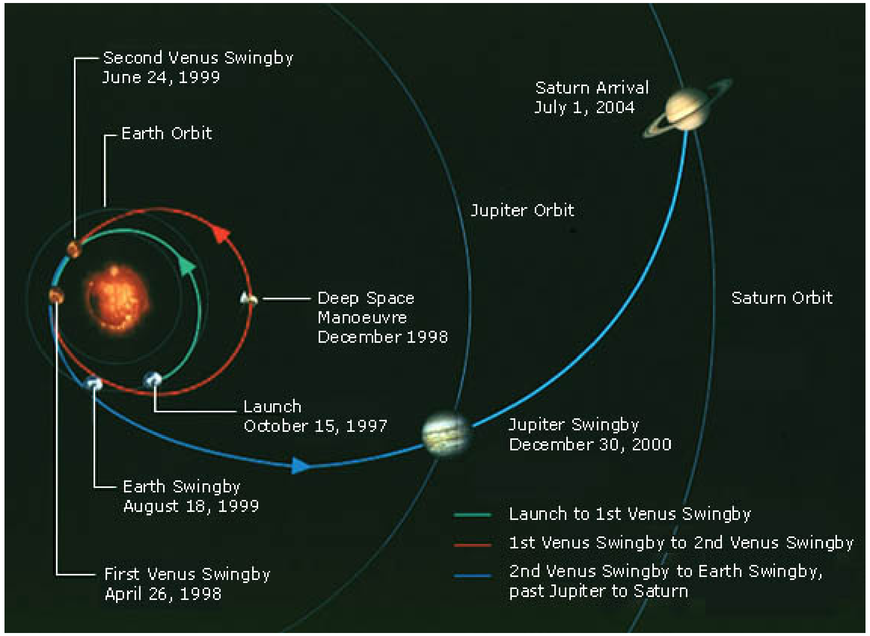
$$t - t_0 = \frac{1}{2}\sqrt{\frac{p^3}{\mu}}\left[D + \frac{1}{3}D^3 - \left(D_0 + \frac{D_0^3}{3}\right)\right]$$

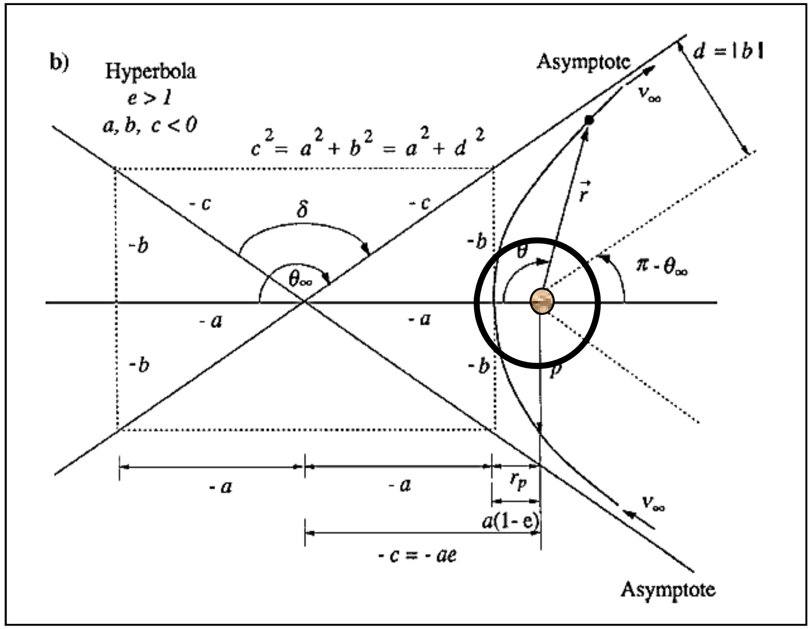
$$t_0 \neq t_P$$

$$\boxed{\Delta t = 229.76 TU_{\oplus} = 2.14\, days}$$

Kepler's Equation

Problem: Suppose we want to make a flyby of Jupiter. The **relative** velocity at approach is v_∞ = 10 km/s. To achieve the proper turning angle, we need an eccentricity of e = 1.07. Radiation limits our time within radius r = 100, 000 km to 1 hour (radius of Jupiter is 71,000km). Will the spacecraft survive the flyby?





$$
\begin{cases}
e = 1.07 \\
v_\infty = 10 Km/\sec \\
\mu_{JUP} = 1.267 \times 10^8 Km^3/\sec^2
\end{cases}
$$

- From energy equation:

$$
E = \frac{v_\infty^2}{2} - \frac{\mu_{JUP}}{r_\infty} = -\frac{\mu_{JUP}}{2a}
$$

$$
\begin{cases}
a = -1.267 \times 10^6 Km \\
r_p = a(1-e) = 88,690 Km
\end{cases}
$$

Kepler's Equation

- From position equation:

$$r = \frac{p}{1 + e\cos f} \Rightarrow p = 1.8359 \times 10^{5}\, Km$$

We need to find the time between $r_1 = 100,000km$ and $r_2 = 100,000km$. Find f at each of these points.

$$f_{1,2} = \cos{-1}\left(\frac{1}{e} - \frac{r_{1,2}}{p}\right) = \pm 64.8\,\deg$$

- To find the times t1 and t2 we need to find the hyperbolic eccentric anomaly and the apply Kepler's equation. Then we can find t2 only and double the time for symmetry.

$$H_2 = \tanh^{-1}\left(\sqrt{\frac{e-1}{e+1}}\tan\left(\frac{f_2}{2}\right)\right) = .1173$$

- Mean Anomaly: $M_2 = e\sinh(H_2) - H_2 = .0085$

$$t_2 = M_2\sqrt{\frac{-a^3}{\mu_{jup}}} = 1,076.6\,\sec \qquad \Delta t = 2 \cdot t_2 = 35\,\min$$

Solve Kepler's equation for the eccentric anomaly E given the eccentricity e and the mean anomaly M_e. See Appendix D.11 for the implementation of this algorithm in MATLAB.

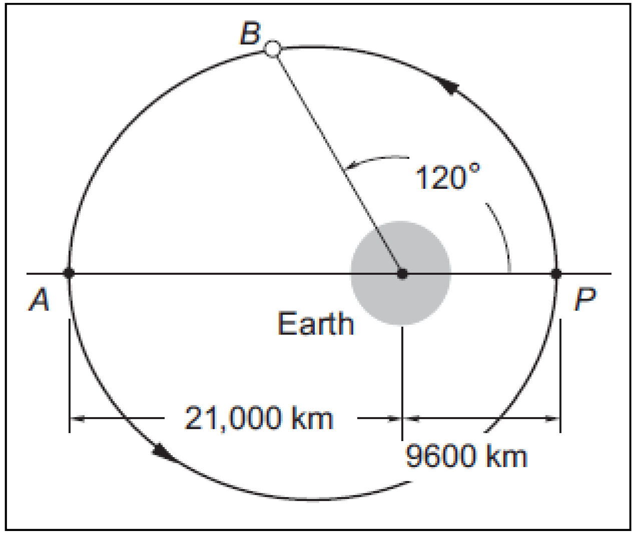
1. Choose an initial estimate of the root E as follows (Prussing and Conway, 1993). If $M_e < \pi$, then $E = M_e + e/2$. If $M_e > \pi$, then $E = M_e - e/2$. Remember that the angles E and M_e are in radians. (When using a hand held calculator, be sure that it is in the radian mode.)
2. At any given step, having obtained E_i from the previous step, calculate $f(E_i) = E_i - e\sin E_i - M_e$ and $f'(E_i) = 1 - e\cos E_i$.
3. Calculate $\text{ratio}_i = f(E_i)/f'(E_i)$.
4. If $|\text{ratio}_i|$ exceeds the chosen tolerance (e.g., 10^{-8}), then calculate an updated value of E,

$$E_{i+1} = E_i - \text{ratio}_i$$

   Return to Step 2.
5. If $|\text{ratio}_i|$ is less than the tolerance, then accept E_i as the solution to within the chosen accuracy.

Kepler's Equation

- **Problem:** A geocentric elliptical orbit has a perigee radius of 9600 km and an apogee radius of 21,000 km, as shown in Figure. Calculate the time to fly from perigee P to a true anomaly of 120 degrees.



1. Before anything else, let us find the primary orbital parameters e and h. The eccentricity is readily obtained from the perigee and apogee radii:

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{21{,}000 - 9600}{21{,}000 + 9600} = 0.37255$$

$$9600 = \frac{h^2}{398{,}600} \frac{1}{1 + 0.37255\cos(0)} \Rightarrow h = 72{,}472 \text{ km}^2/\text{s}$$

2. The orbital period T is given by:

$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1-e^2}}\right)^3 = \frac{2\pi}{398{,}600^2}\left(\frac{72{,}472}{\sqrt{1-0.37255^2}}\right)^3 = 18{,}834 \text{ s}$$

3. Compute the eccentric anomaly E:

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2} = \sqrt{\frac{1-0.37255}{1+0.37255}}\tan\frac{120°}{2} = 1.1711 \Rightarrow E = 1.7281 \text{ rad}$$

4. Use Kepler's equation to find the mean anomaly M:

$$M_e = 1.7281 - 0.37255 \sin 1.7281 = 1.3601 \text{ rad}$$

5. From which we find the time of perigee passage: $t = \frac{M_e}{2\pi} T = \frac{1.3601}{2\pi} 18{,}834 = \boxed{4077\text{s} \quad (1.132\,\text{h})}$

- **Find the true anomaly at three hours after perigee passage (10,800 sec.)**

  1. Since the time (10,800 s) is greater than one-half the period, the true anomaly must be greater than 180 deg. Compute the mean anomaly M:

  $$M_e = 2\pi \frac{t}{T} = 2\pi \frac{10{,}800}{18{,}830} = 3.6029 \text{ rad}$$

  2. Kepler's equation is then employed to find the eccentric anomaly. This transcendental equation will be solved using the previous Algorithm with an error tolerance of 10^{-6}. Since $M > \pi$, a good starting value for the iteration is

  $$E_0 = M_e - e/2 = 3.4166.$$

$E_0 = 3.4166$

$f(E_0) = -0.085124$

$f'(E_0) = 1.3585$

$\text{ratio} = \frac{-0.085124}{1.3585} = -0.062658$

$|\text{ratio}| > 10^{-6}$, so repeat.

$E_1 = 3.4166 - (-0.062658) = 3.4793$

$f(E_1) = -0.0002134$

$f'(E_1) = 1.3515$

$\text{ratio} = \frac{-0.0002134}{1.3515} = -1.5778 \times 10^{-4}$

$|\text{ratio}| > 10^{-6}$, so repeat.

$E_2 = 3.4793 - \left(-1.5778 \times 10^{-4}\right) = 3.4794$

$f(E_2) = -1.5366 \times 10^{-9}$

$f'(E_2) = 1.3515$

$\text{ratio} = \frac{-1.5366 \times 10^{-9}}{1.3515} - 1.137 \times 10^{-9}$

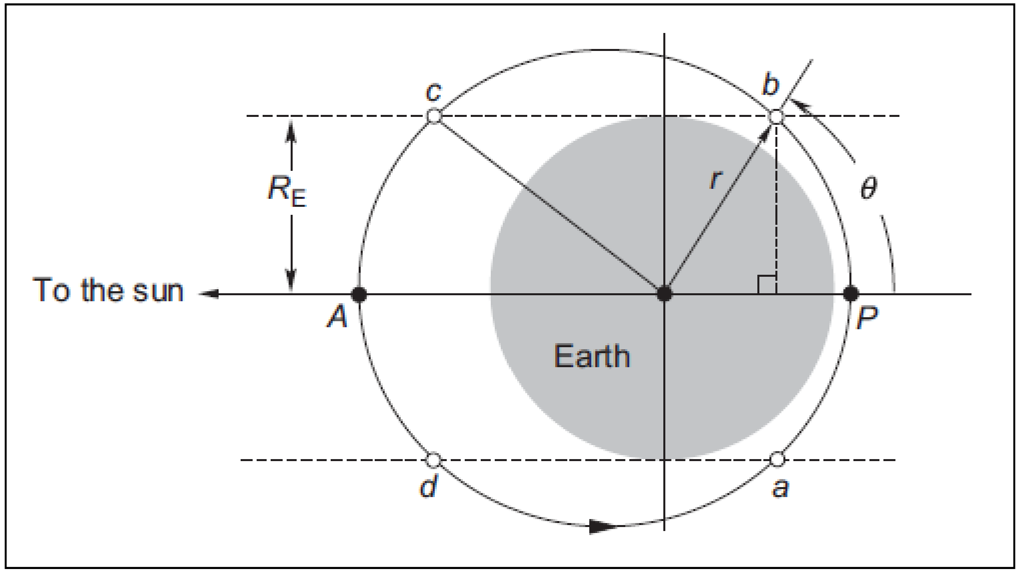
$|\text{ratio}| < 10^{-6}$, so accept $E = 3.4794$ as the solution.

Kepler's Equation

6. From the value of eccentric anomaly E, find the true anomaly θ:

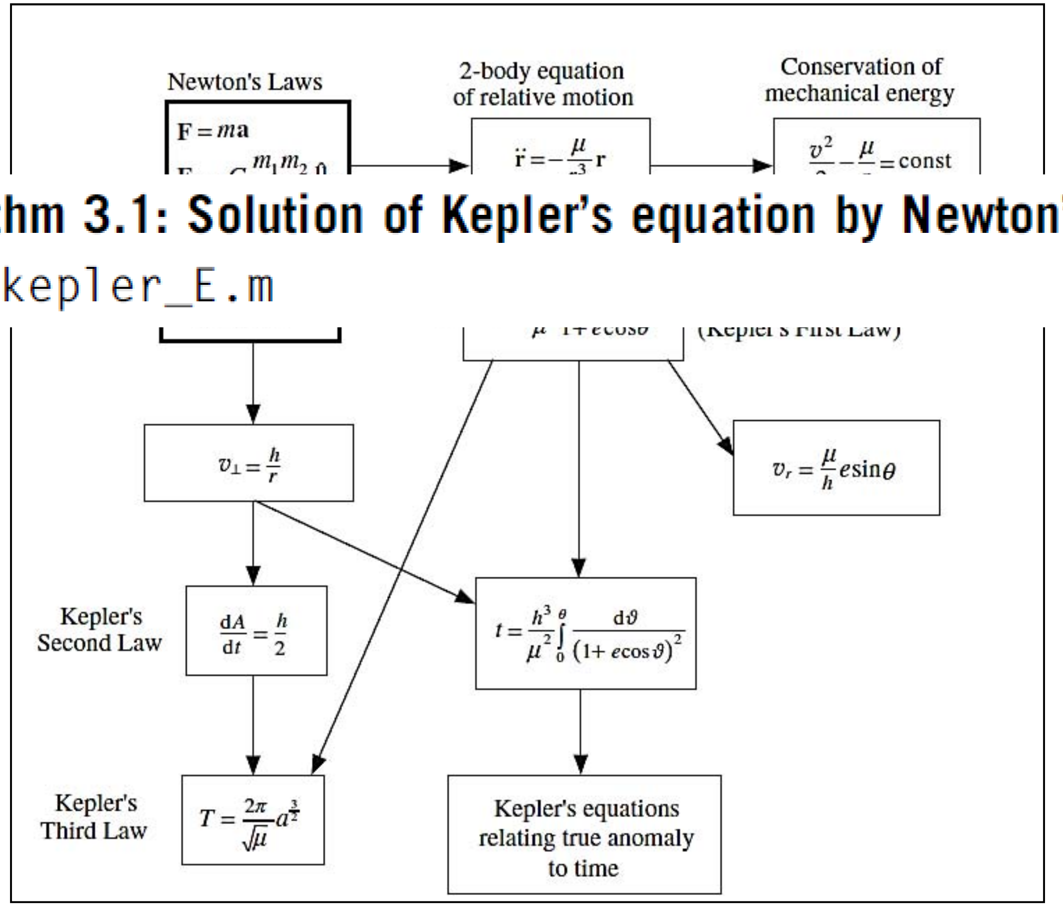
$$\tan\frac{\theta}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}=\sqrt{\frac{1+0.37255}{1-0.37255}}\tan\frac{3.4794}{2}=-8.6721\Rightarrow\boxed{\theta=193.2^{\circ}}$$

- **Exercise:** Let a satellite be in a 500 km by 5000 km orbit with its apse line parallel to the line from the earth to the sun, as shown in Figure. Find the time that the satellite is in the earth's shadow if: (a) the apogee is toward the sun and (b) the perigee is toward the sun.



Kepler’s Equation

- Summary:



D.11 Algorithm 3.1: Solution of Kepler’s equation by Newton’s method

Function file: `kepler_E.m`

https://www.youtube.com/watch?v=obRLUC_o_HQ

https://www.youtube.com/watch?v=wJf8B0SUdYM&t=2324s