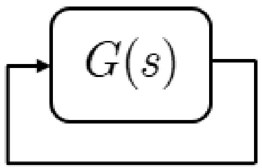
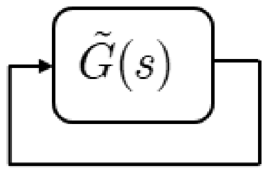
- **The previous results describe the stability robustness conditions for a particular location and type of bounded unstructured uncertainty in the control loop. In the early 1980's theorems were developed (<u>Lehtomaki, Sandell, Doyle, IEEE-TR-AC-2/1981</u>) to derive similar conditions as function of the uncertainty location.**

  - **Nomenclature:** Consider a unity feedback loop with a full state feeback structure. Let the nominal model and the perturbed system be given respectively by:

$$G(s) \Leftrightarrow \begin{pmatrix} A, & B, & K \end{pmatrix}$$



$$\tilde{G}(s) = G(s, \varepsilon), \varepsilon = [0,1] \Leftrightarrow \begin{pmatrix} \tilde{A}, & \tilde{B}, & K \end{pmatrix}$$



Open Loop and Closed Loop Model Characteristic Polynomials

$$\Phi_{OL}(s) = \det[sI - A]$$
$$\Phi_{CL}(s) = \det[sI - A + BK]$$

$$\det\left[I + G(s)\right] = \frac{\Phi_{CL}(s)}{\Phi_{OL}(s)}$$

Open Loop and Closed Loop Perturbed System Characteristic Polynomials

$$\tilde{\Phi}_{OL}(s) = \det[sI - \tilde{A}]$$
$$\tilde{\Phi}_{CL}(s) = \det[sI - \tilde{A} + \tilde{B}K]$$

$$\det\left[I + \tilde{G}(s)\right] = \frac{\tilde{\Phi}_{CL}(s)}{\tilde{\Phi}_{OL}(s)}$$

❑ Computational Details (properties of determinants):

- Consider the nominal system with loop transfer matrix $L(s)$

$$L(s) = K(sI - A)^{-1}B$$

1. Determinant of closed loop characteristic polynomial

$$\phi_{cl}(s) = \det[sI - A + BK]$$
$$= \underbrace{\det[sI - A]}_{\phi_{ol}(s)} \det\left[I + (sI - A)^{-1}BK\right]$$

2. Use the identity:

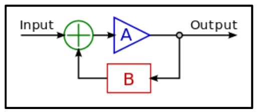
$$\det\left[I_n + \underbrace{F}_{n\times m}\underbrace{G}_{m\times n}\right] = \det\left[I_m + \underbrace{G}_{m\times n}\underbrace{F}_{n\times m}\right]$$

3. Substitute in 1:

$$\phi_{cl}(s) = \phi_{ol}(s)\det\left[I + (sI - A)^{-1}\mathrm{BK}\right]$$
$$= \phi_{ol}(s)\det\left[I + \underbrace{K(sI - A)^{-1}B}_{L(s)}\right]$$
$$= \phi_{ol}(s)\det[I + L(s)]$$

$$\boxed{\det[I + L(s)] = \frac{\phi_{cl}(s)}{\phi_{ol}(s)}}$$

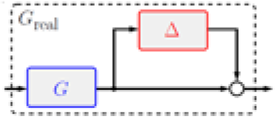
$$\boxed{\det\left[I + \tilde{G}(s)\right] = \frac{\tilde{\Phi}_{CL}(s)}{\tilde{\Phi}_{OL}(s)}}$$



$$\Delta_{ROUT}(s) = \left[\tilde{G}(s) - G(s)\right] G^{-1}(s)$$

- **Multiplicative output uncertainty:**
  $G_{\text{real}}(s) = (I + \Delta(s))G(s)$

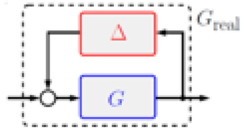
  Useful for: HF plant dynamics, sensor errors, uncertain zeros



$$\Delta_{A}^{INVERSE}(s) = \tilde{G}^{-1}(s) - G^{-1}(s)$$

- **Inverse additive uncertainty:**
  $G_{\text{real}}(s) = G(s)(I - \Delta(s)G(s))^{-1}$

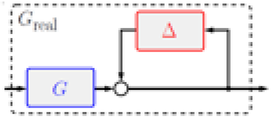
  Useful for:
  low-freq. (LF) parameter errors,
  uncertain poles



$$\Delta_{ROUT}^{INVERSE}(s) = G(s)\left[\tilde{G}^{-1}(s) - G^{-1}(s)\right]$$

- **Inverse mult. output uncertainty:**
  $G_{\text{real}}(s) = (I - \Delta(s))^{-1}G(s)$

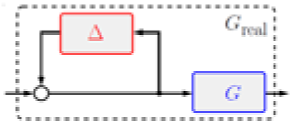
  Useful for:
  low-freq. (LF) parameter errors,
  uncertain poles

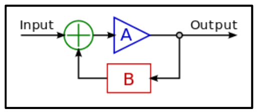


$$\Delta_{RINP}^{INVERSE}(s) = \left[\tilde{G}^{-1}(s) - G^{-1}(s)\right] G(s)$$

- **Inverse mult. input uncertainty:**
  $G_{\text{real}}(s) = G(s)(I - \Delta(s))^{-1}$

  Useful for:
  low-freq. (LF) parameter errors,
  uncertain poles





❑ Stability Margins Implications

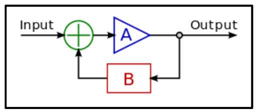
- **Basic Idea:** if the deformation of the Nyquist diagram from the model to the system is achieved without changing the number of encirclements, then there is no instability due to the perturbation.

  - **Note:** the Nyquist Criterion <u>does not require any information on the location of the unstructured uncertainty</u> in the loop, and it provides (formally) information about absolute stability.

  - **Fundamental Robustness Theorem (Theorem 2 – Lehtomaki – IEEE-TR-AC- 02/1981)**

    The closed loop perturbed system is stable, that is $\tilde{\Phi}_{CL}(s)$ has NO CRHP zeros if:

    1. $\tilde{\Phi}_{OL}(s)$ and $\Phi_{OL}(s)$ have the same number of CRHP zeros
    2. $\Phi_{CL}(s)$ Has no CRHP zeros. That is the closed loop model is stable $\Rightarrow \det[I+G] \neq 0$
    3. If $\tilde{\Phi}_{OL}(j\omega_0) = 0$ then $\Phi_{OL}(j\omega_0) = 0$
    4. $\boxed{\det\left[I + G(s,\varepsilon)\right] \neq 0, \forall (s,\varepsilon) \in D_R \times [0,1]}$



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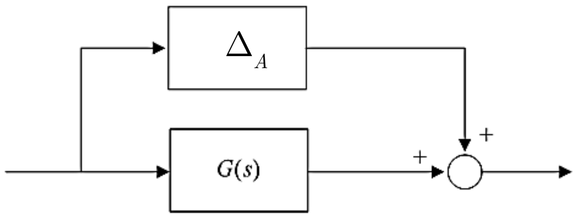
❑ Computational details

- **Theorem 3**

Given the structure in the figure, the closed loop Perturbed system is stable, that is $\tilde{\Phi}_{CL}(s)$ has NO CRHP zeros if:



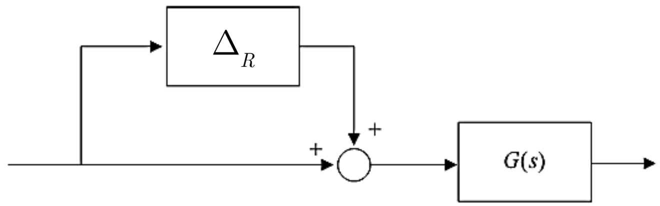
A. Points 1, 2, 3 of Theorem 2 are satisfied
B. The following holds:

$$\boxed{\sigma_{\min}\left[I + G(s)\right] > \sigma_{\max}(\Delta_A)}$$

$$\forall (s,\varepsilon) \in D_R \times [0,1]$$

- **Theorem 4**

Given the structure in the figure, the closed loop Perturbed system is stable, that is $\tilde{\Phi}_{CL}(s)$ has NO CRHP zeros if:



A. Points 1, 2, 3 of Theorem 2 are satisfied
B. The following holds:

$$\boxed{\sigma_{\min}\left[I + G^{-1}(s)\right] > \sigma_{\max}(\Delta_R)}$$

$$\forall (s,\varepsilon) \in D_R \times [0,1]$$

- **Sketch of Proof:** we are interested in verifying if and when the return difference matrix for the perturbed system becomes singular , that is:

$$\begin{cases} \det\left[I + G(s,\varepsilon)\right] = 0, \forall (s,\varepsilon) \in D_R \times [0,1] \\ G(s,\varepsilon) = (1-\varepsilon)G(s) + \varepsilon \tilde{G}(s) \end{cases}$$

- For the multiplicative error characterization:

$$I + G(s,\varepsilon) = I + G(s) + \varepsilon\Delta_R(s)G(s)$$

- But $[I + G(s)]$ is not singular, i.e. $\det[I + G(s)] \neq 0$ by assumption, therefore we must find:

$$\begin{cases} \sigma_{\min}\left[A + B\right] = 0 \\ A = I + G(s);\ B = \varepsilon\Delta_R(s)G(s) \end{cases}$$

- Assume $(A + B)$ to be singular, then $(A + B)$ is rank deficient. This implies:

$$\exists\left\{\boldsymbol{\nu} \neq 0, \left\|\boldsymbol{\nu}\right\|_2 = 1\right\} \Leftrightarrow (A + B)\boldsymbol{\nu} = 0 \in \mathcal{N}(A + B)$$

- Equivalently:

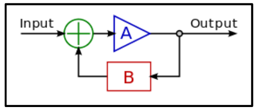
$$A\boldsymbol{\nu} = -B\boldsymbol{\nu} \Leftrightarrow \left\|\mathrm{A}\boldsymbol{\nu}\right\|_2 = \left\|B\boldsymbol{\nu}\right\|_2$$

- Using singular values:

$$\min_{\boldsymbol{\nu}\neq 0}\frac{\left\|\mathrm{A}\boldsymbol{\nu}\right\|_2}{\left\|\boldsymbol{\nu}\right\|_2} = \sigma_{\min}(A) \leq \left\|\mathrm{A}\boldsymbol{\nu}\right\|_2 = \left\|B\boldsymbol{\nu}\right\|_2 \leq \sigma_{\max}(B) = \max_{\boldsymbol{\nu}\neq 0}\frac{\left\|B\boldsymbol{\nu}\right\|_2}{\left\|\boldsymbol{\nu}\right\|_2}$$

- Which yields:

$$\det(A + B) = 0 \Rightarrow \sigma_{\min}(A) \leq \sigma_{\max}(B)$$



- In conclusion:

$$\sigma_{\min}(A) > \sigma_{\max}(B) \Rightarrow \det(A+B) \neq 0$$

$$\boxed{\sigma_{\min}\left[I + G(s)\right] > \sigma_{\max}(\Delta_A)}$$

- This proves Theorem 3

- **Similarly we can write for Theorem 4**

$$I + G^{-1}(s,\varepsilon) = G(s)\left[I + G^{-1}(s) + \varepsilon\Delta_R(s)\right]; \det\left[G(s)\right] \neq 0$$

- Therefore we have the implication:

$$\det\left[I + G(s,\varepsilon)\right] \neq 0 \Leftrightarrow \sigma_{\min}\left[I + G^{-1}(s)\right] > \sigma_{\max}(\varepsilon\Delta_R)$$

$$\boxed{\sigma_{\min}\left[I + G^{-1}(s)\right] > \left|\varepsilon\right|\sigma_{\max}(\Delta_R) > \sigma_{\max}(\Delta_R)}$$

- This proves Theorem 4

$$\begin{cases} \sigma_{\min}\left[I + G(s)\right] > \sigma_{\max}(\Delta_A) \\ \\ \sigma_{\min}\left[I + G^{-1}(s)\right] > \sigma_{\max}(\Delta_R) \end{cases}$$

- Input/Output Additive direct
- Input/Output Multiplicative direct

- **Summary:** The above are the sufficient conditions described by theorems 3 and 4. This result is conservative in the sense that:
    1. It considers the worst case $\varepsilon = 1$
    2. It considers full complex uncertainty (unstructured)

- **For additive and multiplicative <u>inverse</u> errors:**

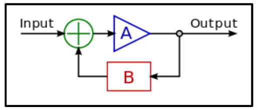
$$\Delta_A^{INVERSE}(s) = \tilde{G}^{-1}(s) - G^{-1}(s)$$
$$\Delta_{RINP}^{INVERSE}(s) = \left[\tilde{G}^{-1}(s) - G^{-1}(s)\right] G(s)$$

- Similarly to the previous derivation:

$$G(s,\varepsilon) = \left[\varepsilon \Delta_A(s) + G^{-1}(s)\right]^{-1}$$
$$G(s,\varepsilon) = G(s)\left[1 + \varepsilon \Delta_R(s)\right]^{-1}$$

- from which

$$G(s,\varepsilon) = \left[(1-\varepsilon)\,G^{-1}(s) + \varepsilon \tilde{G}^{-1}(s)\right]^{-1}$$



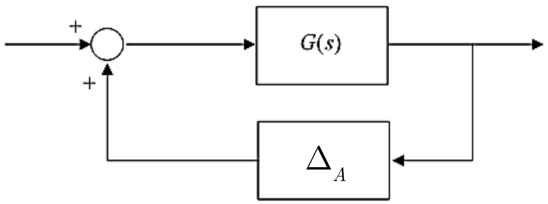
- The inverse uncertainty in its multiplicative form is then:

$$L(s)=\left[I+\Delta_R(s)\right]^{-1}$$

- This yields the robustness conditions for inverse perturbations expressed in multiplicative form:

Theorem 5

Given the structure in the figure, the closed loop Perturbed system is stable, that is $\tilde{\Phi}_{CL}(s)$ has NO CRHP zeros if:

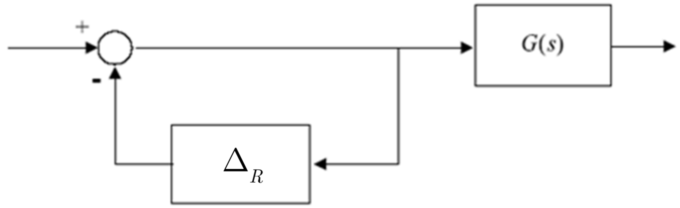


A. Points 1, 2, 3 of Theorem 2 are satisfied
B. The following holds:

$$\boxed{\sigma_{\min}\left[I+G^{-1}(s)\right]>\sigma_{\max}(\Delta_A)}$$
$$\forall(s,\varepsilon)\in D_R\times[0,1]$$

Theorem 6

Given the structure in the figure, the closed loop Perturbed system is stable, that is $\tilde{\Phi}_{CL}(s)$ has NO CRHP zeros if:



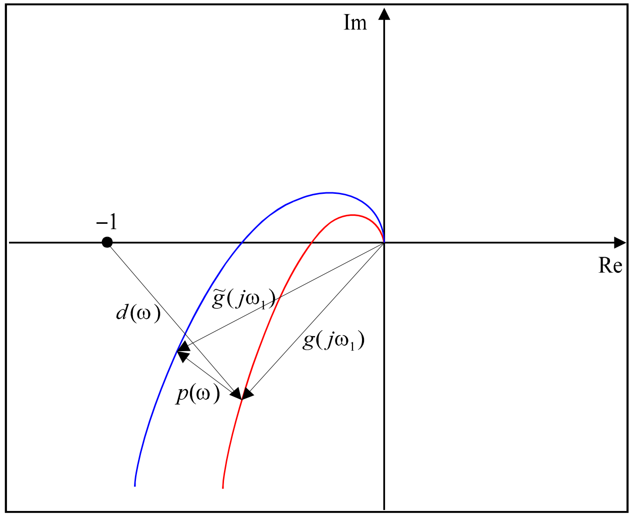
A. Points 1, 2, 3 of Theorem 2 are satisfied
B. The following holds:

$$\boxed{\sigma_{\min}\left[I+G(s)\right]>\sigma_{\max}(\Delta_R)}$$
$$\forall(s,\varepsilon)\in D_R\times[0,1]$$

- Depending on the uncertainty type (additive, multiplicative) and location, robustness theorems lead to the following inequalities:

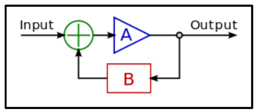
$$\boxed{\sigma_{\min}\left[I+G(s)\right] > \sigma_{\max}(\Delta_A)} \qquad \boxed{\sigma_{\min}\left[I+G^{-1}(s)\right] > \sigma_{\max}(\Delta_M)}$$

- **Question:** Can we extend the concept of Stability Margins to the MIMO Case using the above relationships?



- **Recall the SISO case: NO changes** in the Nyquist diagram encirclement of (-1, 0) occur if:

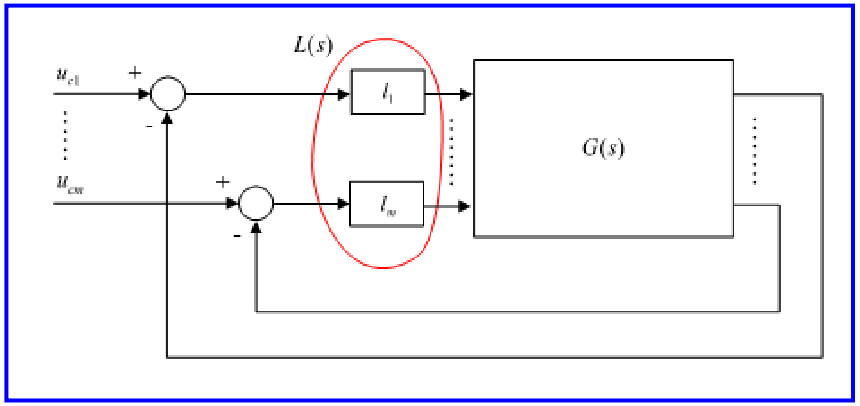
$$\begin{cases} \left|1+g^{-1}(s)\right| > \left|L_M(s)-1\right| = \left|\Delta_M\right| \\ \left|1+g(s)\right| > \left|L_A^{-1}(s)-1\right| = \left|\Delta_A\right| \end{cases}$$



- We can extend the constraints to the MIMO case by introducing the singular values of the appropriate transfer function matrices:

$$
\begin{cases}
\sigma_{\min}\left[I+G^{-1}\right] > \sigma_{\max}\left[L_M - I\right] = \sigma_{\max}\left[\Delta_M\right] \\
\sigma_{\min}\left[I+G\right] > \sigma_{\max}\left[L_A^{-1} - I\right] = \sigma_{\max}\left[\Delta_A\right]
\end{cases}
$$

- The uncertainty must appear simultaneously in all channels either as gain perturbation or phase perturbation (this is a conservative result due to the use of singular values as measure of size of a matrix).



MIMO Gain Margin: is the pair of real (c_1, c_2) that defines the largest closed interval such that when $l_i(j\omega)$ are all real constant that satisfy

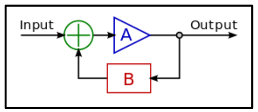
$$c_1 < l_i < c_2 \quad i = 1,..,m$$

the closed loop system remains stable.

MIMO Phase Margin: is the pair of real $(-c_1, c_1)$ that defines the largest closed interval such that when $l_i(j\omega) = e^{j\phi_i(\omega)}$, and ϕ_i satisfy

$$-c_1 < \phi_i < c_1 \quad i = 1,..,m$$

the closed loop system remains stable.



1. Consider the uncertainty present as <u>gain perturbation</u> from nominal:

$$\boxed{\left|L(j\omega)\right| = diag\left\{\left|l_i(j\omega)\right|\right\}}$$

- Assume the error to be bounded by some value α: $\min_{\omega}\left\{\sigma_{\min}\left[I + G^{-1}\right]\right\} = \alpha$

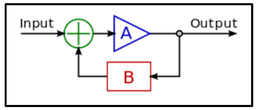
- From the robust stability requirement:

$$\begin{cases} \sigma_{\min}\left[I + G^{-1}\right] > \sigma_{\max}\left[L_M - I\right] = \sigma_{\max}\left[\Delta(s)\right] \\ \sigma_i\left[\Delta(s)\right] = \sigma_i\left[L - I\right] = \left|l_i - 1\right| \end{cases}$$

- If the largest $|l_i - 1|$ is less than α, then we have the following result:

$$\boxed{1 - \alpha \le l_i \le 1 + \alpha}$$

- Which guarantees a gain margin interval: $[\ 1 - \alpha,\ 1 + a\ ]$



2. Consider the uncertainty present as phase perturbation from nominal: $\boxed{\measuredangle L(j\omega) = diag\left\{\measuredangle l_i(j\omega)\right\}}$

- This can be rewritten as: $\left|l_i - 1\right| = \left|e^{j\phi_i(\omega)} - 1\right| \leq \beta$

- Using Euler formulas:

$$\left|e^{j\phi_i(\omega)} - 1\right| = \left|\cos\left[\phi_i(\omega)\right] - 1 + j\sin\left[\phi_i(\omega)\right]\right| \leq \beta$$

$$\boxed{\begin{aligned} |\varepsilon_i - 1| = \left|e^{j\phi_i(\omega)} - 1\right| &\leq \beta_\sigma \\ = |\cos(\phi_i(\omega)) - 1 + j\sin(\phi_i(\omega))| &\leq \beta_\sigma \\ = \left(\cos^2(\phi_i(\omega)) - 2\cos(\phi_i(\omega)) + 1 + \sin^2(\phi_i(\omega))\right)^{\frac{1}{2}} &\leq \beta_\sigma \\ = (2(1 - \cos(\phi_i(\omega))))^{\frac{1}{2}} &\leq \beta_\sigma \\ = \left(4\sin^2\left(\frac{\phi_i(\omega)}{2}\right)\right)^{\frac{1}{2}} &\leq \beta_\sigma \end{aligned}}$$

- Which guarantees a phase margin interval: $\boxed{\pm 2\sin^{-1}\left(\frac{\beta}{2}\right)}$

Corollary 1 (Th. 4): If $\Phi_{CL}(j\omega)$ has no CRHP zeros, and $\underline{\sigma}\left[I+G^{-1}\right]>\alpha$ for $\omega\in\Omega_R$, then the stability margins are limited by:

$$GM\in\left[1-\alpha,1+\alpha\right]$$
$$PM\in\left[-2\sin^{-1}\frac{\alpha}{2},2\sin^{-1}\frac{\alpha}{2}\right]$$

Corollary n° 2 (Th. 6): If $\Phi_{CL}(j\omega)$ has no CRHP zeros, and $\underline{\sigma}\left[I+G\right]>\alpha$ for $\omega\in\Omega_R$, then the stability margins are limited by:

$$GM\in\left[\frac{1}{1-\alpha},\frac{1}{1+\alpha}\right]$$
$$PM\in\left[-2\sin^{-1}\frac{\alpha}{2},2\sin^{-1}\frac{\alpha}{2}\right]$$

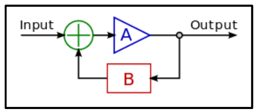
$$\mathrm{GM}=\mathrm{GM}_{I+L}\cup\mathrm{GM}_{I+L^{-1}},\qquad \mathrm{PM}=\mathrm{PM}_{I+L}\cup\mathrm{PM}_{I+L^{-1}}$$

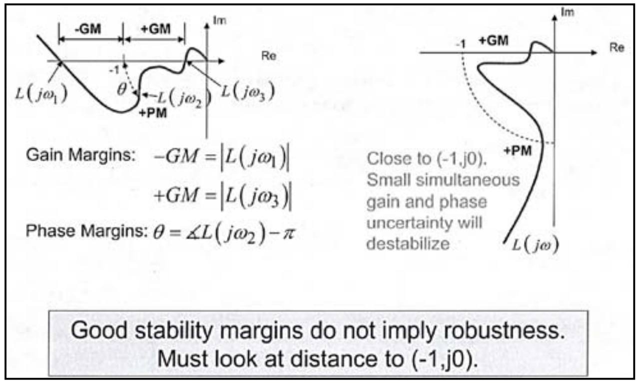
- **Note # 1:** from the previous corollaries, the best minimum singular value is obtained when $\alpha=1$. This yields:

$$\begin{cases}\sigma_{\min}\left[I+G\right]=1\Rightarrow GM_{I+G}=\left[\frac{1}{2},+\infty\right]=\left[-6dB,+\infty\right]\\ \sigma_{\min}\left[I+G^{-1}\right]=1\Rightarrow GM_{I+G^{-1}}=\left[0,2\right]=\left[-\infty,+6dB\right]\end{cases}$$

$$\begin{cases}PM_{I+G}=\left[\pm2\sin^{-1}\left(\frac{1}{2}\right)\right]=\left[-60^0,+60^0\right]\\ PM_{I+G^{-1}}=\left[\pm2\sin^{-1}\left(\frac{1}{2}\right)\right]=\left[-60^0,+60^0\right]\end{cases}$$

- **Note # 2:** Stability Margins are a conservative measure of robustness. They may fail in the case of ill-conditioned systems:





Example: Consider the linear-time-invariant (LTI) ***short period*** pitch-plane dynamic approximation of an unmanned aircraft

Translation along Zb

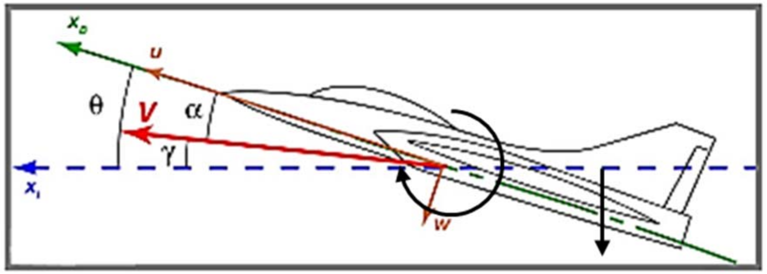
$$\dot{w} \approx Z_w w + Z_{\delta_E}\delta_E \Rightarrow \dot{\alpha} = \frac{Z_\alpha}{V}\alpha + \frac{Z_{\delta_E}}{V}\delta_E$$

Rotation about Yb

$$\dot{q} = \ddot{\theta} = M_\alpha \alpha + M_{\delta_E}\delta_E$$

Center of Mass Acceleration along Zb

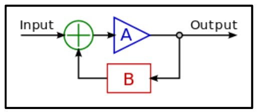
$$Az = \dot{w} - Vq \approx \dot{w} = Z_\alpha \alpha + Z_{\delta_E}\delta_E$$



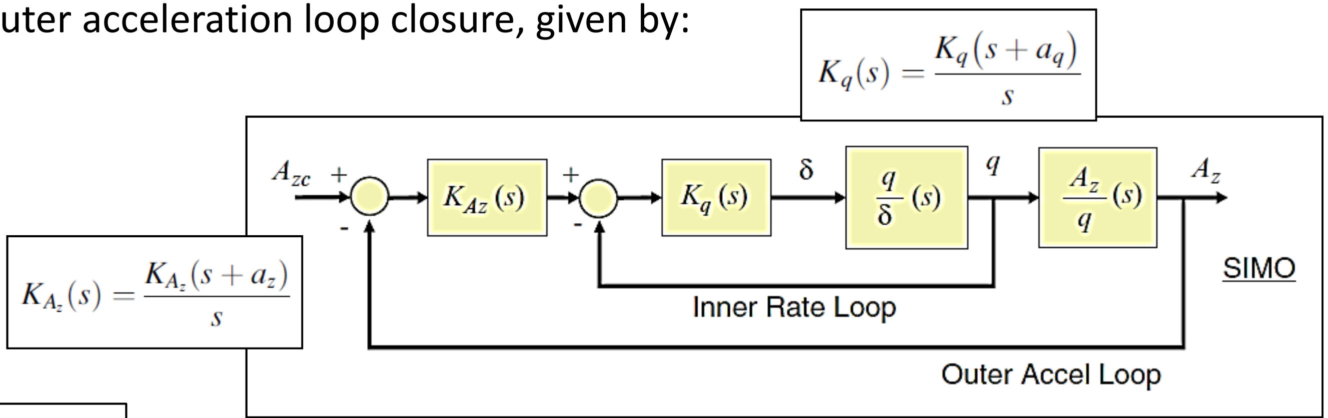
$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{Z_\alpha}{V} & 1 \\ M_\alpha & 0 \end{bmatrix}}_{A} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{Z_\delta}{V} \\ M_\delta \end{bmatrix}}_{B} \delta_e$$

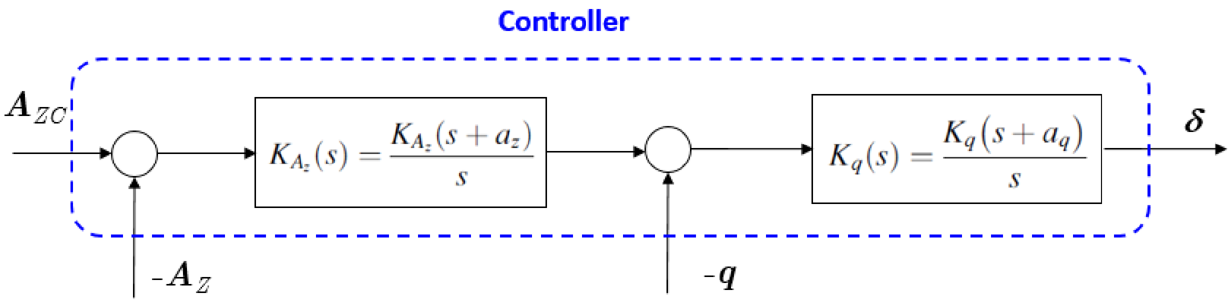
$$\begin{bmatrix} A_z \\ q \end{bmatrix} = \underbrace{\begin{bmatrix} Z_\alpha & 0 \\ 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \underbrace{\begin{bmatrix} Z_\delta \\ 0 \end{bmatrix}}_{D} \delta_e$$

$$G(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} \frac{A_z}{\delta_e} \\ \frac{q}{\delta_e} \end{bmatrix}$$



- The system has 1 input and 2 outputs (SIMO). The autopilot (controller) for this plant contains proportional - plus-integral control elements in the inner rate loop closure and outer acceleration loop closure, given by:

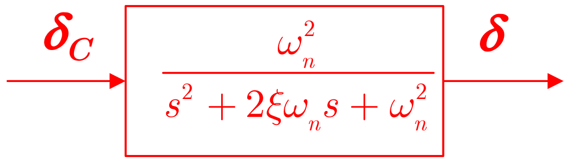




- Add a second order actuator model and consider a flight condition corresponding to unstable dynamics

$$
\begin{aligned}
\dot{\alpha} &= \frac{Z_\alpha}{V}\alpha + \frac{Z_\delta}{V}\delta_e + q \\
\dot{q} &= M_\alpha \alpha + M_\delta \delta_e \\
\ddot{\delta}_e &= -2\zeta\omega_n\delta_e - \omega_n^2(\delta_e - \delta_c)
\end{aligned}
$$

$$
\begin{cases}
\zeta = 0.6 \\
\omega_n = 113\, rad / \sec
\end{cases}
$$



- Rewrite the model in state space format:

$$\dot{x} = A_p x + B_p u$$
$$y = C_p x + D_p u$$

$$\begin{bmatrix} A_p & B_p \\ C_p & D_p \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -1.3046 & 1.0 & -0.21420 & 0 \\ 47.711 & 0 & -104.83 & 0 \\ 0 & 0 & 0 & 1.0 \\ 0 & 0 & -1.2769.0 & -1.356 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 12769.0 \end{bmatrix} \\ \begin{bmatrix} -1156.9 & 0 & -189.95 & 0 \\ 0 & 1.0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Rewrite the controller in state space format ($\boldsymbol{y}$ is the model state vector, $\boldsymbol{r}$ is the commanded Acceleration $\boldsymbol{A}_{ZC}$):

- Using the following numerical values:

$$K_{az} = -0.0015$$
$$K_{q=} - 0.32$$
$$a_q = 2.0$$
$$a_z = 6.0$$

$$\dot{x}_c = A_c x_c + B_{c1} y + B_{c2} r$$
$$u = C_c x_c + D_{c1} y + D_{c2} r$$

$$\begin{bmatrix} A_c & B_{c1} & B_{c2} \\ C_c & D_{c1} & D_{c2} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ -1.92 & 0 \end{bmatrix} & \begin{bmatrix} 0.0030 & 0 \\ -0.0029 & 1.92 \end{bmatrix} & \begin{bmatrix} -0.0030 \\ 0.0029 \end{bmatrix} \\ [-0.32 \quad 1.0] & [-0.0005 \quad 0.32] & [0.00048] \end{bmatrix}$$

- Build the complete state space model:

$$u = C_c x_c + D_{c1} y + D_{c2} r$$
$$u = C_c x_c + D_{c1}\left(C_p x + D_p u\right) + D_{c2} r$$
$$\underbrace{\left(I - D_{c1} D_p\right)}_{Z} u = C_c x_c + D_{c1} C_p x + D_{c2} r$$
$$u = Z^{-1}\left(C_c x_c + D_{c1} C_p x + D_{c2} r\right)$$

- **NOTE:** the matrix Z must be invertible for the feedback problem to be well-posed!

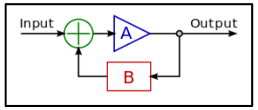
$$(I - D_{c1} D_p) = 1 - \begin{bmatrix} -0.0005 & 0.32 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$$

- Build the complete closed loop system:

$$\dot{x} = \left(A_p + B_p Z^{-1} D_{c1} C_p\right) x + B_p Z^{-1} C_c x_c + B_p Z^{-1} D_{c2} r$$
$$\dot{x}_c = \left(A_c + B_{c1} D_p Z^{-1} C_c\right) x_c + B_{c1}\left(I + D_p Z^{-1} D_{c1}\right) C_p x + \left(B_{c2} + B_{c1} D_p Z^{-1} D_{c2}\right) r$$

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \underbrace{\begin{bmatrix} A_p + B_p Z^{-1} D_{c1} C_p & B_p Z^{-1} C_c \\ B_{c1}\left(I + D_p Z^{-1} D_{c1}\right) C_p & A_c + B_{c1} D_p Z^{-1} C_c \end{bmatrix}}_{A_{cl}} \begin{bmatrix} x \\ x_c \end{bmatrix} + \underbrace{\begin{bmatrix} B_p Z^{-1} D_{c2} \\ B_{c2} + B_{c1} D_p Z^{-1} D_{c2} \end{bmatrix}}_{B_{cl}} r$$

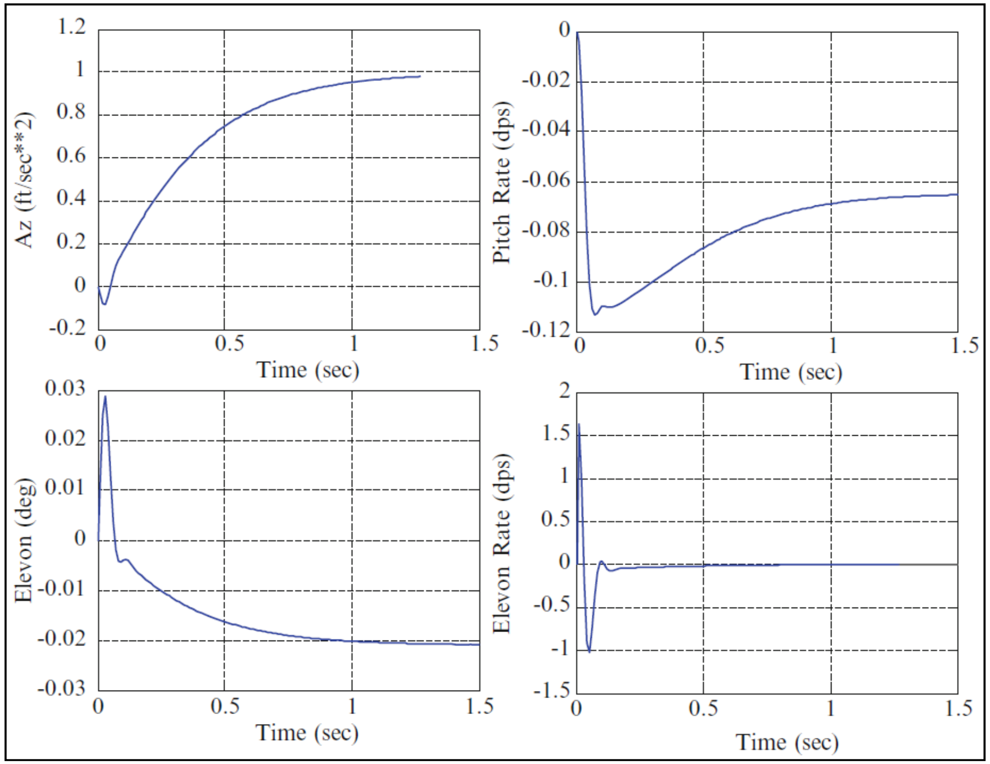
$$\begin{aligned} y &= C_p x + D_p u \\ &= C_p x + D_p Z^{-1}\left(C_c x_c + D_{c1} C_p x + D_{c2} r\right) \\ &= \begin{bmatrix} C_p + D_p Z^{-1} D_{c1} C_p & D_p Z^{-1} C_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} D_p Z^{-1} D_{c2} \end{bmatrix} r \\ & \underbrace{\begin{bmatrix} \left(I + D_p Z^{-1} D_{c1}\right) C_p & D_p Z^{-1} C_c \end{bmatrix}}_{C_{cl}} \begin{bmatrix} x \\ x_c \end{bmatrix} + \underbrace{\begin{bmatrix} D_p Z^{-1} D_{c2} \end{bmatrix}}_{D_{cl}} r \end{aligned}$$



- In summary:

$$\dot{x}_a = A_{cl}\,x_a + B_{cl}\,r$$
$$y = C_{cl}\,x_a + D_{cl}\,r$$

- Closed Loop Time Histories

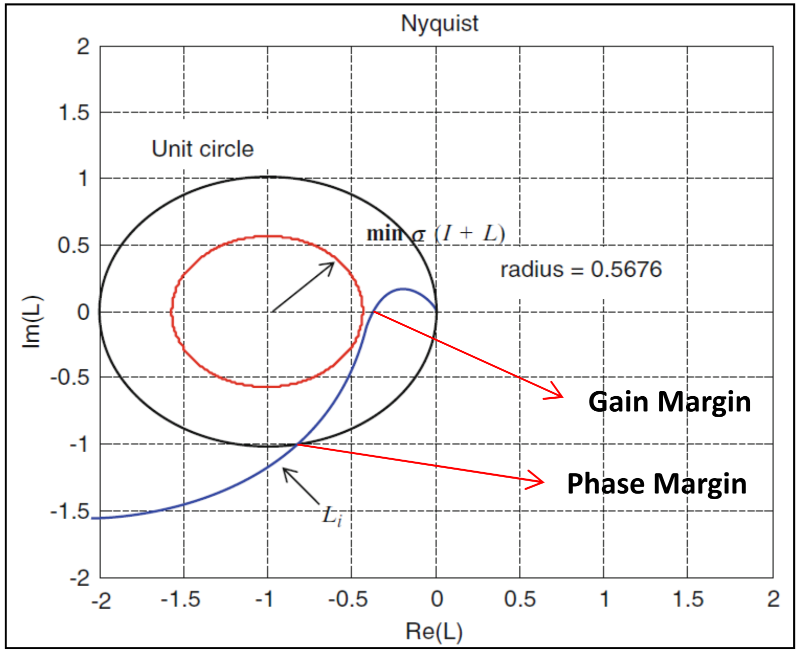


- **Exercise:** Derive the loop transfer function matrices at input to the plant $\boldsymbol{L_i}(s)$ and at the output of the plant $\boldsymbol{L_o}(s)$ respectively, in literal form:

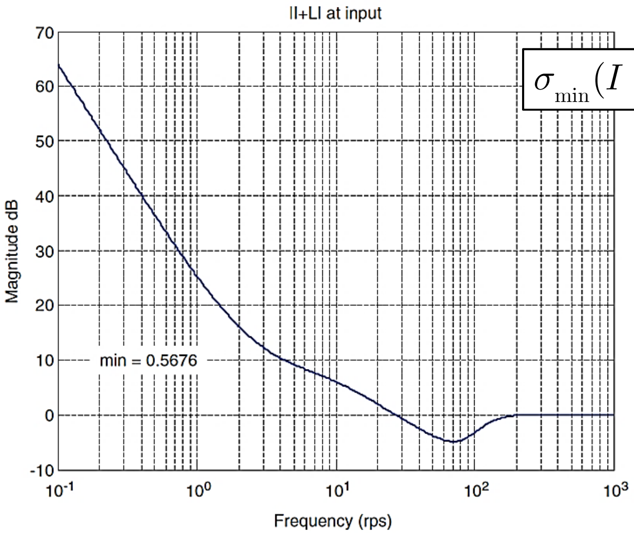
$$\begin{cases} L_i(s) = C_{Li}(s)\left[sI - A_{Li}\right]^{-1} B_{Li} + D_{Li} \\ L_o(s) = C_{Lo}(s)\left[sI - A_{Lo}\right]^{-1} B_{Lo} + D_{Lo} \end{cases}$$

- **Consider the loop broken at the input and compute the classical stability margins:**

$$\begin{cases} GM \approx 8.8dB \\ PM \approx 50^0 \end{cases}$$

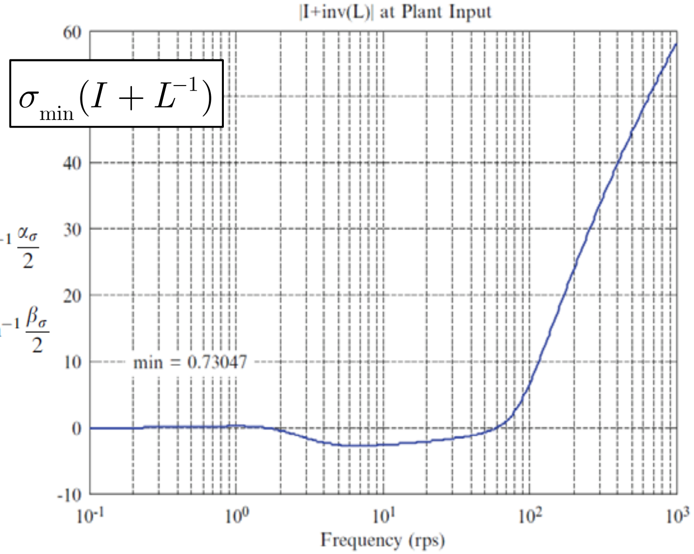


- Multivariable Gain and Phase Margins:



$$\mathrm{GM}_{I+L} = \left[\frac{1}{1+\alpha_\sigma}, \frac{1}{1-\alpha_\sigma}\right]; \quad \mathrm{PM}_{I+L} = \pm 2\sin^{-1}\frac{\alpha_\sigma}{2}$$

$$\mathrm{GM}_{I+L^{-1}} = [1-\beta_\sigma, 1+\beta_\sigma]; \quad \mathrm{PM}_{I+L^{-1}} = \pm 2\sin^{-1}\frac{\beta_\sigma}{2}$$



- The minimum values can be computed:

$$\alpha_\sigma = \min \underline{\sigma}(I+L) = 0.5676; \quad \beta_\sigma = \min \underline{\sigma}(I+L^{-1}) = 0.7305$$

$$\begin{cases} \sigma_{\min}\left[I+L\right] = 1 \Rightarrow GM_{I+L} = \left[0.6379, 2.3127\right] = \left[-3.9, 7.28\right] dB \\ \sigma_{\min}\left[I+L^{-1}\right] = 1 \Rightarrow GM_{I+L^{-1}} = \left[0.2695, 1.7305\right] = \left[-11.4, 4.7\right] dB \end{cases}$$

$$\begin{cases} PM_{I+L} = \left[\pm 2\sin^{-1}\left(0.5676\right)\right] = \left[\pm 32.97^0\right] \\ PM_{I+L^{-1}} = \left[\pm 2\sin^{-1}\left(0.7305\right)\right] = \left[\pm 42.84^0\right] \end{cases}$$

$$\boxed{GM = [\ \text{-}11.4 \quad 7.28]\ \text{dB}; \qquad PM = \ \pm 42.84\ \text{deg}}$$

$$\boxed{GM \approx 8.8dB,\ PM \approx 50^0}$$

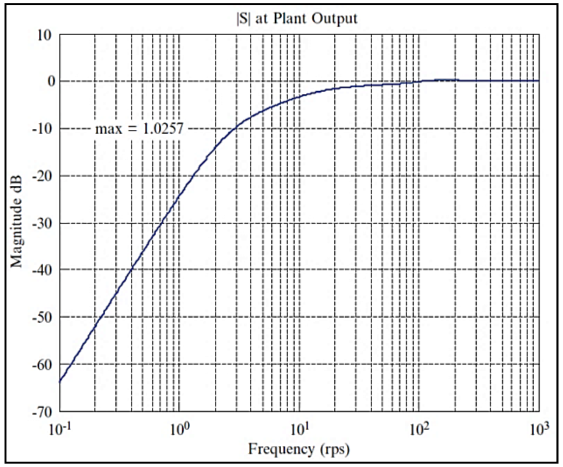
$$\boxed{GM = [\ -11.4 \quad 7.28]\ \text{dB}; \qquad PM = \ \pm 42.84\ \text{deg}}$$

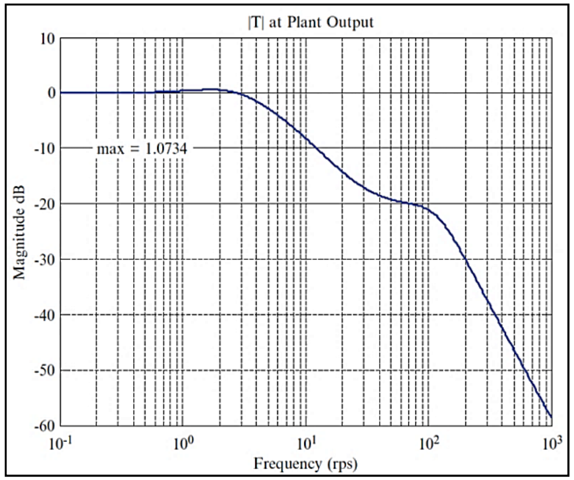
- Note that the stability margins computed via singular values are **more conservative** than those using classical control.

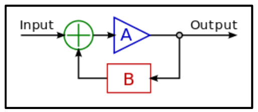
- **Consider the loop broken at the output and compute the sensitivity and complementary sensitivity functions**

$$\begin{cases} S(s) = \left[I + L_o(s)\right]^{-1} \\ T(s) = L_o(s)\left[I + L_o(s)\right]^{-1} \end{cases}$$

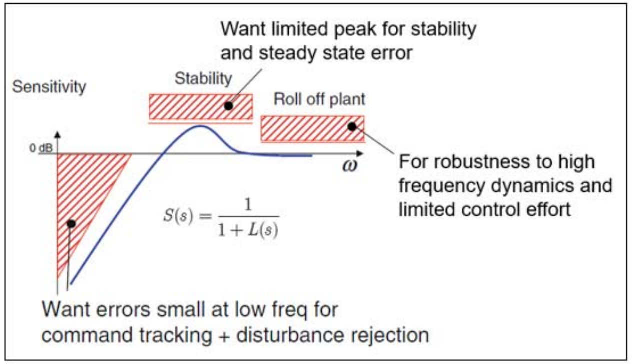
$$\begin{cases} \left\|S(s)\right\|_\infty = \sup\left\{\sigma_{\max}\left[I + L_o(s)\right]^{-1}\right\} \\ \left\|T(s)\right\|_\infty = \sup\left\{\sigma_{\max} L_o(s)\left[I + L_o(s)\right]^{-1}\right\} \end{cases}$$

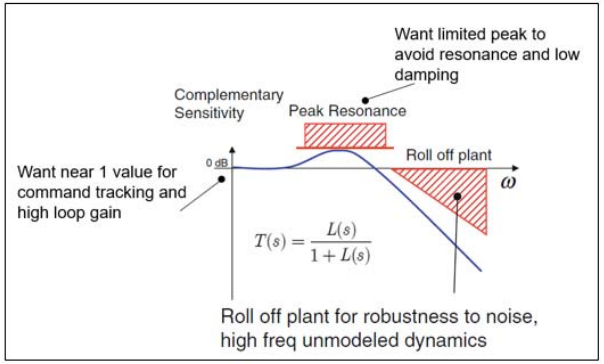






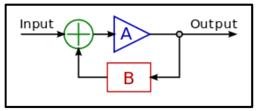
$$\begin{cases} \left\| S(s) \right\|_{\infty} = \sup\left\{ \sigma_{\max}\left[I + L_o(s) \right]^{-1} \right\} = 1.0257 \\ \left\| T(s) \right\|_{\infty} = \sup\left\{ \sigma_{\max} L_o(s)\left[I + L_o(s) \right]^{-1} \right\} = 1.0734 \end{cases}$$



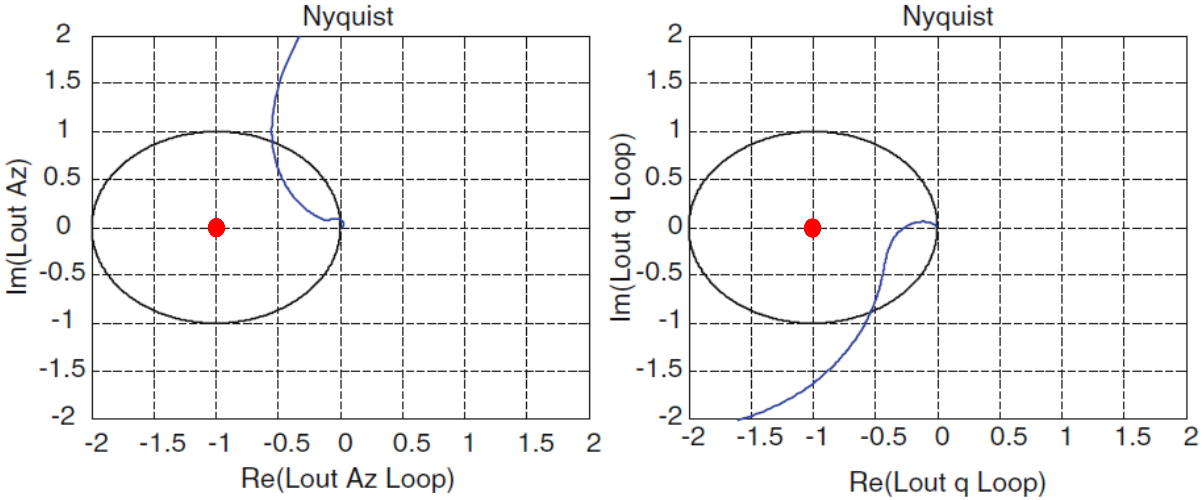


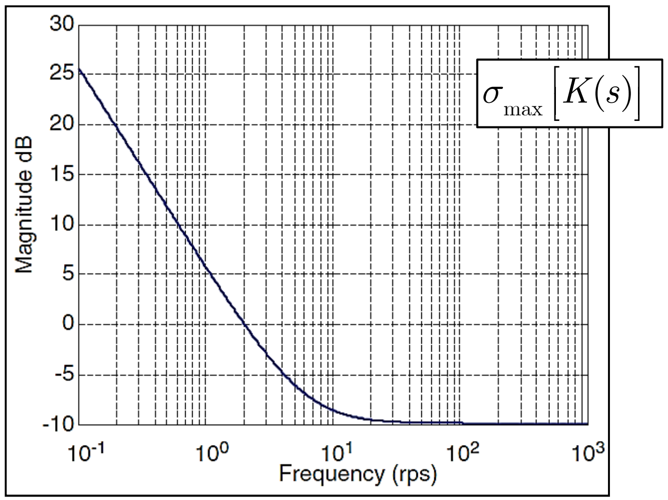
- **Comments:**
    - The sensitivity peak 1.0257 is very small peak indicating good stability margins at the plant output.
    - The complementary sensitivity represents the acceleration closed-loop transfer function. The sup at 1.0734 is a measure of the peak resonance in the acceleration loop. This is also small value also indicating good margins in this loop.
    - It is important to verify both input and output breaking points for stability margins

- In some multivariable systems, the margins at the plant input will be adequate, but at the plant output they could be low. It is always prudent to **check margins at all loop break points** to make sure no sensitivity problems exist.

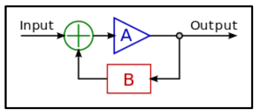


- The Nyquist plots for both loops open at the plant output are shown next, indicating good stability margins:

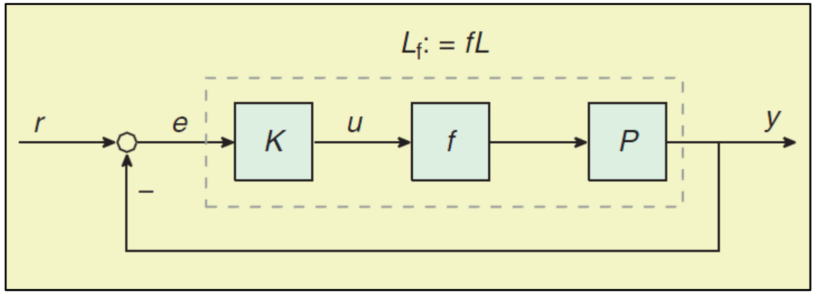




- Controller Frequency Response
  - The proportional – Integral behavior is evident from the figure
  - If Noise rejection is not satisfactory, additional poles are necessary (low pass filtering), however in this case stability margins may decrease due to the increased phase lag.



- Disk margins are robust stability measures that account for simultaneous gain and phase perturbations. They also provide additional information regarding the impact of model uncertainty at various frequencies.



- Given a nominal loop transfer function $L = PK$, consider a scalar complex parameter f defining a perturbation.

$$\boxed{f = ge^{-j\varphi}} \qquad L_f(j\omega) = fL(j\omega)$$

$$\begin{cases} \left|L_f(j\omega)\right| = \left|fL(j\omega)\right| = \left|gL(j\omega)\right| \\ \measuredangle L_f(j\omega) = \measuredangle fL(j\omega) = \measuredangle L(j\omega) - \varphi \end{cases}$$

- Magnitude and Phase of the perturbed loop transfer function.

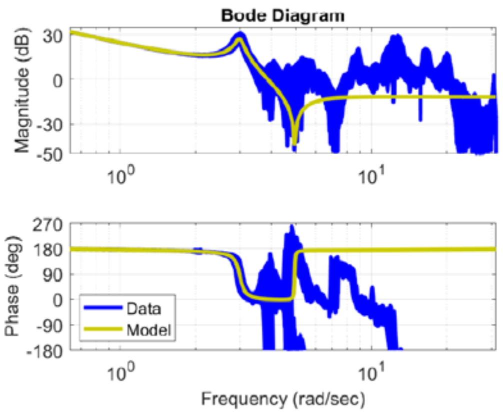
- Rewrite the definitions of classical stability margins:
  - The gain margin $f = g$ measures the amount of allowable perturbation in the plant gain.
  - The phase margin $f = e^{-j\phi}$ is the amount of allowable variation in the plant phase before the closed loop becomes unstable.

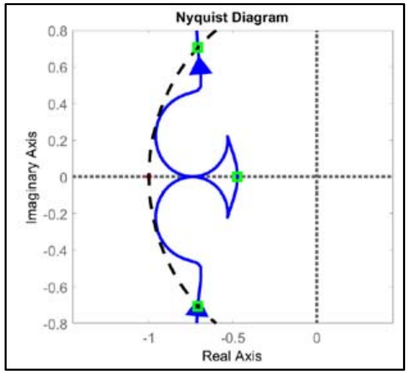
Definition 1
The *gain margins* consist of an upper limit $g_U > 1$ and a lower limit $g_L < 1$ such that
1) the closed loop is stable and well posed for all positive gain variations g in the range $g_L < g < g_U$
2) the closed loop is unstable or ill posed for gain variations $g = g_U$ (if $g_U < \infty$) and $g = g_L$ (if $g_L > 0$).

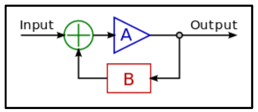
Definition 2
The *phase margin* consists of an upper limit $\phi_U \geq 0$ such that
1) the closed loop is stable and well posed for all phase variations ϕ in the range $-\phi_U < \phi < \phi_U$
2) the closed loop is unstable or ill posed for $\phi = \phi_U$ (if $\phi_U < \infty$).

- Real systems may have simultaneous gain and phase perturbations not well described by the above definitions

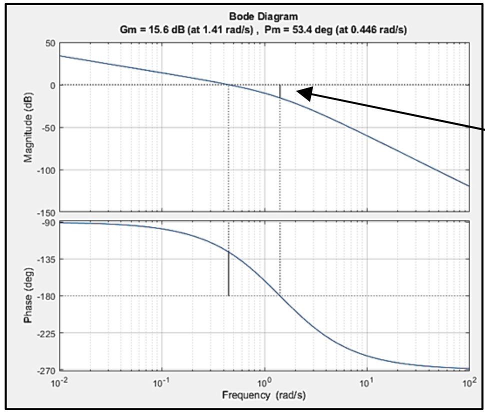
Courtesy of Seagate





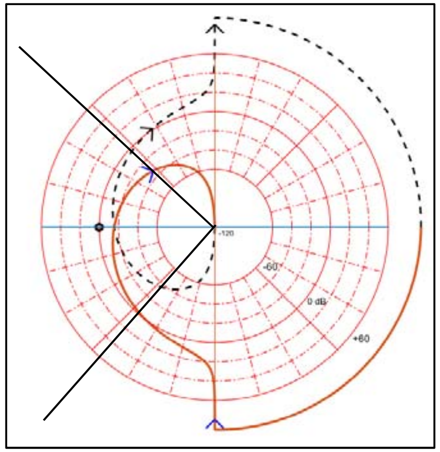


- Consider the open loop system: $L(s) = \dfrac{k}{s(s+1)(s+2)}, k = 1$

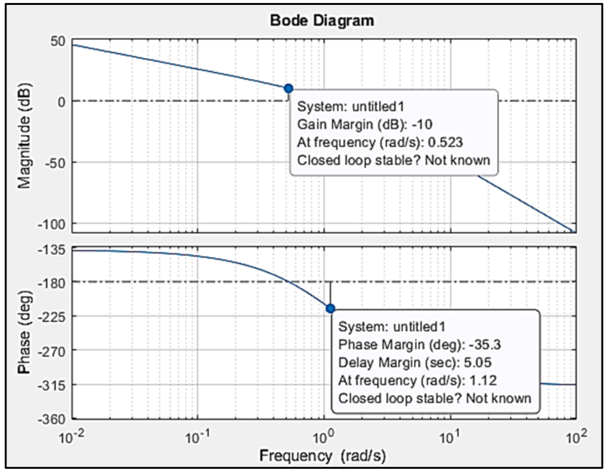


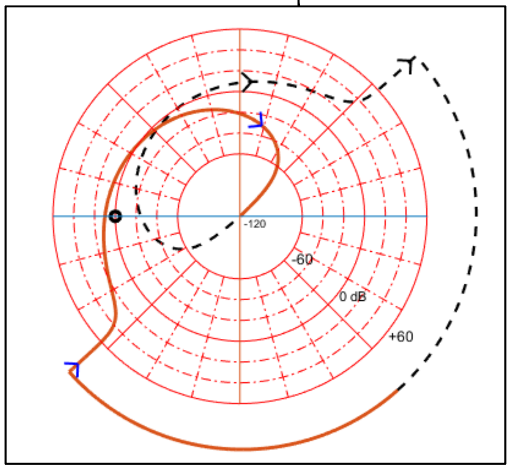
$K_{cr} = 6 =>> 15.6\ dB$

$PM = 53.4\ deg$

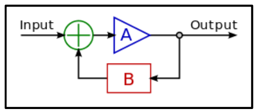


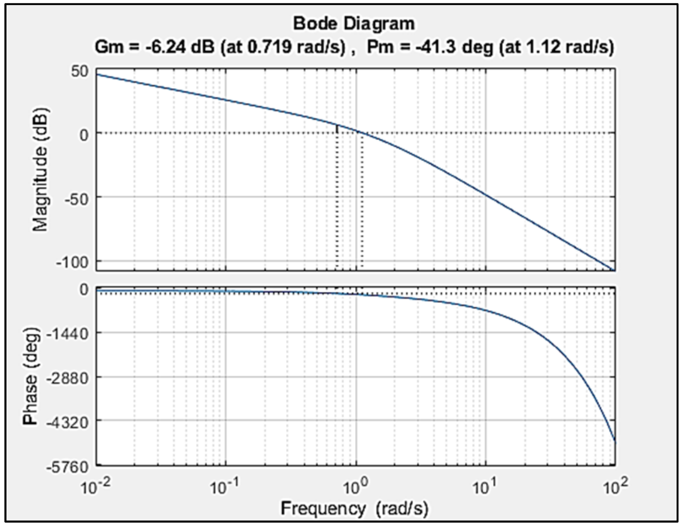
- Assume a complex perturbation $f = 2.6 - 2.86j$ $\begin{cases} \left|f\right| = 11.7dB \\ \measuredangle f = -48\,\text{deg} \end{cases}$

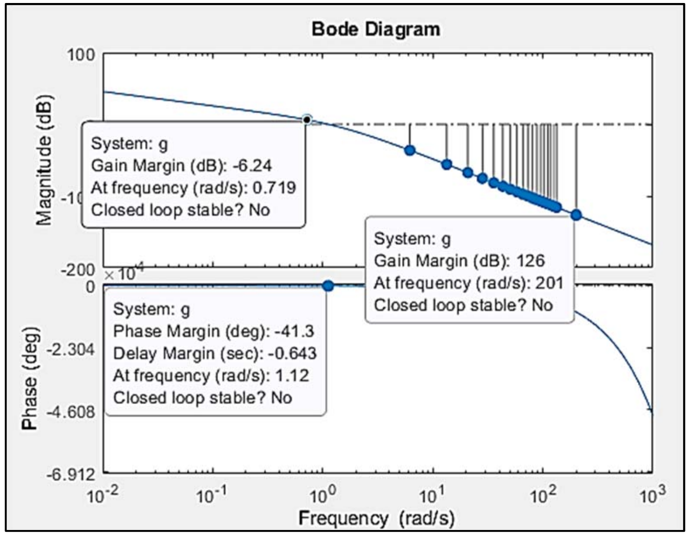


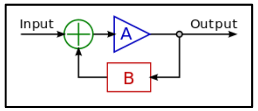


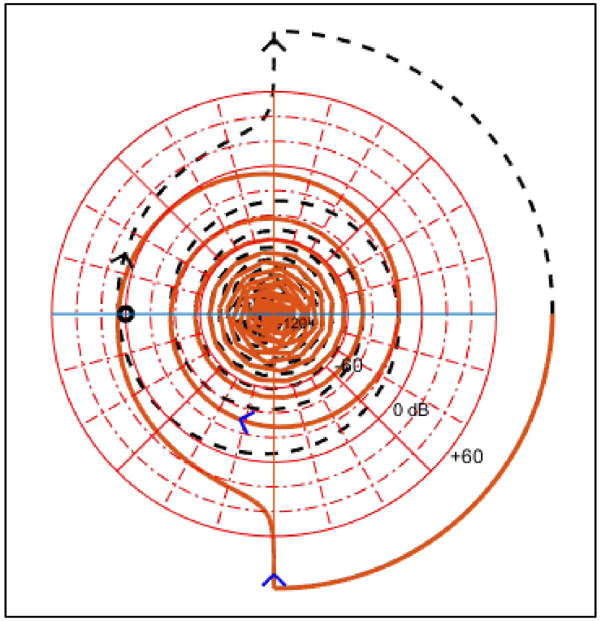

```
          GainMargin: [0.4028 2.4829]
         PhaseMargin: [-46.1246 46.1246]
          DiskMargin: 0.8515
          LowerBound: 0.8515
          UpperBound: 0.8515
           Frequency: 0.6436
  WorstPerturbation: [1×1 ss]
```





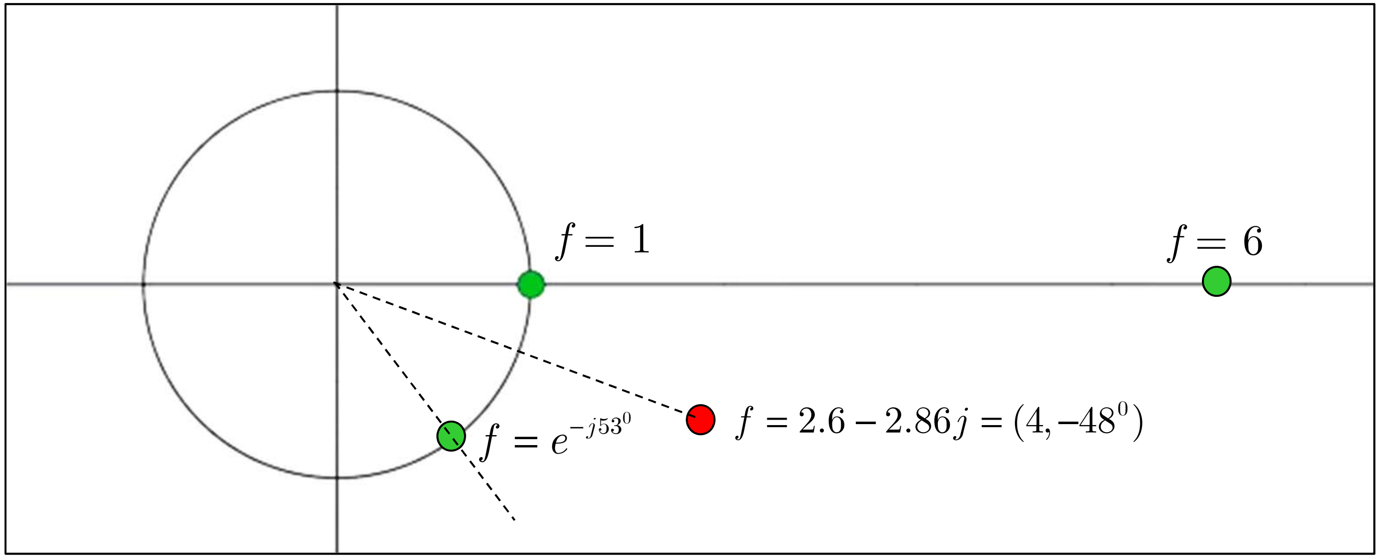







```
Number of poles in RHP of open-loop system: 0
Number of net encirclements around the -1 point:   2
=> Number of poles in RHP of closed-loop system:   2
=> Closed-loop-system is unstable
```

Analysis using a generic complex perturbation f



- **Main Idea:** define a disk, which can handle simultaneous complex perturbations as well

$$f \in D(\alpha, a, b) = \left\{ \frac{1 + a\delta}{1 - b\delta} : \delta \in \mathbb{C} \text{ with } |\delta| < \alpha \right\}$$

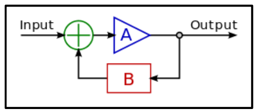
where a; b; α are real parameters that define the set of perturbations

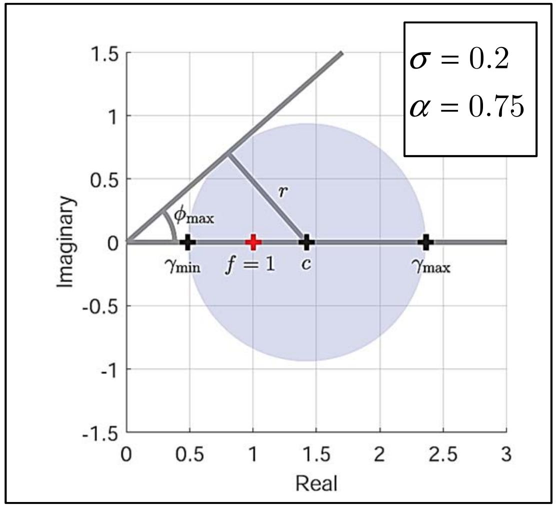
$$a + b = 1$$

$$a = \frac{1}{2}(1 - \sigma); b = \frac{1}{2}(1 + \sigma)$$

$$f \in D(\alpha, \sigma) = \left\{ \frac{1 + \frac{1-\sigma}{2}\delta}{1 - \frac{1+\sigma}{2}\delta} : \delta \in \mathbb{C} \text{ with } |\delta| < \alpha \right\}$$

The sets D are circles with center on the real axis with “size” α and “skew” (deviazione) σ.

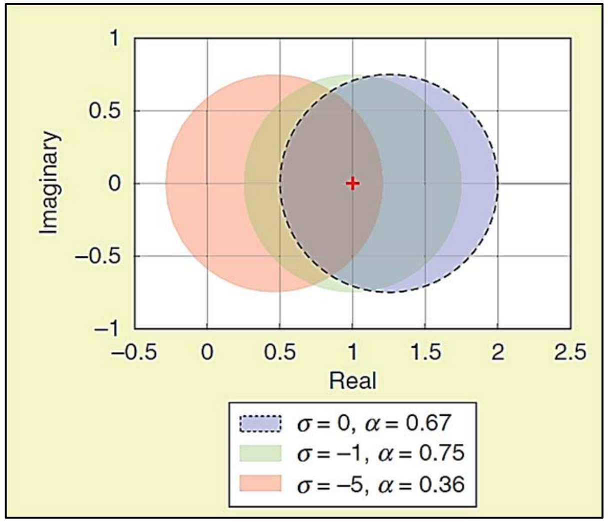


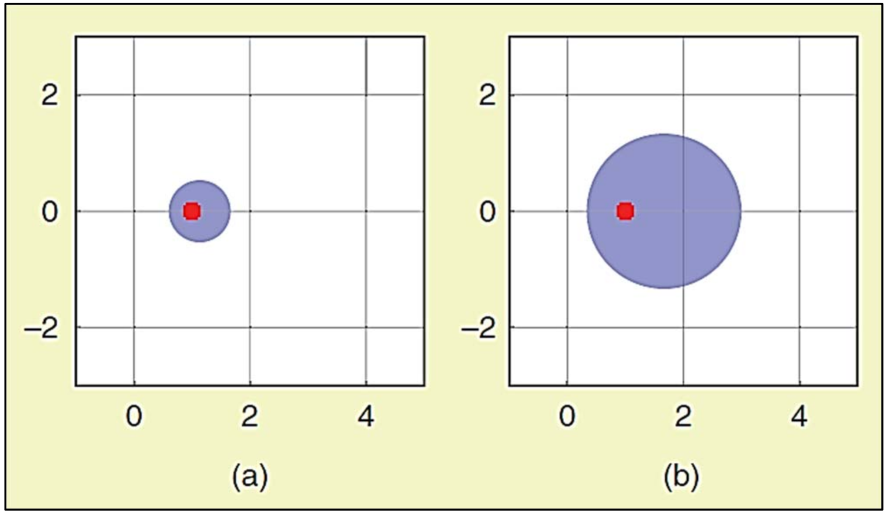


$$\gamma_{\min} = \frac{2 - \alpha(1-\sigma)}{2 + \alpha(1+\sigma)} \text{ and } \gamma_{\max} = \frac{2 + \alpha(1-\sigma)}{2 - \alpha(1+\sigma)}$$

$$c = \frac{1}{2}(\gamma_{\min} + \gamma_{\max}) \text{ and } r = \frac{1}{2}(\gamma_{\max} - \gamma_{\min})$$

- **Key Point:** it is helpful to think of α as controlling the amount of gain and phase variation while σ captures the difference between the amount of relative gain increase and decrease.





- How do we relate disk margins to closed loop stability?

- The first approach is to select σ and compute the largest value of α for which closed-loop stability is maintained. This yields a stability margin, formally defined that can be used to estimate the degree of robustness for a feedback loop

- **Definition 1**: For a given skew σ, the disk margin α_{max} is the largest value of α such that the closed loop with fL is well posed and stable for all complex perturbations $f \in D(\alpha_{MAX}, \sigma)$

- **Alternative approach**:
  - Uses $D(\alpha, \sigma)$ to cover known gain and phase variations, e.g. neglected actuator or sensor dynamics. This approach requires some knowledge of the plant modeling errors specified in terms of gain and phase variations. Then α and σ are selected to give the smallest set $D(\alpha, \sigma)$ that covers these known variations.
  - The goal is then to assess the robustness of the closed-loop with respect to this set of variations.

This second analysis approach can be performed by computing the disk margin α_{max} associated with the chosen skew σ. If $\alpha_{max} > \alpha$ then the closed-loop is stable for all variations in $D(\alpha, \sigma)$ and hence the system is robust to the known modeling errors

Comments on the computation of $D(\alpha, \sigma)$:

- Recall the definition of $D(\alpha, \sigma)$: $f \in D(\alpha,\sigma) = \left\{ \dfrac{1+\frac{1-\sigma}{2}\delta}{1-\frac{1+\sigma}{2}\delta} \;:\; \delta \in \mathbb{C} \text{ with } |\delta| < \alpha \right\}$ **(*)**

- Starting from the nominal stability condition, which means $f = 1$, as f changes, stability of the closed loop system changes when its poles $s = j\omega_0$ cross the imaginary axis, for some f_0. That is:

$$s = j\omega_0 \in I\Re \Leftrightarrow 1 + f_0 L(j\omega_0) = 0$$

- From (*), It exists a f_0 such that:

$$f_0 = \frac{2+(1-\sigma)\delta_0}{2-(1+\sigma)\delta_0} \quad : \; \delta_0 \in \mathbb{C} \text{ with } |\delta| < \alpha$$

- Stability transition can be rewritten (after some algebra):

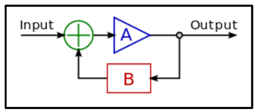
$$\left(S(j\omega_0) + \frac{\sigma - 1}{2} \right) \delta_0 = 1$$

- From the above, the following theorem holds:

  - **Theorem:** Let σ be a given skew parameter defining the disk margin. Assume the closed-loop is well-posed and stable with the nominal, SISO loop transfer function L. Then the disk margin is given by:

$$\alpha_{MAX} = \frac{1}{\left\| S(j\omega) + \dfrac{\sigma - 1}{2} \right\|_\infty}$$

- Note that for $\sigma = 0$ the condition becomes: $\alpha_{MAX} = \dfrac{1}{\left\| \dfrac{S(j\omega) - T(j\omega)}{2} \right\|_\infty}$



- **Summary**:

  - Disk margins are related to classical gain and phase margins providing a more comprehensive analysis of robustness.

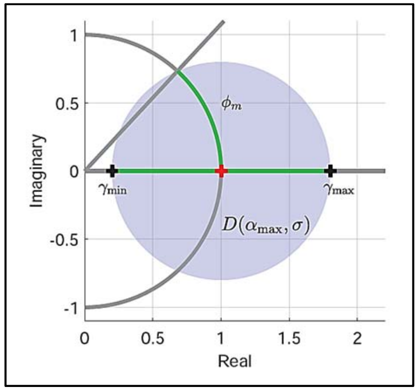
  - They account for simultaneous changes in gain and phase. Perturbations in phase only are located on the unit circle $|f| = 1$; perturbations in gain only are located on the real axis $f \in \mathbb{R}$

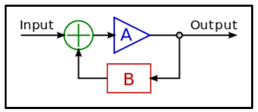
  - The margin α_{MAX} is used to compute guaranteed gain and phase margins in the figure

  - With $\delta = \pm\alpha_{MAX}$, we obtain, using geometry and trigonometry:

$$\gamma_{\min} = \frac{2 - \alpha_{\max}(1-\sigma)}{2 + \alpha_{\max}(1+\sigma)} \qquad \gamma_{\max} = \frac{2 + \alpha_{\max}(1-\sigma)}{2 - \alpha_{\max}(1+\sigma)} \qquad \cos\phi_m = \frac{1 + c^2 - r^2}{2c} = \frac{1 + \gamma_{min}\gamma_{max}}{\gamma_{min} + \gamma_{max}}.$$

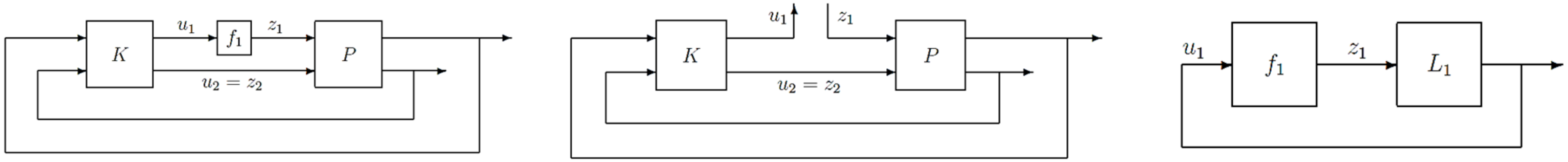
$$[\gamma_{MIN}, \gamma_{MAX}]; [-\phi_m, \phi_m]$$





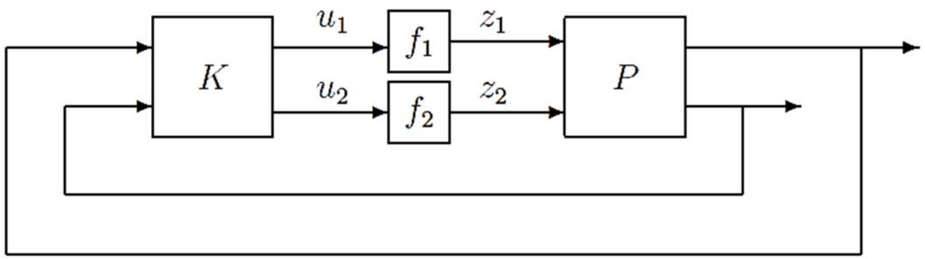
- **MIMO Extension**:

- One loop at the time application: This introduces perturbations in a single channel while holding all other channels fixed. This can be overly optimistic as it fails to capture the effects of simultaneous perturbations in multiple channels.



- Multi-loop disk margins capture the effects of simultaneous perturbations in multiple channels.

- The multi-loop disk margin is a single number α_{MAX} defining the largest generalized disk of perturbations f1 and f2 for which the closed-loop in Figure is well-posed and stable. It is emphasized that the perturbations f1 and f2 are allowed to vary independently, i.e. they are not necessarily equal.

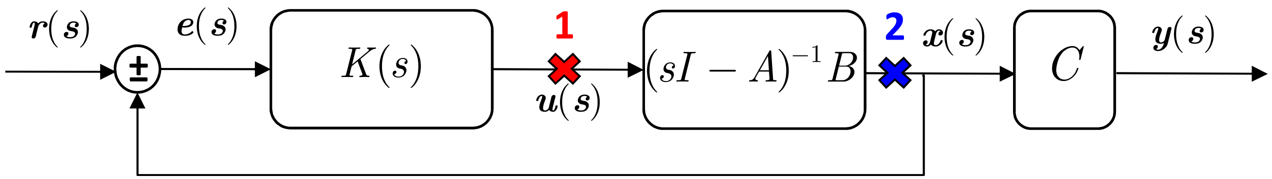


https://it.mathworks.com/help/robust/ug/mimo-stability-margins-for-spinning-satellite.html

Brian Douglas RC 2 Brian Douglas RC 3

AUBURN UNIVERSITY

- Consider the general unity feedback structure of a **LQR controller**:



$$\left\| I + L(s) \right\|_2 \geq 1$$

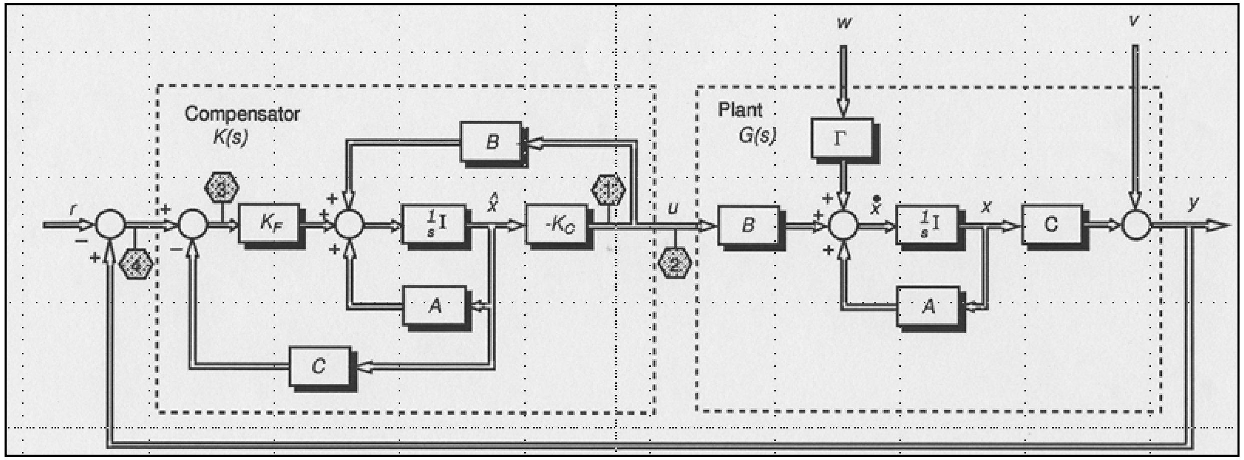
- We can prove the Kalman's Inequality for the **Kalman Filter** as well, since it is a feedback loop itself:

$$L(s) = -C(sI - A)^{-1} K_f$$

- **Implications:** The Nyquist plot of $L(j\omega)$ will always be outside the unit circle centered at (-1,0).

$$GM = \left[\frac{1}{2} \quad \infty\right], \; PM = \left[-60^0 \quad +60^0\right]$$

❑ Stability Margins for the LQG compensator



$$\boxed{K_{LQG}(s) = -K_C\left(sI - A + BK_C + K_F C\right)^{-1} K_F}$$

$$\dot{\boldsymbol{q}} = \begin{bmatrix} \dot{\boldsymbol{e}} \\ \dot{\hat{\boldsymbol{x}}} \end{bmatrix} = \left[\begin{array}{c|c} A - K_F C & 0 \\ \hline K_F C & A - BK_c \end{array}\right]\boldsymbol{q} + \left[\begin{array}{c|c} D & -K_F \\ \hline 0 & K_F \end{array}\right]\begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{v} \end{bmatrix}$$

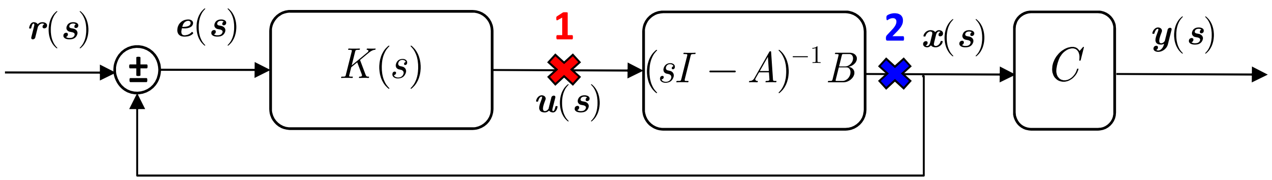
Guaranteed Margins for LQG Regulators

JOHN C. DOYLE

Abstract—There are none.

- **Linear Optimal Control and Estimation** play a fundamental role in the design of compensators for Multivariable linear systems. Their Robustness properties (in terms of stability margins) are therefore important to evaluate.

  - Consider the general unity feedback structure of a **LQR controller**:



$$\begin{cases} G(s) = C\left(sI - A\right)^{-1} B \\ K(s) = K_c = R^{-1}B^T P \qquad P \geq 0 \end{cases}$$

$$\begin{cases} L(s) = K_c\left(sI - A\right)^{-1} B \\ I + K_c\left(sI - A\right)^{-1} B = I + L(s) \end{cases}$$

$$J = \int_0^\infty \left[\boldsymbol{y}^T \boldsymbol{y} + \boldsymbol{u}^T R \boldsymbol{u}\right] dt = \int_0^\infty \left[\boldsymbol{x}^T C^T C \boldsymbol{x} + \boldsymbol{u}^T R \boldsymbol{u}\right] dt$$

$$\boldsymbol{u} = -K_c \boldsymbol{x}$$

- Recall the **Single Input Case**. From the Riccati equation we obtain the Kalman Inequality, leading to Guaranteed Gain and Phase Margins.

$$\left|1 + K_c (sI - A)^{-1} b\right| \geq 1 \Rightarrow \left|1 + L(s)\right| \geq 1$$

- **Kalman's inequality can be extended to the MIMO case:**

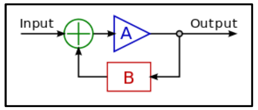
$$R + B^T(-sI - A^T)^{-1} Q (sI - A)^{-1} B = [I + B^T(-sI - A^T)^{-1} K_c^T] R [I + K_c (sI - A)^{-1} B] \geq R$$

  - Perform Spectral Factorization $\Phi(s) = W^T(-s) W(s)$

$$\Phi(s) = R + B^T(-sI - A^T)^{-1} Q (sI - A)^{-1} B$$
$$W(s) = R[I + K_c (sI - A)^{-1} B] \geq R$$

  - Substituting:

$$[I + \sqrt{R} K_c (sI - A)^{-1} B \sqrt{R}]^T [I + \sqrt{R} K_c (sI - A)^{-1} B \sqrt{R}] \geq I$$
$$[W(-s)\sqrt{R}]^T [W(-s)\sqrt{R}] \geq I$$



- Let: $R = \rho I$

  $L(s) = K_c (sI - A)^{-1} B$

- Kalman's Inequality can be written as:

$$[I + L(-s)]^T [I + L(s)] \geq I$$

- Which is satisfied if and only if:

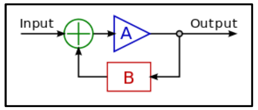
$$\sigma_{\min}[I + L(s)] = \frac{1}{\sigma_{\max}\left[\left(I + L(s)\right)^{-1}\right]} \geq 1 \Leftrightarrow \sigma_{\max}[S] \leq 1$$

- From the definition of MIMO stability margins, LQR gives at the input of the plant:

$$GM = \begin{bmatrix} \frac{1}{2} & \infty \end{bmatrix}, \; PM = \begin{bmatrix} -60^0 & +60^0 \end{bmatrix}$$

- This holds also for any diagonal input weight matrix $\boldsymbol{R}$ (see Athans 1978):

$$\boxed{\underline{\sigma}\left[\sqrt{R}\left(I + K_c (sI - A)^{-1} B\right) R^{-\frac{1}{2}}\right] \geq 1}$$



- Consider the general structure of a **Kalman Filter Estimator**:

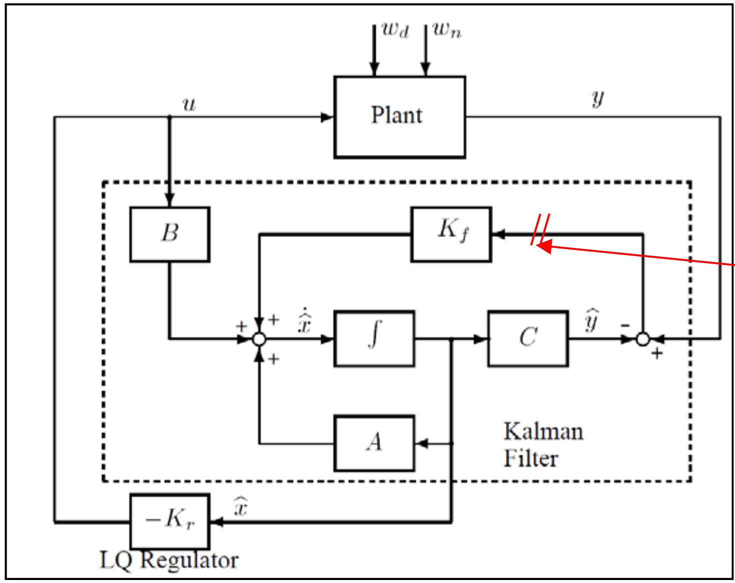
$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} + \boldsymbol{w} \\ \boldsymbol{y} = C\boldsymbol{x} + \boldsymbol{v} \end{cases} \qquad \begin{cases} \boldsymbol{w}(t) \Rightarrow [0, W] \\ \boldsymbol{v}(t) \Rightarrow [0, V] \\ \boldsymbol{x}_0(t) \Rightarrow [\boldsymbol{\eta}_0, Q_0] \end{cases} \qquad \begin{matrix} \dot{\hat{\boldsymbol{x}}} = A\hat{\boldsymbol{x}} + B\boldsymbol{u} + K_F(\boldsymbol{y} - C\hat{\boldsymbol{x}}) \\ \hat{\boldsymbol{x}}(t_0) = \hat{\boldsymbol{x}}_0 \end{matrix}$$

$$J = E\left\{\boldsymbol{e}^T Z \boldsymbol{e}\right\} = E\left\{(\boldsymbol{x} - \hat{\boldsymbol{x}})^T Z (\boldsymbol{x} - \hat{\boldsymbol{x}})\right\}$$

$$K_F = QC^T V^{-1} \qquad AQ + QA^T + W - QC^T V^{-1} CQ = 0$$

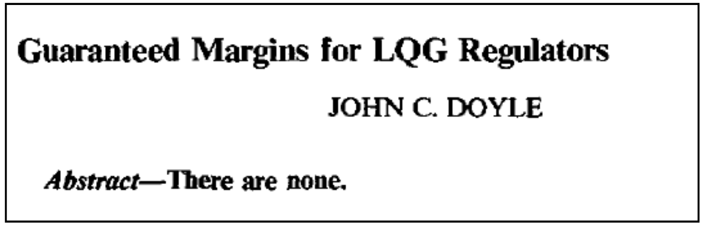
- We can prove the Kalman's Inequality for the Kalman Filter as well, since it is a feedback loop itself:

$$L(s) = -C(sI - A)^{-1} K_f$$



- **Using the Filter Riccati equation**:
- If the measurement noise power spectral density is diagonal, then at the input to the Kalman gain matrix K_f there will be an infinite Gain Margin, a gain reduction margin of 0.5 and a minimum Phase Margin of ± 60 degrees.
- Consequently, for a single-output plant, the Nyquist diagram of the open-loop filter transfer function will lie outside the unit circle with center at -1.

❑ Stability Margins for the LQG compensator


Guaranteed Margins for LQG Regulators

JOHN C. DOYLE

Abstract—There are none.

- The use of a state estimator and/or observer can make the system arbitrarily sensitive, and all Robustness considerations are lost. The reason for this is that the LQG compensator operates on a system, which is not the internal Kalman filter model.

- The sensitivity of LQG with respect to loop uncertainty was proved Doyle (1981) with a simple SISO example:

$$\begin{cases} \dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x} + n \end{cases} \qquad \begin{cases} w = \left(0, \quad \sigma\right) \\ n = \left(0, \quad 1\right) \end{cases}$$

- Choose the following weights:

$$Q = q_0 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, R = 1$$

$$q_0 = \sigma = 1 \Rightarrow K_c = K_f^T$$

$$K_c = R^{-1}B^T P = \left(2 + \sqrt{4 + q_0}\right)\begin{bmatrix} 1 & 1 \end{bmatrix} = f\begin{bmatrix} 1 & 1 \end{bmatrix}$$

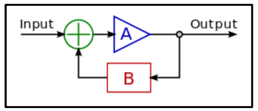
$$K_F = \Sigma C^T V^{-1} = \left(2 + \sqrt{4 + \sigma}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = d\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Consider now the complete system with an arbitrary controller gain $\boldsymbol{m}$ and a unity filter gain:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} - BK_c m\hat{\boldsymbol{x}} + \ldots$$

$$\Rightarrow A_{CL} = \left[\begin{array}{c|c} A & -BK_c m \\ \hline K_f C & A - BK_c - K_f C \end{array}\right]$$

$$\Rightarrow A_{CL} = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & -mf & -mf \\ \hline d & 0 & 1-d & 1 \\ d & 0 & -d-f & 1-f \end{array}\right]$$



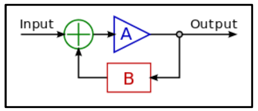
- The characteristic Polynomial is given by:

$$(.)\lambda^4 + (.)\lambda^3 + (.)\lambda^2 + \left[d + f - 4 + 2(m-1)df\right]\lambda + \left[1 + (1-m)df\right]$$

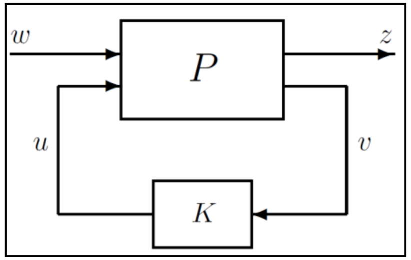
- From Algebra:

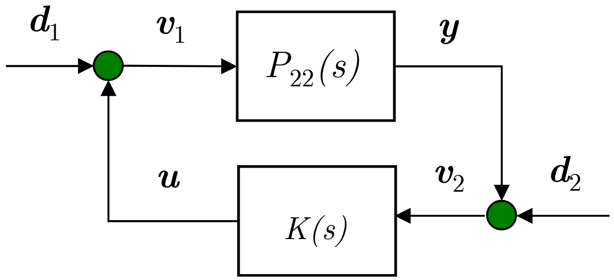
$$\begin{cases} \left[d + f - 4 + 2(m-1)df\right] > 0 \\ \left[1 + (1-m)df\right] > 0 \end{cases}$$

- From which we can infer that for large enough $\boldsymbol{d}$ and $\boldsymbol{f}$, $\boldsymbol{df} > \boldsymbol{d} + \boldsymbol{f}$, and the closed loop system becomes unstable for arbitrarily small values of m.



- In the previous chapter **lower linear fractional transformation** was used to represent a nominal closed loop system in a convenient 2 – Block structure



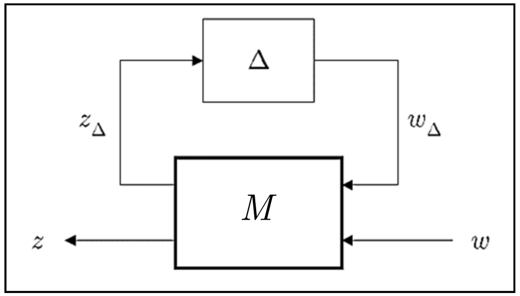


$$\left\{P_{11}+P_{12}K(I-P_{22}K)^{-1}P_{21}\right\}=T_{11}(s)$$

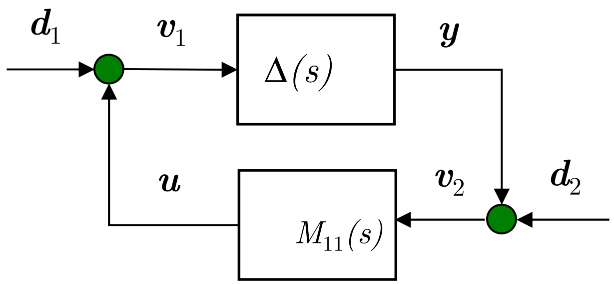
$$\begin{cases}L_o = -P_{22}K\\ L_i = -KP_{22}\end{cases} \qquad \begin{cases}S_o = (I+L_o)^{-1}\\ S_i = (I+L_i)^{-1}\end{cases}$$

- In the following, we will use the **upper linear fractional transformation** to incorporate uncertainty.
  - **Advantages:** Computer mechanization, account for unstructured (full complex) as well as structured (real diagonal) uncertainties.

Upper LFT representation



$$M(s) = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right]$$



$$\boxed{\left[M_{22} + M_{21}\Delta(I - N_{11}\Delta)^{-1}M_{12}\right] = T_{22}(s)}$$

$$\begin{cases} L_o = -M_{22}K \\ L_i = -KM_{22} \end{cases} \qquad \begin{cases} S_o = \left(I + L_o\right)^{-1} \\ S_i = \left(I + L_i\right)^{-1} \end{cases}$$

- Existence of the inverse $(I - M_{11}\Delta)^{-1}$ necessary for well – posed closed loop system as seen by the uncertainty

- **How does it work?**

- **Example:** Consider the standard feedback loop for the controlled system, where $G(s)$ is an uncertain LTI system approximated within 5% by the model $G_0(s)$:

$$G_0(s) = \frac{1}{s^2 + 0.01s + 1}$$

- Therefore we can write

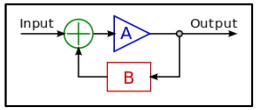
$$G(s) = G_0(s)\left[I + \Delta(s)\right]$$

1. $\Delta(s)$ is a multiplicative error, modelled with any stable LTI system having a gain less than 0.05.

$$\left|\Delta(j\omega)\right| \le 0.05, \forall \omega \in [0, \infty)$$

- Or, using a weighting function

$$\left|\Delta(j\omega)\right| = W_1 \cdot \boldsymbol{\Delta}(\omega), \left\|\boldsymbol{\Delta}(\omega)\right\|_\infty \le 1, \forall \omega \in [0, \infty)$$
$$W_1 = 0.05$$

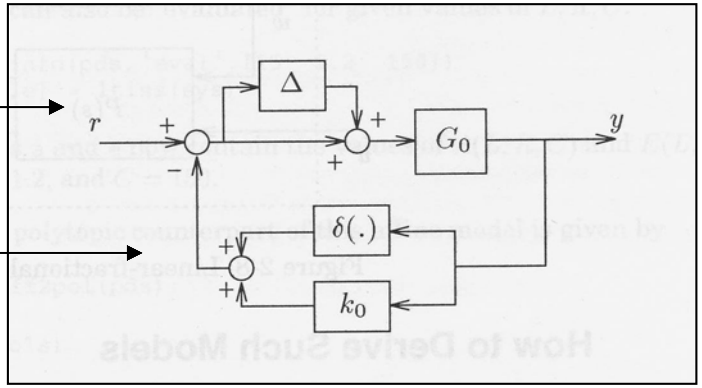


2. In addition, a structured uncertainty is present as a 10% fluctuating feedback gain about the nominal value k_0

$$k(t) = k_0 + \delta(t), \left|\delta(t)\right| \leq 0.1k_0$$

Unstructured multiplicative

Structured parametric



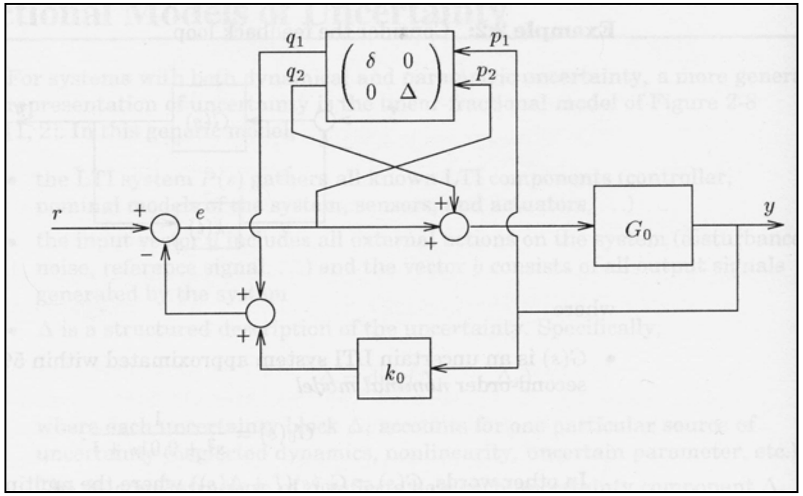
- **Objective:** **Combine all uncertainties in a single block**

Chapter 4: Uncertainty in a 2x2 Block Structure

1. Introduce fictitious inputs and outputs in order to isolate the uncertainties δ and Δ.

$$p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} y \\ e \end{bmatrix} \qquad q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \delta p_1 \\ \Delta p_2 \end{bmatrix}$$

2. Draw the original block diagram including fictitious p and q vectors



- Collect the system and uncertainty representation, with:

$$\begin{cases} G_1(s) = \dfrac{G_0(s)}{1 + k_0 G_0(s)} \\ G_2(s) = k_0 G_1(s) \end{cases}$$

$$\begin{cases} p = M_{11} q + M_{12} r \\ y = M_{21} q + M_{22} r \end{cases} \Rightarrow \begin{bmatrix} p \\ y \end{bmatrix} = M(s) \begin{bmatrix} q \\ r \end{bmatrix}$$

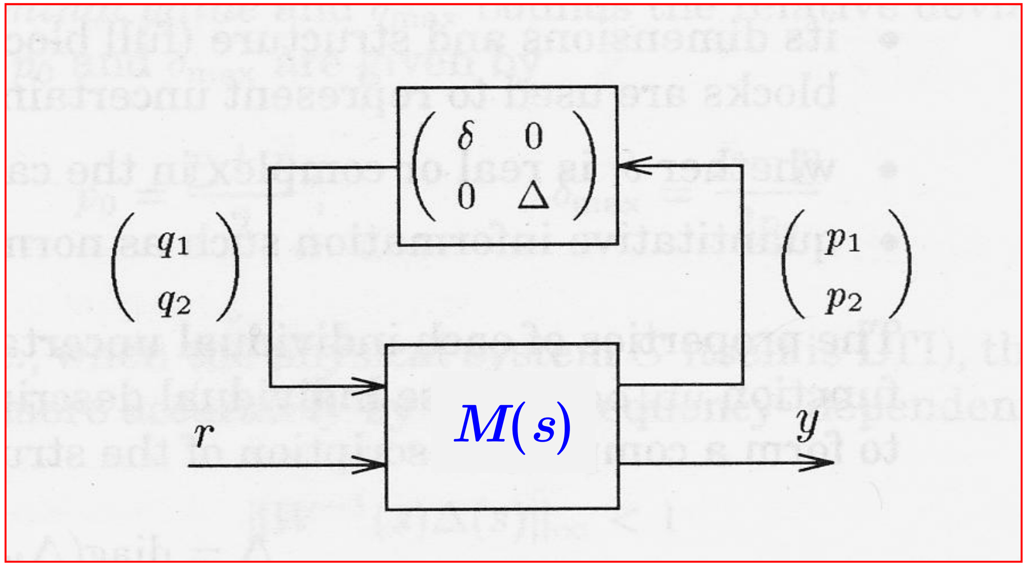
- We obtain:

$$M_{11}(s) = \begin{bmatrix} 1 - G_2 & 1 - G_2 \\ 1 - G_2 & -G_2 \end{bmatrix}$$

$$M_{12}(s) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$M_{21}(s) = \begin{bmatrix} G_1 & G_1 \end{bmatrix}$$

$$M_{22}(s) = G_1$$



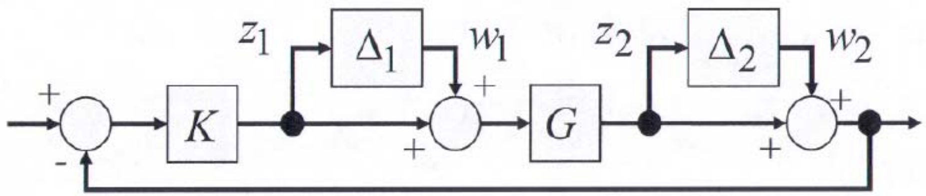
- Correction

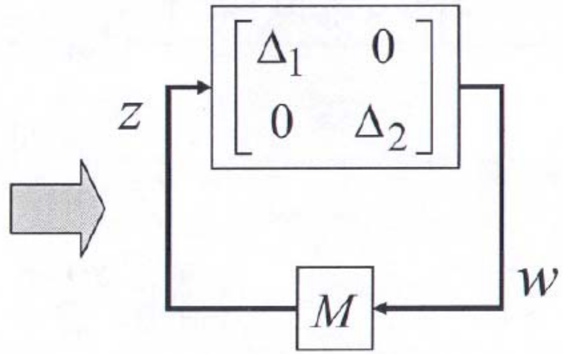
$$M_{11}(s) = \begin{bmatrix} -G_1 & G_1 \\ -(1 - G_2) & -G_2 \end{bmatrix}$$

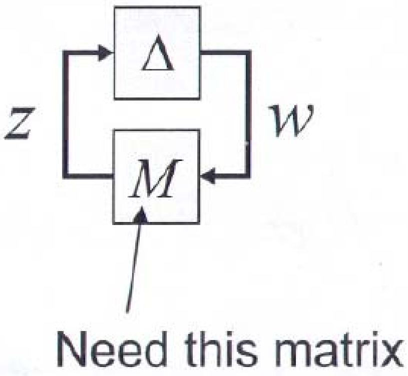
$$M_{12}(s) = \begin{bmatrix} G_1 \\ 1 - G_2 \end{bmatrix}$$

$$M_{21}(s) = \begin{bmatrix} -G_1 & G_1 \end{bmatrix}$$

$$M_{22}(s) = G_1$$







$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$$

$$L = M\Delta$$

$$I + L = I - M\Delta$$

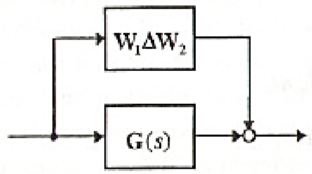
$$z_1 = -K\left(G\left(z_1 + w_1\right) + w_2\right)$$

$$z_2 = G\left(-K\left(z_2 + w_2\right) + w_1\right)$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -(I+KG)^{-1}KG & -(I+KG)^{-1}K \\ (I+GK)^{-1}G & -(I+GK)^{-1}GK \end{bmatrix}}_{M} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

- **Note:** Return Difference Matrix. Non singularity required from Nyquist Criterion.

- **More general weighted representation of unstructured uncertainty**
  - $W_1(s)$ and $W_2(s)$ are used to model the uncertainty, and later on incorporated in the controller analysis and synthesis processes.
  - Δ is fully unknown but stable and bounded in magnitude: $\|\Delta\|_\infty \leq 1$

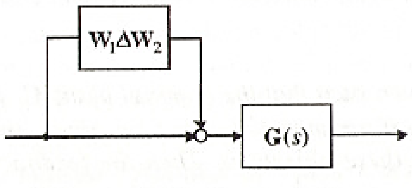


$$\tilde{G}(s) = G(s) + W_1 \Delta W_2$$

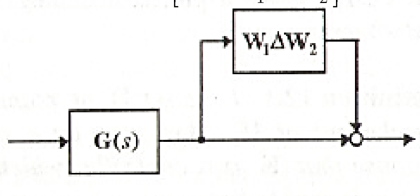
(a) Additive uncertainty

$$\tilde{G}(s) = G(s)\left[I + W_1 \Delta W_2\right]$$

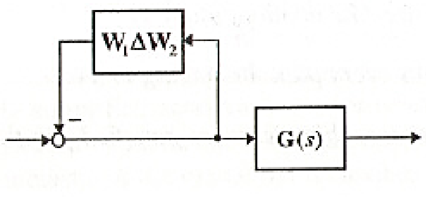
$$\tilde{G}(s) = \left[I + W_1 \Delta W_2\right] G(s)$$



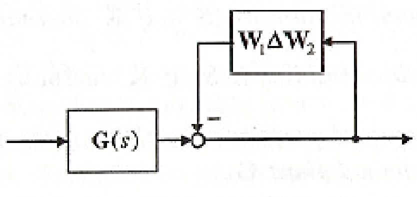
(b) Input-multiplicative uncertainty



(c) Output-multiplicative uncertainty



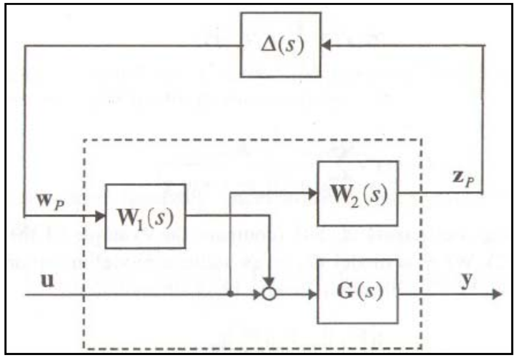
(d) Input feedback uncertainty

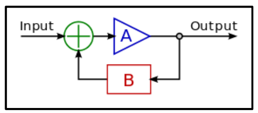


(e) Output feedback uncertainty

$$\tilde{G}(s) = G(s)\left[I + W_1 \Delta W_2\right]^{-1}$$

$$\tilde{G}(s) = \left[I + W_1 \Delta W_2\right]^{-1} G(s)$$





- **The block Δ isolated from unstructured uncertainty is given by a full complex matrix**

- Weighting Matrices can be used to shape desired design requirements **as well as uncertainty.**
  - Recall <u>earlier results</u> for robustness stability with shaping
    - Nominal closed loop system asymptotically stable $\det\left[I + GK\right] \neq 0; \forall \omega \in [0,\infty)$

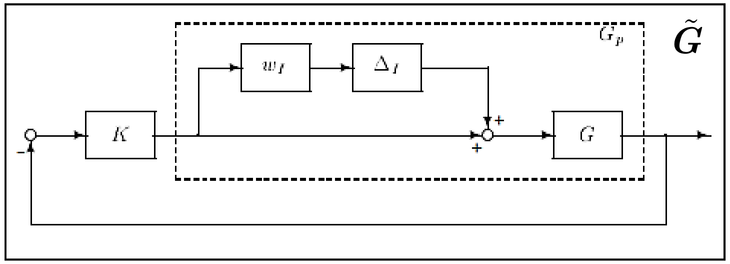
$$\det\left[I + \tilde{G}K\right] = \det\left[I + G(j\omega,\varepsilon)K(j\omega)\right] \neq 0; \forall \omega \in [0,\infty)$$

$$\text{as } \varepsilon \quad 0 \to 1 \qquad \Rightarrow G(j\omega) \Rightarrow \tilde{G}(j\omega)$$

$$\sigma_{\min}\left[I + \left(G(j\omega)K(j\omega)\right)^{-1}\right] > \sigma_{\max}\left[L(j\omega)\right], \forall \omega \in \left(0,\infty\right)$$

With $L(j\omega)$ stable, bounded, minimum phase uncertainty

- For SISO systems $\sigma_{\min}\left[G(s)\right] = \sigma_{\max}\left[G(s)\right] = \left|G(s)\right|$



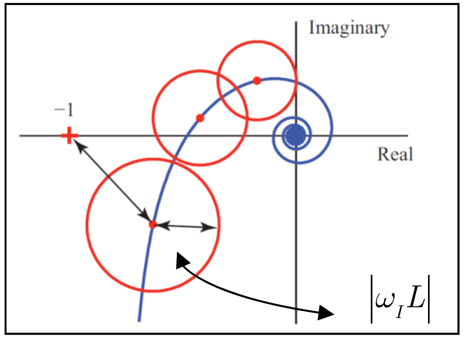
$$\begin{cases} L_{OL}(s) = G(s)K(s) \\ \tilde{L}_{OL}(s) = \tilde{G}(s)K(s) \\ \tilde{G}(s) = [I + \omega_I(s)\Delta_I]G(s) \end{cases}$$

$$\tilde{L}_{OL}(s) = [I + \omega_I(s)\Delta_I]G(s)K(s) = L_{OL}(s) + \omega_I(s)\Delta_I L_{OL}(s); \left|\Delta_I(\omega)\right| \leq 1, \forall \omega$$

- From SISO Nyquist, worst case stability of closed loop real system $G_P(s)$

$$\left|\omega_I(s)L_{OL}(s)\right| < \left|1+L_{OL}(s)\right|;\forall\omega \qquad \left|\frac{\omega_I(s)L(s)}{1+L(s)}\right| < 1;\forall\omega$$

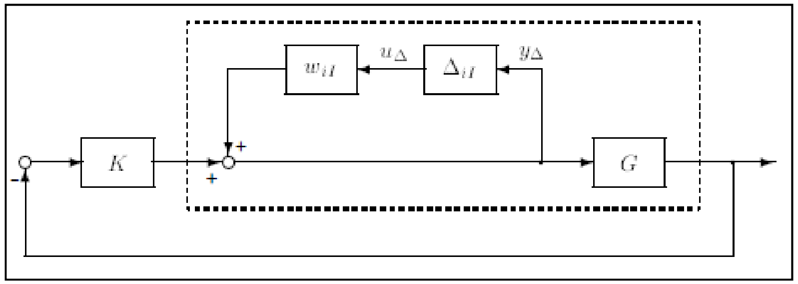
$$\boxed{\left|\omega_I(s)T(s)\right| < 1 \Leftrightarrow \left|T(s)\right| < \left|\frac{1}{\omega_I(s)}\right|;\forall\omega}$$



- For a MIMO system, the above requirement becomes a normed weighted limit on the complementary sensitivity matrix. From the general definition of $\mathcal{H}_\infty$ norm:

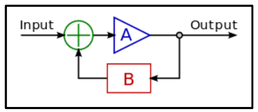
$$\left\|G(j\omega)\right\|_\infty = \sup_{\omega\in\Re}\sigma_{\max}\left[G(j\omega)\right] \qquad \boxed{\left\|\omega_I(s)T(s)\right\|_\infty < 1;\forall\omega}$$

- **A similar result can be obtained using Sensitivity and for other locations of uncertainty**



$$\begin{cases}\tilde{G}(s) = \left[I+\omega_{iI}(s)\Delta_{iI}\right]^{-1}G(s)\\ S(j\omega) = \left[I+L_{OL}(s)\right]^{-1}\\ \left|\Delta_{iI}\right| \le 1;\forall\omega\end{cases}$$

$$\boxed{\left|\omega_{iI}(s)S(s)\right| < 1 \Leftrightarrow \left\|\omega_{iI}(s)S(s)\right\|_\infty < 1}$$



- **Example:** Consider a process control model

$$G(s) = \frac{ke^{-\tau s}}{Ts+1}, \quad 4 \le k \le 9,\ 2 \le T \le 3,\ 1 \le \tau \le 2.$$

Take the nominal model as

$$G_0(s) = \frac{6.5}{(2.5s+1)(1.5s+1)}$$

$$\Delta_a(j\omega) = G(j\omega) - G_0(j\omega)$$

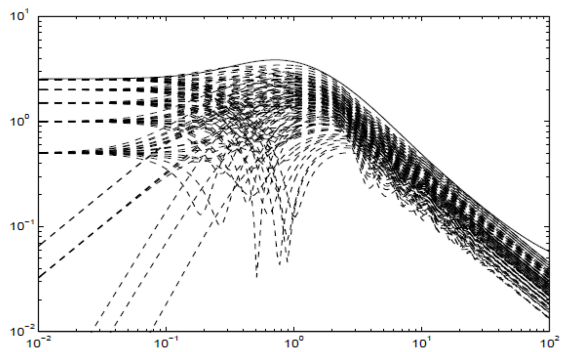
$$W_a(s) = \frac{0.0376(s+116.4808)(s+7.4514)(s+0.2674)}{(s+1.2436)(s+0.5575)(s+4.9508)}$$

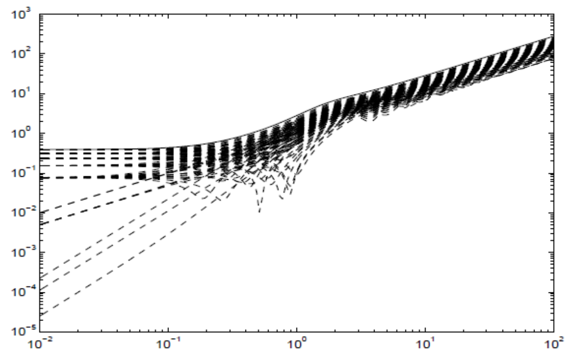
$$|W_a(s)| \ge |\Delta_a(s)|$$

$$\Delta_m(s) := \frac{G(s) - G_0(s)}{G_0(s)}$$

$$W_m = \frac{2.8169(s+0.212)(s^2+2.6128s+1.732)}{s^2+2.2425s+2.6319}$$

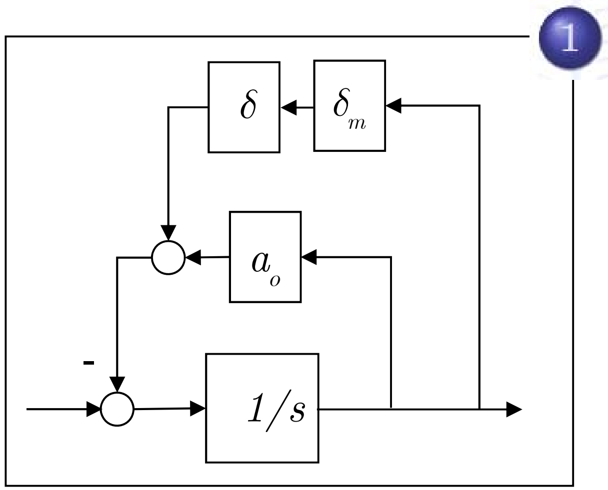
$$|W_m(s)| \ge |\Delta_m(s)|$$

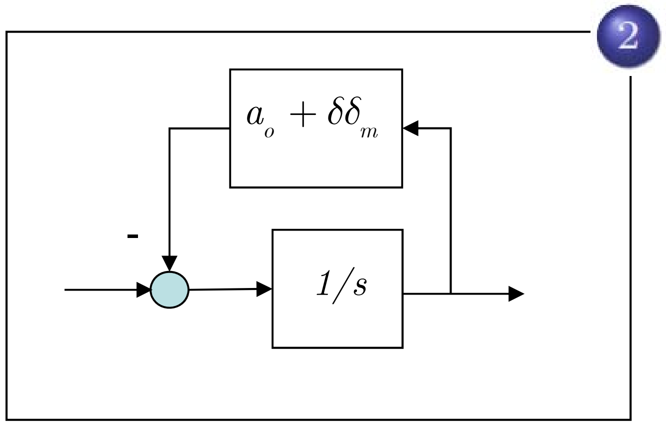


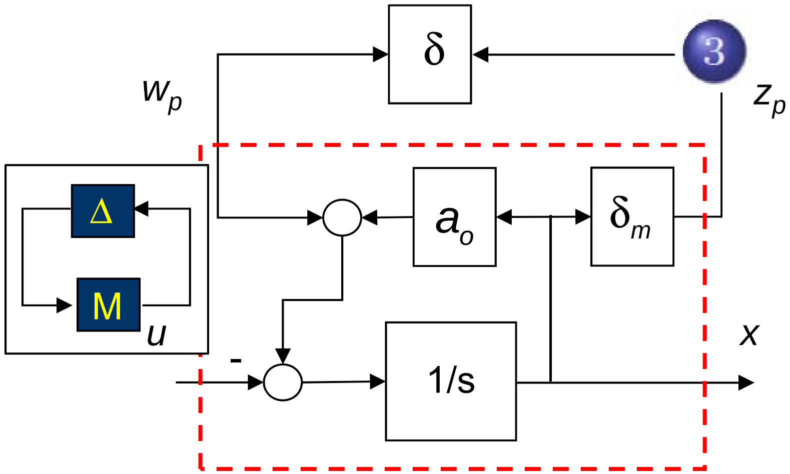


- **2 – Block structure built for direct incorporation of <u>structured uncertainty</u>.**

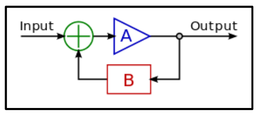
- **Example 1:** $\begin{cases} \dot{x} = -\left(a_o + \delta\delta_m\right)x + u \\ \left|\delta\right| < 1 \end{cases}$







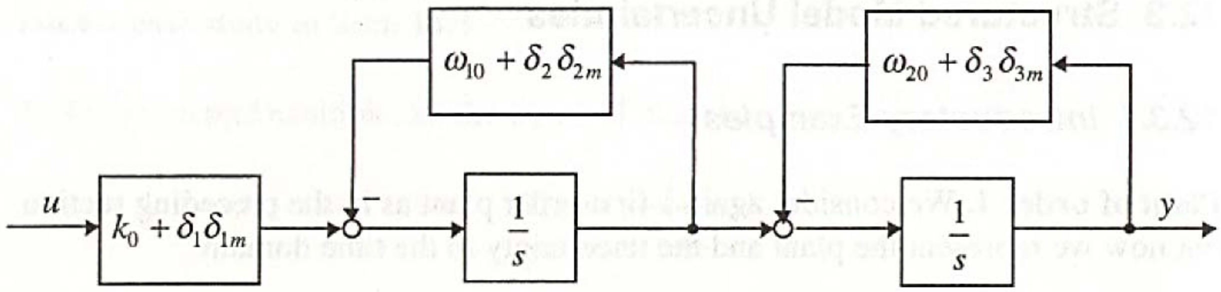
$$\begin{cases} \dot{x} = -a_o x - w_p + u \\ z_p = \delta_m x \end{cases}$$



- **Example 2:** Consider a second order system with gain and corner frequency uncertainties

$$G(s) = \frac{k}{(s+\omega_1)(s+\omega_2)}$$

$$\left|\delta_i\right| \le 1, i = 1,2,3$$



- We can isolate the 3 uncertainties, as before, creating a fictitious input-output pair for each uncertainty

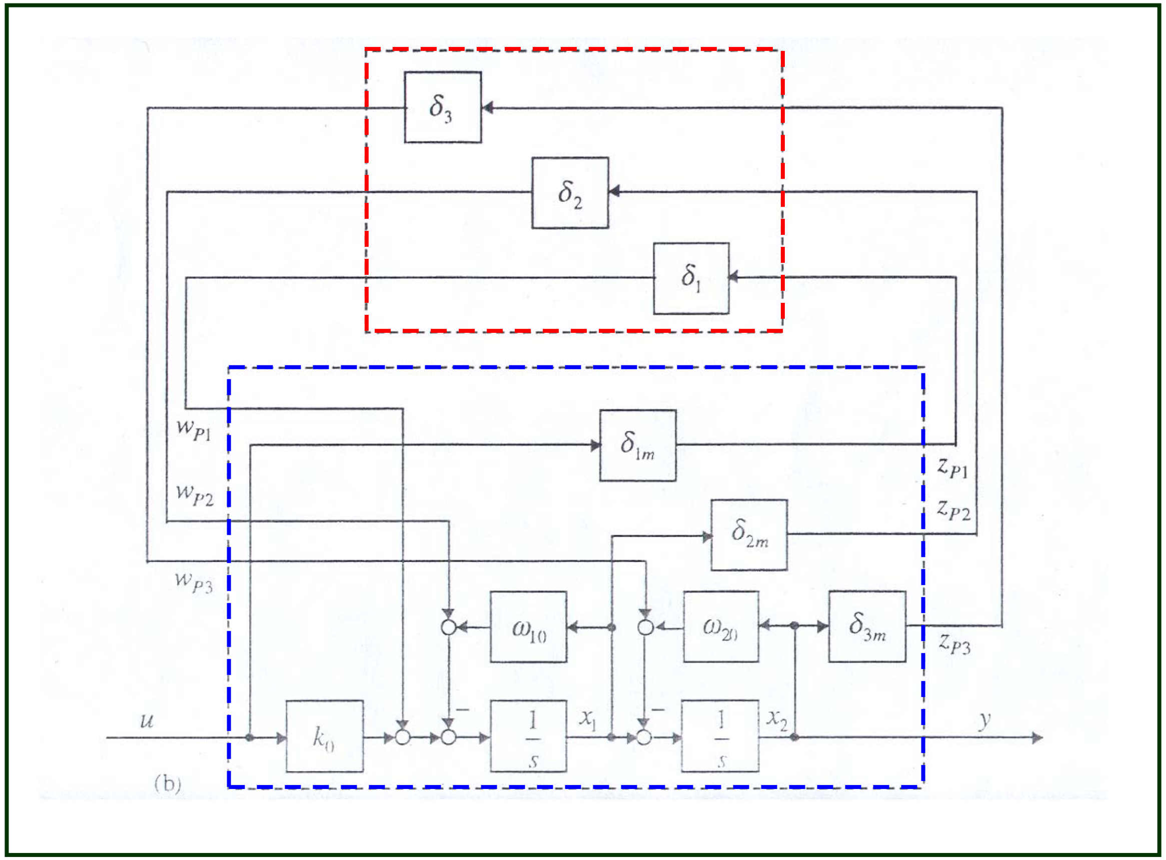
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\omega_{10} & 0 \\ 0 & -\omega_{20} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} k_0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} w_{p1} \\ w_{p2} \\ w_{p3} \end{bmatrix}$$

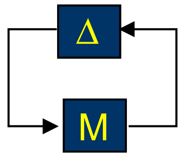
$$\begin{bmatrix} z_{p1} \\ z_{p2} \\ z_{p3} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \delta_{2m} & 0 \\ 0 & \delta_{3m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \delta_{1m} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- And the 3 constant gain uncertainty Block:

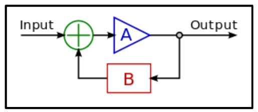
$$\begin{bmatrix} w_{p1} \\ w_{p2} \\ w_{p3} \end{bmatrix} = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{bmatrix} \begin{bmatrix} z_{p1} \\ z_{p2} \\ z_{p3} \end{bmatrix}$$





- **The block Δ isolated from the model has a definite structure (diagonal in this case) , therefore represents a structured uncertainty**

- δ_{im} **constitute the weights**



❑ Uncertainty using state space representation:

$$\begin{cases}\dot{x} = Ax + Bu \\ y = Cx + Du\end{cases}$$

- **<u>Assume as an example</u>** the uncertainty be additive with respect to the system matrices

- Linear dependence on uncertain parameters

$$\begin{pmatrix}\mathbf{A}_\delta & \mathbf{B}_\delta \\ \mathbf{C}_\delta & \mathbf{D}_\delta\end{pmatrix} = \sum_{i=1}^{k} \delta_i \begin{pmatrix}\mathbf{A}_i & \mathbf{B}_i \\ \mathbf{C}_i & \mathbf{D}_i\end{pmatrix}.$$

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_\delta, \quad \mathbf{B} = \mathbf{B}_0 + \mathbf{B}_\delta, \quad \mathbf{C} = \mathbf{C}_0 + \mathbf{C}_\delta, \quad \mathbf{D} = \mathbf{D}_0 + \mathbf{D}_\delta .$$

- In order to separate the uncertainty, we introduce a set of fictitious inputs and outputs

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}_0 \mathbf{x} + \mathbf{B}_0 \mathbf{u} + \mathbf{B}_2 \mathbf{w}_P \\ \mathbf{y} &= \mathbf{C}_0 \mathbf{x} + \mathbf{D}_0 \mathbf{u} + \mathbf{D}_{12} \mathbf{w}_P \\ \mathbf{z}_P &= \mathbf{C}_2 \mathbf{x} + \mathbf{D}_{21} \mathbf{u} .\end{aligned}$$

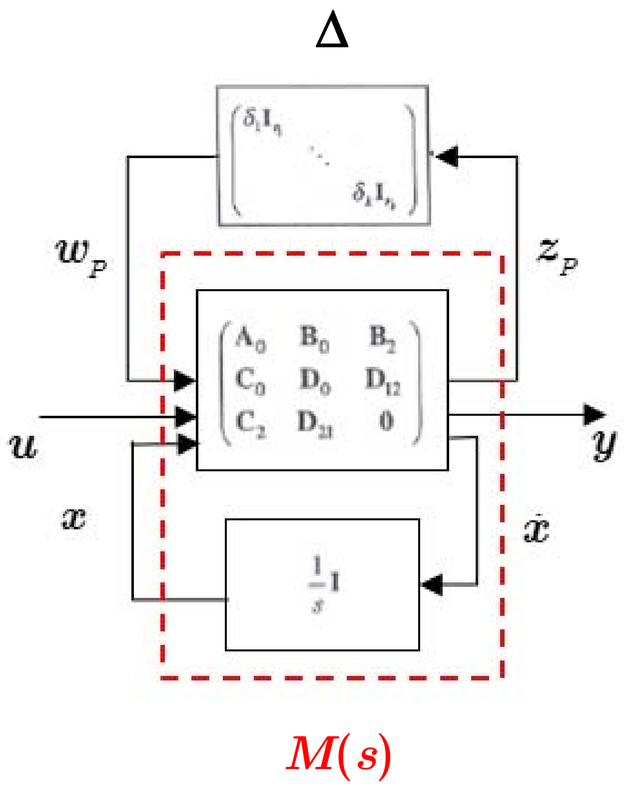
$$q_i = \operatorname{rank}\begin{pmatrix}\mathbf{A}_i & \mathbf{B}_i \\ \mathbf{C}_i & \mathbf{D}_i\end{pmatrix},$$

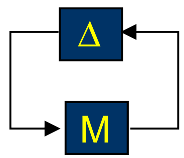
$$\begin{pmatrix}\mathbf{A}_i & \mathbf{B}_i \\ \mathbf{C}_i & \mathbf{D}_i\end{pmatrix} = \begin{pmatrix}\mathbf{L}_i \\ \mathbf{W}_i\end{pmatrix}\begin{pmatrix}\mathbf{R}_i^T & \mathbf{Z}_i^T\end{pmatrix}$$

$$\mathbf{E}_\delta = \begin{pmatrix}\delta_1 \mathbf{I}_{r_1} & & \\ & \ddots & \\ & & \delta_k \mathbf{I}_{r_k}\end{pmatrix}.$$

- Construct $\left(B_2, C_2, D_{12}, D_{21}, D_{22}\right)$ So that the uncertainty enters as a feedback law:

$$w_{pi} = \delta_{iz} z_{pi}, i = 1, \ldots k$$

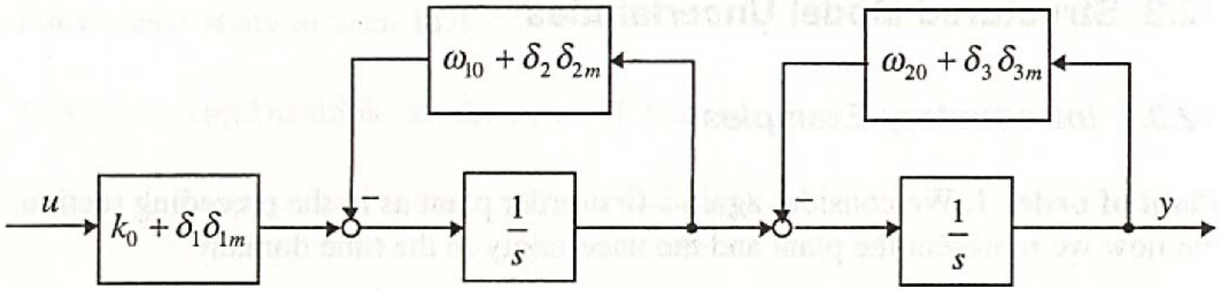




- **Recall Example 2:**

$$G(s) = \frac{k}{(s+\omega_1)(s+\omega_2)}$$

$$\left|\delta_i\right| \leq 1, i = 1,2,3$$



$$\mathbf{A}_0 = \begin{pmatrix} -\omega_{10} & 0 \\ 1 & -\omega_{20} \end{pmatrix}, \quad \mathbf{A}_\delta = \begin{pmatrix} -\delta_2\delta_{2m} & 0 \\ 0 & -\delta_3\delta_{3m} \end{pmatrix}$$

$$\mathbf{B}_0 = \begin{pmatrix} k_0 \\ 0 \end{pmatrix}, \quad \mathbf{B}_\delta = \begin{pmatrix} \delta_1\delta_m \\ 0 \end{pmatrix}$$

$$\mathbf{C}_0 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad \mathbf{C}_\delta = \begin{pmatrix} 0 & 0 \end{pmatrix}, \quad \mathbf{D}_0 = 0, \quad \mathbf{D}_\delta = 0.$$

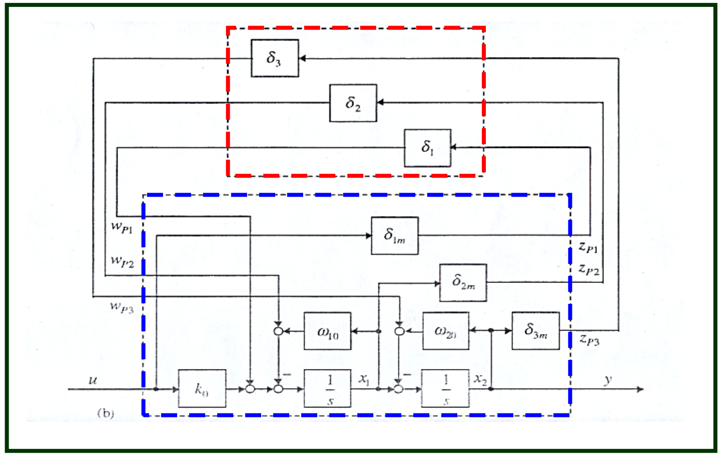
$$\begin{pmatrix} \mathbf{A}_\delta & \mathbf{B}_\delta \\ \mathbf{C}_\delta & \mathbf{D}_\delta \end{pmatrix} = \delta_1 \begin{pmatrix} 0 & 0 & \delta_{1m} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \delta_2 \begin{pmatrix} -\delta_{2m} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \delta_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\delta_{3m} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \delta_1 \begin{pmatrix} 0 & 0 & \delta_{1m} \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \delta_2 \begin{pmatrix} \delta_{2m} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \delta_3 \begin{pmatrix} 0 & \delta_{3m} & 0 \end{pmatrix}.$$

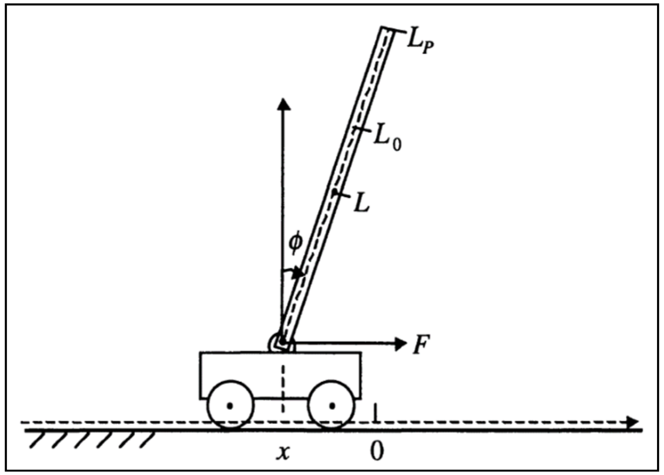
$$
\mathbf{L}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{L}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \mathbf{L}_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix},
$$
$$
\mathbf{W}_1 = 0, \quad \mathbf{W}_2 = 0, \quad \mathbf{W}_3 = 0,
$$
$$
\mathbf{R}_1^T = \begin{pmatrix} 0 & 0 \end{pmatrix}, \quad \mathbf{R}_2^T = \begin{pmatrix} \delta_{2m} & 0 \end{pmatrix}, \quad \mathbf{R}_3^T = \begin{pmatrix} 0 & \delta_{3m} \end{pmatrix},
$$
$$
\mathbf{Z}_1 = \delta_{1m}, \quad \mathbf{Z}_2 = 0, \quad \mathbf{Z}_3 = 0 .
$$

This implies that $\mathbf{D}_{12} = \mathbf{0}$ and

$$
\mathbf{B}_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{C}_2 = \begin{pmatrix} 0 & 0 \\ \delta_{2m} & 0 \\ 0 & \delta_{3m} \end{pmatrix}, \quad \mathbf{D}_{21} = \begin{pmatrix} \delta_{1m} \\ 0 \\ 0 \end{pmatrix},
$$



Exercise: Inverted Pendulum Example

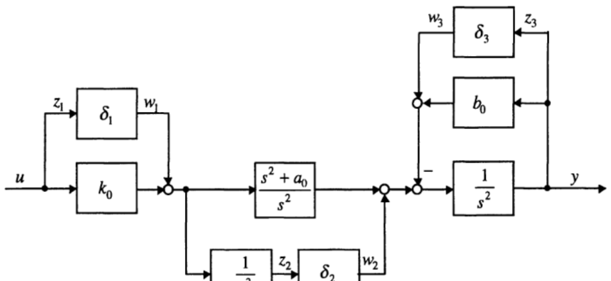


- Neglect DC motor dynamics at the hinge, and compute the transfer function between armature current and cart position:

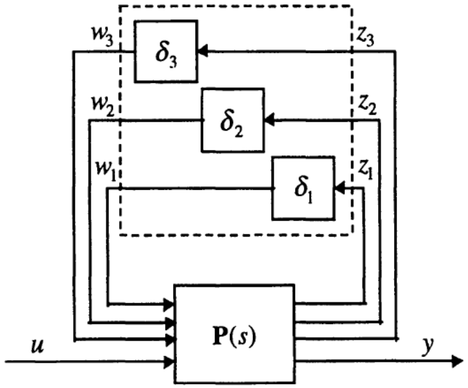
$$m_P \ddot{x}_P = F_x$$
$$m_P \ddot{z}_P = F_z - m_P g$$
$$J\ddot{\phi} = LF_z \sin\phi - LF_x \cos\phi$$

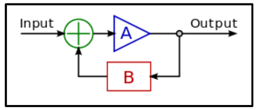
$$G(s) = k\frac{s^2 + a}{s^2(s^2 + b)}$$

Let all three parameters be uncertain: $k = k_0 + \delta_1$, $a = a_0 + \delta_2$, $b = b_0 + \delta_3$. In order to pull out the Δ s, one has to draw a block diagram where again the uncertain parameters $\delta_1, \delta_2, \delta_3$ are isolated in the form of additional blocks.



(a)

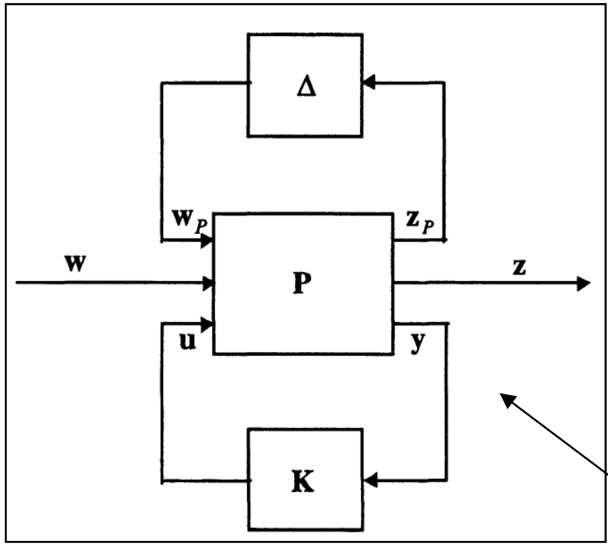




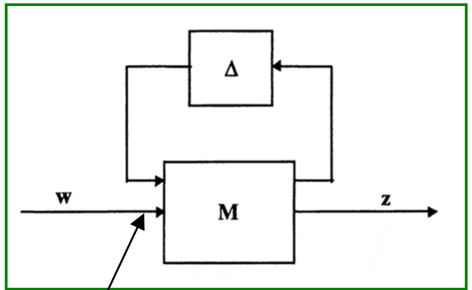
Chapter 4: Uncertainty in a 2x2 Block Structure

Summary

- The basic idea is to isolate uncertainty (structured and unstructured) in a separate block, to be connected in feedback to the plant, via additional input – output pair.



$$\begin{pmatrix} \mathbf{z}_P \\ \mathbf{z} \\ \mathbf{y} \end{pmatrix} = \mathbf{P}(s)\begin{pmatrix} \mathbf{w}_P \\ \mathbf{w} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{P}_{11}(s) & \mathbf{P}_{12}(s) & \mathbf{P}_{13}(s) \\ \mathbf{P}_{21}(s) & \mathbf{P}_{22}(s) & \mathbf{P}_{23}(s) \\ \mathbf{P}_{31}(s) & \mathbf{P}_{32}(s) & \mathbf{P}_{33}(s) \end{pmatrix}\begin{pmatrix} \mathbf{w}_P \\ \mathbf{w} \\ \mathbf{u} \end{pmatrix}.$$

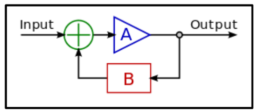


The uncertain plant is

$$\mathbf{G} = \mathcal{F}_u(\mathbf{P}, \Delta)\ .$$

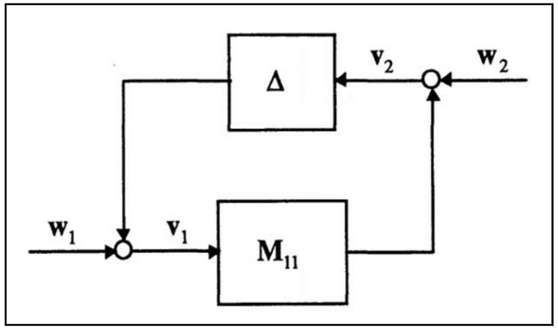
Here, $\mathcal{F}_u(\mathbf{P}, \Delta)$ denotes the feedback connection of $\mathbf{P}$ and Δ (similar to $\mathcal{F}_l(\mathbf{P}, \mathbf{K})$, cf. Sect. 8.2.3), and the uncertain feedback system is

$$\mathbf{M}(s) = \mathcal{F}_l(\mathbf{P}(s), \mathbf{K}(s)) = \begin{pmatrix} \mathbf{M}_{11}(s) & \mathbf{M}_{12}(s) \\ \mathbf{M}_{21}(s) & \mathbf{M}_{22}(s) \end{pmatrix}.$$



❑ Road to the Small Gain Theorem

❑ **Statement:** Given an uncertain 2-Block structure with stable uncertainty transfer matrix $\Delta(s)$, furthermore let the uncertainty bounded in magnitude $\|\Delta\|_\infty \le \gamma$: stability robustness requires finding the largest γ, such that the feedback system below is internally stable.

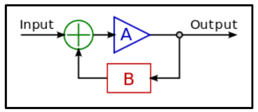


The uncertainty Δ can be composed of several components. Normally we assume that it has the following block diagonal form:

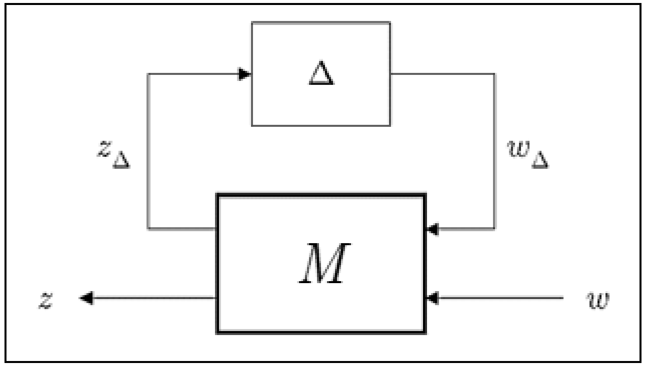
$$\Delta = \text{diag}\left(\delta_1 \mathbf{I}_{r_1}, \ldots, \delta_S \mathbf{I}_{r_S}, \Delta_1, \ldots, \Delta_F\right)$$

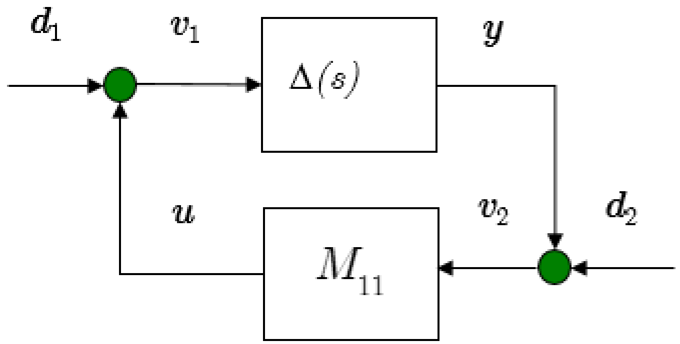
with $\Delta_i(s) \in \mathcal{RH}_\infty$. The quantities δ_i may be real numbers or transfer functions, $\delta_i(s) \in \mathcal{RH}_\infty$. This will be discussed in the next chapter. By suitable weights which have to be added to the plant, we may assume that

$$|\delta_i| \le 1 \quad (\text{or } \|\delta_i\|_\infty \le 1) \text{ and } \|\Delta_i\|_\infty \le 1 .$$



- Recall the Internal Stability Requirements for MIMO Systems:





$$\left[M_{22} + M_{21}\Delta(\boldsymbol{I} - \boldsymbol{M_{11}\Delta})^{-1}M_{12}\right] = T_{22}(s)$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -(I+KG)^{-1}KG & -(I+KG)^{-1}K \\ (I+GK)^{-1}G & -(I+GK)^{-1}GK \end{bmatrix}}_{M} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

- **Definition 1 : An upper Fractional transformation (LFT) $\mathcal{F}(M, \Delta_U)$ is said to be well posed if $(I - M_{11}\Delta_U)^{-1}$ exists**

- **Definition 2 : An upper Fractional transformation (LFT) $\mathcal{F}(M, \Delta_U)$ is said to be well posed if $(I - M_{11}\Delta_U)$ has a proper and stable inverse**

- **Small Gain Theorem (Zhou, p. 218):** Let a nominal system $M(s)$ be internally stable. Moreover, let $\gamma > 0$ (also in the presence of stable weights); then :

  - The perturbed system is well-posed and Internally stable for all stable perturbations for which:

$$(a)\ \|\Delta\|_\infty \leq 1/\gamma \ \textit{if and only if}\ \|M(s)\|_\infty < \gamma;$$

$$(b)\ \|\Delta\|_\infty < 1/\gamma \ \textit{if and only if}\ \|M(s)\|_\infty \leq \gamma.$$

- Iff when $\gamma = 1$

  - Proof of Part (a) with $\gamma = 1$:

(Sufficiency) It is clear that $M(s)\Delta(s)$ is stable since both $M(s)$ and $\Delta(s)$ are stable. Thus by Theorem 5.7 (or Corollary 5.6) the closed-loop system is stable if $\det(I - M\Delta)$ has no zero in the closed right-half plane for all $\Delta \in \mathcal{RH}_\infty$ and $\|\Delta\|_\infty \leq 1$. Equivalently, the closed-loop system is stable if

$$\inf_{s \in \overline{\mathbb{C}}_+} \underline{\sigma}\left(I - M(s)\Delta(s)\right) \neq 0$$

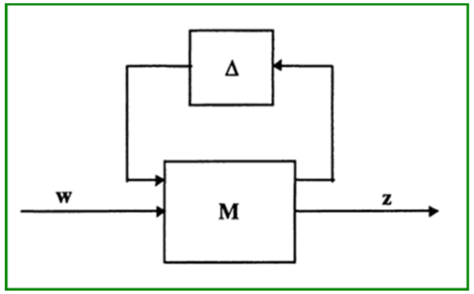
for all $\Delta \in \mathcal{RH}_\infty$ and $\|\Delta\|_\infty \leq 1$. But this follows from

$$\inf_{s \in \overline{\mathbb{C}}_+} \underline{\sigma}\left(I - M(s)\Delta(s)\right) \geq 1 - \sup_{s \in \overline{\mathbb{C}}_+} \bar{\sigma}\left(M(s)\Delta(s)\right) = 1 - \|M(s)\Delta(s)\|_\infty \geq 1 - \|M(s)\|_\infty > 0.$$

(Necessity) This will be shown by contradiction. Suppose $\|M\|_\infty \geq 1$. We will show that there exists a $\Delta \in \mathcal{RH}_\infty$ with $\|\Delta\|_\infty \leq 1$ such that $\det(I - M(s)\Delta(s))$ has a zero on the imaginary axis, so the system is unstable.

- The Small Gain Theorem has several versions, and it is an original extension of the MIMO Nyquist Criterion

❑ Small Gain Theorem (Lavretsky, 2013, p. 134):



- Given a block diagonal, bounded, stable uncertainty matrix $\Delta(s)$, and a stable nominal system $M(s)$, robust stability requires the return difference matrix to be nonsingular from Nyquist.

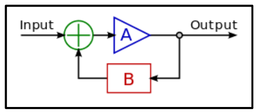
$$\det(I - \Delta M) \neq 0 \Rightarrow \sigma_{\min}(I - \Delta M) > 0$$

- A bound on the norm of Δ(s) can be found recalling the A + B argument (see slide 55).

- From $\det(I\text{-}\Delta M) = 0$ we have: $\sigma_{\min}(I) \leq \sigma_{\max}(\Delta M)$

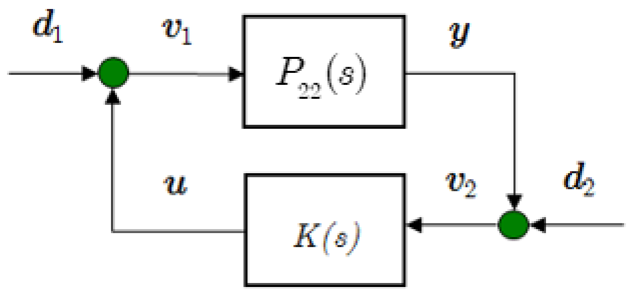
$$1 = \sigma_{\min}(I) \leq \sigma_{\max}(\Delta M) < \sigma_{\max}(\Delta) \cdot \sigma_{\max}(M) \Rightarrow \sigma_{\max}(\Delta) > \frac{1}{\sigma_{\max}(M)}$$

$$\boxed{\det(I - \Delta M) \neq 0 \Rightarrow \sigma_{\max}(\Delta) < \frac{1}{\sigma_{\max}(M)}}$$



- Sufficient condition due to conservatism of inequality in red **(*)**

❑ Small Gain Theorem (Mackenroth, 2004, p. 388):



$L_o = -P_{22}K$

Given $P_{22}(s) \in \mathcal{RH}_\infty$, $K(s) \in \mathcal{RH}_\infty$, and additionally $\left\|L_o(s)\right\|_\infty < 1$ then the feedback system is well – posed and internally stable

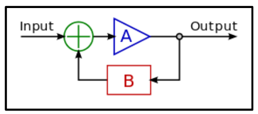
- Well posedeness and internal stability require S_o to exist and be stable (MIMO Nyquist) $\quad S_o = (I + L_o)^{-1}$

$$\left\|L_o(s)\right\|_\infty < 1 \Rightarrow \left\|D_{22}D_K\right\|_\infty < 1 \Rightarrow \det(I - D_{22}D_K) \neq 0 \qquad S_o \text{ is well posed.}$$

- Internal stability requires no unstable transmission zeros of $(I + L_o)$, i.e.

$$\inf_{s\in\bar{\mathbb{C}}_+} \underline{\sigma}(\mathbf{I}+\mathbf{L}_o(s)) \neq 0 \qquad\qquad \inf_{s\in\bar{\mathbb{C}}_+} \underline{\sigma}(\mathbf{I}+\mathbf{L}_o(s)) \geq 1 - \sup_{s\in\bar{\mathbb{C}}_+} \bar{\sigma}(\mathbf{L}_o(s)) > 0$$

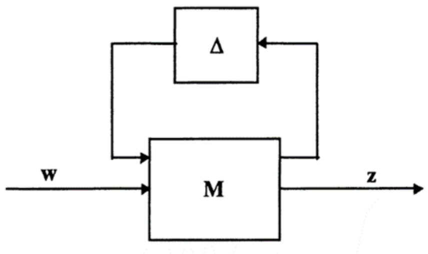
(*) The Small Gain theorem gives a necessary and sufficient condition for unstructured uncertainty, and only a sufficient condition for structured uncertainty.

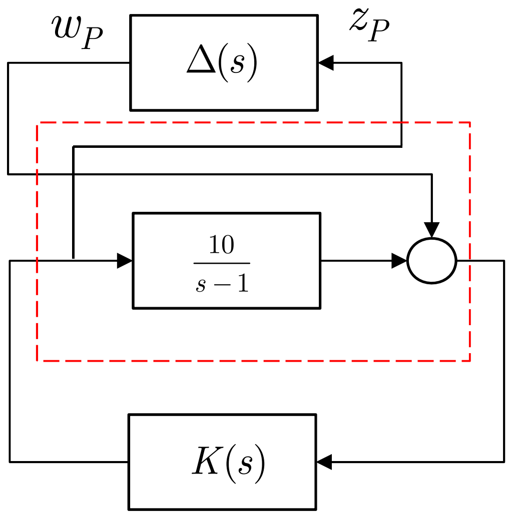


- **Example:** Consider a SISO model given by:

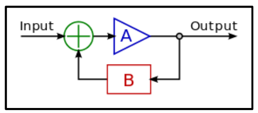
$$\begin{cases} G_0(s) = \dfrac{10}{s-1} \\ \left\|\Delta(s)\right\|_\infty \le 1 \\ K(s) = \dfrac{1}{2} \end{cases}$$

- Nominal Plant
- Input additive uncertainty
- Controller





- The uncertain system is unstable



- Compute the transfer function from w_P to z_P

$$M_{11}(s) = \frac{-K(s)}{1+K(s)G_0(s)} = \frac{-\frac{1}{2}(s-1)}{s+4}$$

- The nominal system is asymptotically stable
- The maximum singular value of $M_{11}(s)$ is:

$$\sigma_{\max}\left[M_{11}(j\omega)\right] = \left|M_{11}(j\omega)\right| = \frac{1}{2}\frac{\sqrt{\omega^2+1}}{\sqrt{\omega^2+16}}$$

- The $\mathcal{H}_\infty$ is given by:

$$\left\|M_{11}(j\omega)\right\|_\infty = \sup_\omega \sigma_{\max}\left[\frac{1}{2}\frac{\sqrt{\omega^2+1}}{\sqrt{\omega^2+16}}\right] = \frac{1}{2}$$

- **Preview of Approach to robust stability and robust performance using SISO systems** (Skogestad, Multivariable Feedback Control, Chapter 7.)

- A control system is robust if it is insensitive to differences between the actual system and the model of the system which was used to design the controller. These differences are referred to as model/plant mismatch or simply model uncertainty. The key idea in the H_∞ robust control paradigm is to check whether the design specifications are satisfied even for the "worst-case" uncertainty.

  - The approach can then be described as follows:

1. **Determine the uncertainty set:** find a mathematical representation of the model uncertainty ("clarify what we know about the system and what we don't know").
2. **Check Robust stability (RS):** determine whether the system remains stable for all plants in the uncertainty set.
3. **Check Robust performance (RP):** if RS is satisfied, determine whether the performance specifications are met for all plants in the uncertainty set.

- It should also be appreciated that model uncertainty is not the only concern when it comes to robustness. **Other considerations** include <u>sensor and actuator failures</u>, physical constraints, <u>changes in control objectives</u>, the opening and closing of loops, etc. Furthermore, if a control design is based on an optimization, then robustness problems may also be caused by the mathematical <u>objective function not properly describing</u> the real control problem. Also, the <u>numerical design algorithms</u> themselves may not be robust.

- **Remark:** Another strategy for dealing with model uncertainty is to approximate its effect on the feedback system by adding fictitious disturbances or noise. For example, this is the only way of handling model uncertainty within the so-called LQG approach to optimal control. **Is this an acceptable strategy?** In general, the answer is no. This is easily illustrated for linear systems where the addition of disturbances does not affect system **stability**, whereas model uncertainty combined with feedback may easily create instability.

- Uncertainty in the plant model may have several origins:

1. There are always parameters in the linear model which are only known approximately or are simply in error.
2. The parameters in the linear model may vary due to nonlinearities or changes in the operating conditions.
3. Measurement devices have imperfections. This may even give rise to uncertainty on the manipulated inputs, since the actual input is often measured and adjusted in a cascade manner. For example, this is often the case with valves where a flow controller is often used. In other cases limited valve resolution may cause input uncertainty.
4. At high frequencies even the structure and the model order is unknown, and the uncertainty will always exceed 100% at some frequency.
5. Even when a very detailed model is available we may choose to work with a simpler (low-order) nominal model and represent the neglected dynamics as "uncertainty".
6. Finally, the controller implemented may differ from the one obtained by solving the synthesis problem. In this case one may include uncertainty to allow for controller order reduction and implementation inaccuracies.

Chapter 4: Summary

- How do we combine all the uncertainties in a mathematically tractable set?

1. **Parametric uncertainty.** Here the structure of the model (including the order) is known, but some of the parameters are uncertain.

   - **Structured Uncertainty**

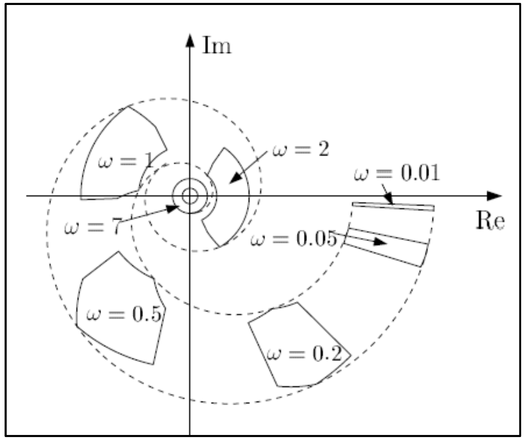
2. **Neglected and unmodelled dynamics uncertainty.** Here the model is in error because of missing dynamics, usually at high frequencies, either through deliberate neglect or because of a lack of understanding of the physical process. Any model of a real system will contain this source of uncertainty.

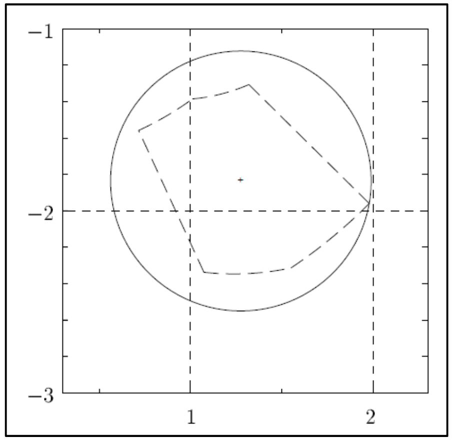
   - **Unstructured Uncertainty**

3. **Lumped uncertainty.** Here the uncertainty description represents one or several sources of parametric and/or unmodelled dynamics uncertainty combined into a single lumped perturbation of a chosen structure.

   - **Structured Uncertainty** + - **Unstructured Uncertainty**

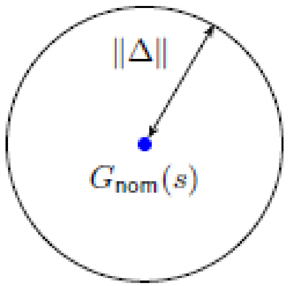
- Uncertainty approximation as a frequency dependent complex function (or matrix in MIMO settings disc (conservative)





Model sets

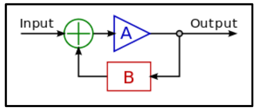
$$G(s) \in \{\, G_{\text{nom}}(s) + \Delta \mid \|\Delta\| \le \gamma \,\}$$



$G_{\text{nom}}(s)$ = Nominal plant

Δ = unknown, but bounded, perturbation (i/o operator)

Typically, Δ is stable, causal and satisfies, $\|\Delta\|_{\mathcal{H}_\infty} \le \gamma$.

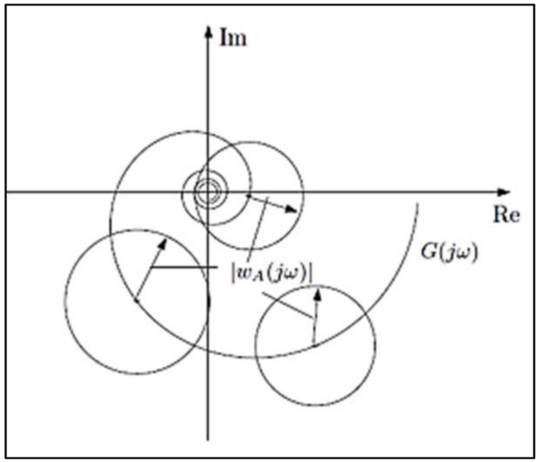


- Disc shape representation of a complex additive uncertainty

$$G_p(s) = G(s) + w_A(s)\Delta_A(s); \quad |\Delta_A(j\omega)| \le 1 \; \forall\omega$$

- $\Delta_A(s)$ is any stable transfer function, at each frequency no larger than one. At each frequency then $G_P(s)$ generates a disc-shaped region with radius $|w_A(j\omega)|$ centered at $G(s)$.

- $w_A(j\omega)$ can be thought as a weight introduced to normalize the perturbation to be less than one at each frequency. A typical selection is to be stable and minimum phase.



- **Question:** What happens in the case of a MIMO system?

1. Mathematically nothing except for

$$\left\|\Delta_A(j\omega)\right\|_\infty \le 1$$

2. Graphically we have no equivalent

- Disc shape representation of a complex multiplicative uncertainty

$$G_p(s) = G(s)(1 + w_I(s)\Delta_I(s)); \quad \underbrace{|\Delta_I(j\omega)| \leq 1 \, \forall\omega}_{||\Delta_I||_\infty \leq 1}$$

- Note that for SISO systems both uncertainties are equivalent as long as:

$$|w_I(j\omega)| = |w_A(j\omega)|/|G(j\omega)|$$

- However, multiplicative (relative) weights are often preferred because their numerical value is more informative. At frequencies where $|w_I(j\omega)| > 1$ the uncertainty exceeds 100% and the Nyquist curve may pass through the origin.

$$G(1 + w_I\Delta_I) = (1 + w_O\Delta_O)G \quad \text{with } \Delta_I(s) = \Delta_O(s) \text{ and } w_I(s) = w_O(s)$$

- How do we generate the weights $w(s)$ so that we have a single lumped complex perturbation Δ_A or Δ_I ?

  - The objective is then to find a suitable set of frequency dependent weighting functions **$w_A(j\omega)$** and **$w_I(j\omega)$**.

1. Select a nominal model $G(s)$.
2. $\boxed{\textit{Additive uncertainty.}}$ At each frequency find the smallest radius $l_A(\omega)$ which includes all the possible plants Π:

$$l_A(\omega) = \max_{G_p \in \Pi} |G_p(j\omega) - G(j\omega)|$$

If we want a rational transfer function weight, $w_A(s)$ (which may not be the case if we only want to do analysis), then it must be chosen to cover the set, so

$$|w_A(j\omega)| \geq l_A(\omega) \quad \forall\omega \qquad |w_A(j\omega)| \geq \bar{\sigma}_{\omega}[G_p(s) - G(s)] = \|G_p(s) - G(s)\|_\infty$$

Usually $w_A(s)$ is of low order to simplify the controller design. Furthermore, an objective of frequency-domain uncertainty is usually to represent uncertainty in a simple straightforward manner.

3. *Multiplicative (relative) uncertainty.* This is often the preferred uncertainty form, and we have

$$l_I(\omega) = \max_{G_p \in \Pi} \left| \frac{G_p(j\omega) - G(j\omega)}{G(j\omega)} \right|$$

and with a rational weight

$$|w_I(j\omega)| \geq l_I(\omega), \forall \omega$$

$$l_I(\omega) = \max_{G_p \in \Pi} \bar{\sigma}\left(G^{-1}(G_p - G)(j\omega)\right); \quad |w_I(j\omega)| \geq l_I(\omega) \quad \forall \omega$$

$$l_O(\omega) = \max_{G_p \in \Pi} \bar{\sigma}\left((G_p - G)G^{-1}(j\omega)\right); \quad |w_O(j\omega)| \geq l_O(\omega) \quad \forall \omega$$

- **Notes:**
  - Lumping uncertainty either at the output or the input does not work well. This is because one cannot in general shift a perturbation from one location in the to another location without introducing candidate plants which were not present in the original set.
  - Changing the position of the uncertainty, changes the bounds especially if the condition number of the system is high.

$$G(1 + w_I\Delta_I) = (1 + w_O\Delta_O)G \quad \text{with} \;\; \Delta_I(s) = \Delta_O(s) \text{ and } w_I(s) = w_O(s)$$

Is not valid !!!

Chapter 4: Summary

- Assume: $w_I(j\omega) = w_O(j\omega); \; \left\|\Delta_I(s)\right\|_\infty = \left\|\Delta_O(s)\right\|_\infty \le 1$

- At the input: $G_p = G(I + E_I) \qquad E_I = w_I(j\omega)\,\Delta_I(s)$

$$l_I(\omega) = \max_{E_I} \bar{\sigma}(G^{-1}(G_p - G)) = \max_{E_I} \bar{\sigma}(E_I)$$

- At the output: $G_P = (I + E_I)G \qquad E_I = E_O$

$$l_O(\omega) = \max_{E_I} \bar{\sigma}((G_p - G)G^{-1}) = \max_{E_I} \bar{\sigma}(GE_IG^{-1})$$

$$\bar{\sigma}(GE_IG^{-1}) \le \bar{\sigma}(GE_I)\bar{\sigma}(G^{-1}) = \bar{\sigma}(GE_I)\frac{1}{\underline{\sigma}(G)} \le \bar{\sigma}(G)\bar{\sigma}(E_I)\frac{1}{\underline{\sigma}(G)} = \frac{\bar{\sigma}(G)}{\underline{\sigma}(G)}\bar{\sigma}(E_I) = \gamma(G)\bar{\sigma}(E_I)$$

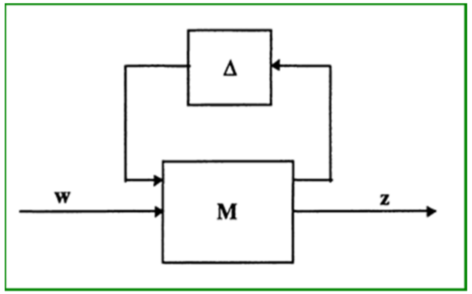
- For a large condition number of the system: $\gamma(G) = \left[\dfrac{\bar{\sigma}(G)}{\underline{\sigma}(G)}\right]$

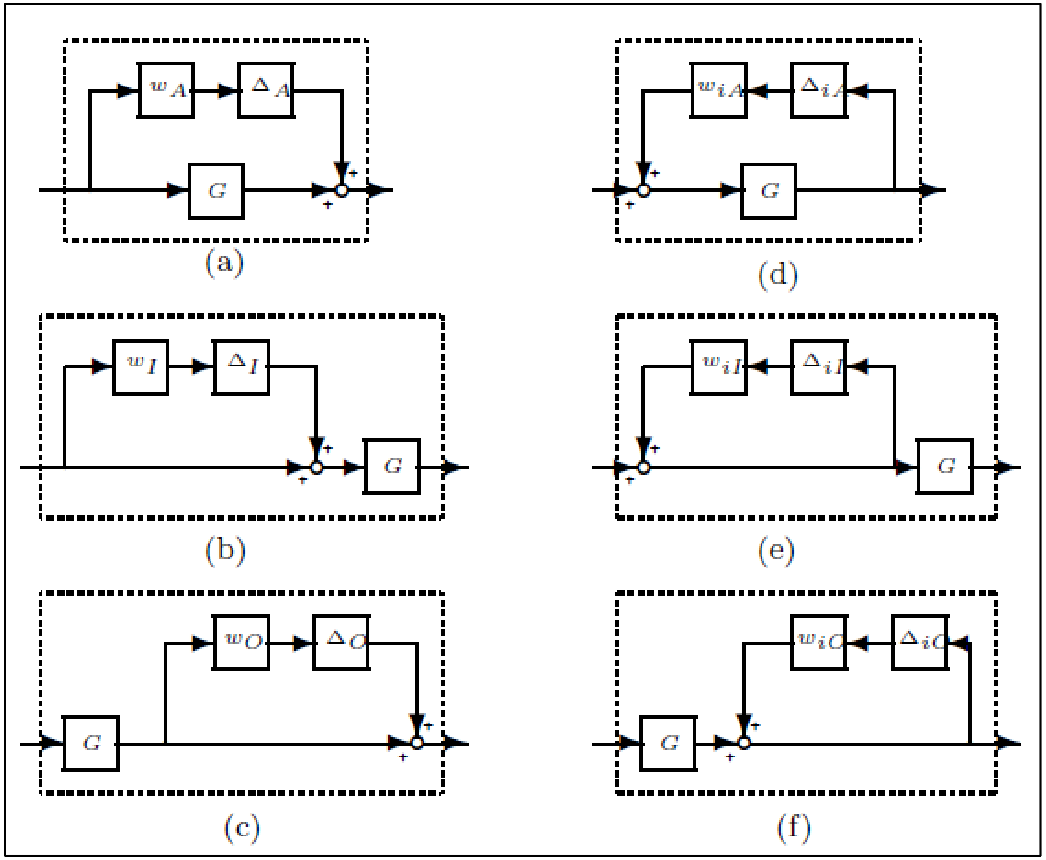
$$\boxed{l_O(\omega) \gg l_I(\omega)}$$

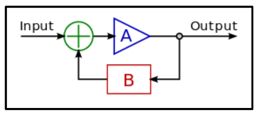
SISO Robust Stability

- We have seen that uncertainties may occur in different parts of the feedback loop, depending on the type of perturbation

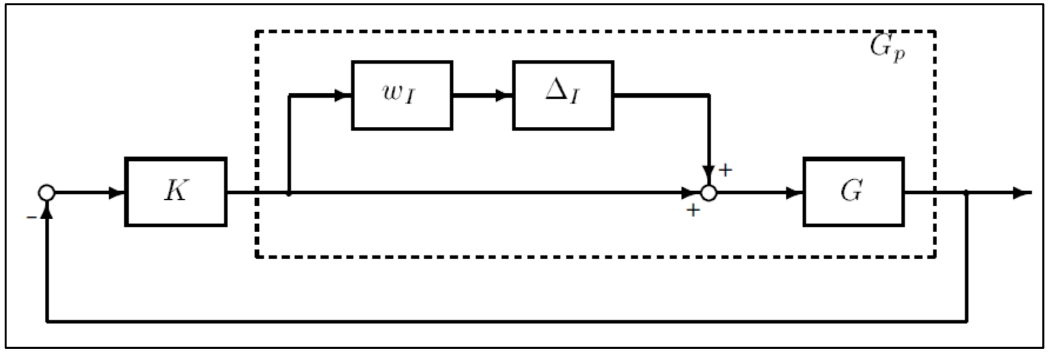
- We know how to combine the uncertainties in a single block set







- Consider a system with an **input multiplicative** perturbation from nominal



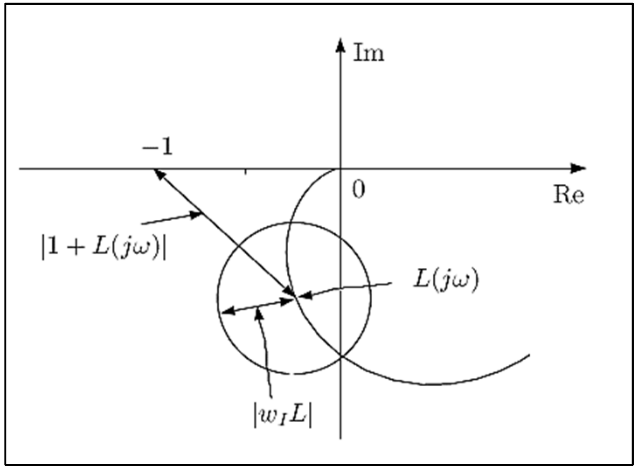
We want to determine the stability of the uncertain feedback system in Figure when there is multiplicative (relative) uncertainty of magnitude $|w_I(j\omega)|$. With uncertainty the loop transfer function becomes

$$L_p = G_p K = GK(1 + w_I \Delta_I) = L + w_I L \Delta_I, \quad |\Delta_I(j\omega)| \leq 1, \forall\omega$$

As always, we assume (by design) stability of the nominal closed-loop system (i.e. with $\Delta_I = 0$). For simplicity, we also assume that the loop transfer function L_p is stable. We now use the Nyquist stability condition to test for robust stability of the closed-loop system.

- **Graphical Derivation (valid for SISO systems)**

$|-1-L| = |1+L|$ is the
distance from the point -1 to the centre of the disc representing L_p



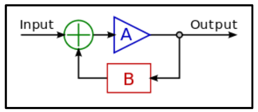
$$\text{RS} \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \text{System stable } \forall L_p$$

$$\Leftrightarrow L_p \text{ should not encircle the point } -1, \quad \forall L_p$$

$$\begin{aligned} \text{RS} \quad &\Leftrightarrow \quad |w_I L| < |1+L|, \quad \forall \omega \\ &\Leftrightarrow \quad \left| \frac{w_I L}{1+L} \right| < 1, \ \forall \omega \quad \Leftrightarrow \quad |w_I T| < 1, \ \forall \omega \\ &\boxed{\stackrel{\text{def}}{\Leftrightarrow} \quad \|w_I T\|_\infty < 1} \end{aligned}$$

Note that for SISO systems $w_I = w_O$ and $T = T_I = GK(1+GK)^{-1}$, so the condition could equivalently be written in terms of $w_I T_I$ or $w_O T$. Thus, the requirement of robust stability for the case with multiplicative uncertainty gives an upper bound on the complementary sensitivity:

$$\boxed{\text{RS} \Leftrightarrow |T| < 1/|w_I|, \quad \forall \omega}$$



- **Algebraic Derivation**
  - Since L_P and L are both stable (as an example), then

$$
\begin{aligned}
\mathrm{RS} \quad &\Leftrightarrow \quad |1+L_p| \neq 0, \quad \forall L_p, \forall\omega \\
&\Leftrightarrow \quad |1+L_p| > 0, \quad \forall L_p, \forall\omega \\
&\Leftrightarrow \quad |1+L+w_I L\Delta_I| > 0, \quad \forall|\Delta_I| \leq 1, \forall\omega
\end{aligned}
$$

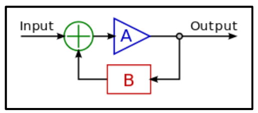
At each frequency the last condition is most easily violated (the worst case) when the complex number $\Delta_I(j\omega)$ is selected with $|\Delta_I(j\omega)| = 1$ and with phase such that the terms $(1+L)$ and $w_I L\Delta_I$ have opposite signs (point in the opposite direction). Thus

$$
\mathrm{RS} \Leftrightarrow |1+L| - |w_I L| > 0, \quad \forall\omega \quad \Leftrightarrow |w_I T| < 1, \quad \forall\omega
$$

- **NOTES:**
  - We must reduce $T(j\omega)$ for frequencies where $w_I > 1$
  - The condition is CNS for all plants for which perturbations are such that

$$
|\Delta_I(s)| \leq 1
$$

  - The result holds even if L and LP are unstable as long as the uncertainty does not add unstable poles



Example (Skogestad pp. 272 – 273)

$$G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)}$$

$$K(s) = K_c \frac{12.7s+1}{12.7s}$$

- Assume the real plant:

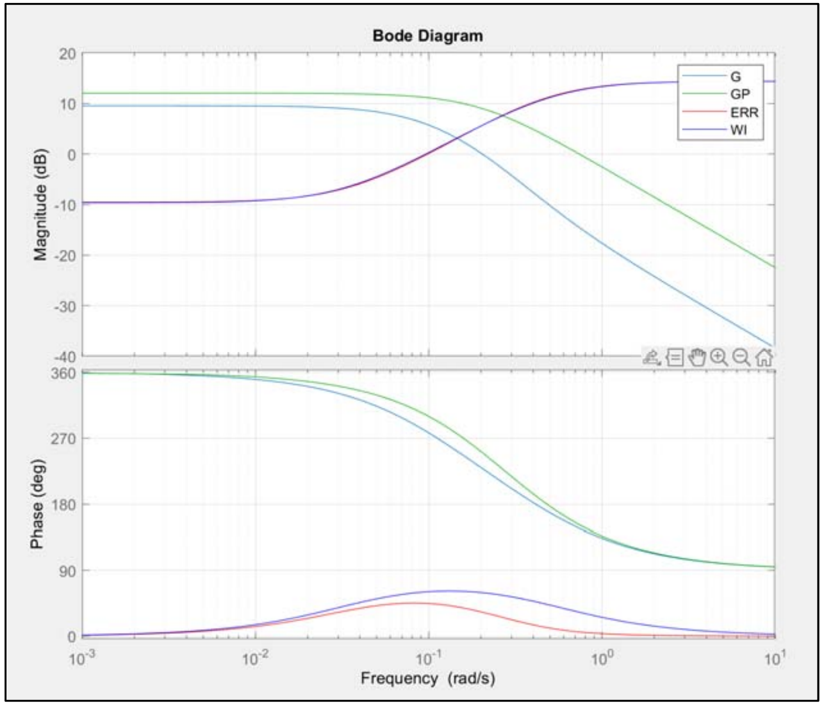
$$G_P(s) = \frac{4(-2s+1)}{(4s+)^2}$$

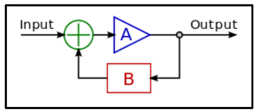
- Relative error from frequency response:

$$\frac{G_P - G}{G} = \begin{cases} 0.33, \omega \ll 0.1 \\ 1, \omega \simeq 0.1 \\ 5.25, \omega \gg 0.1 \end{cases}$$

- Relative error from frequency response:

$$w_I(s) = \frac{10s+0.33}{(10/5.25)s+1}$$

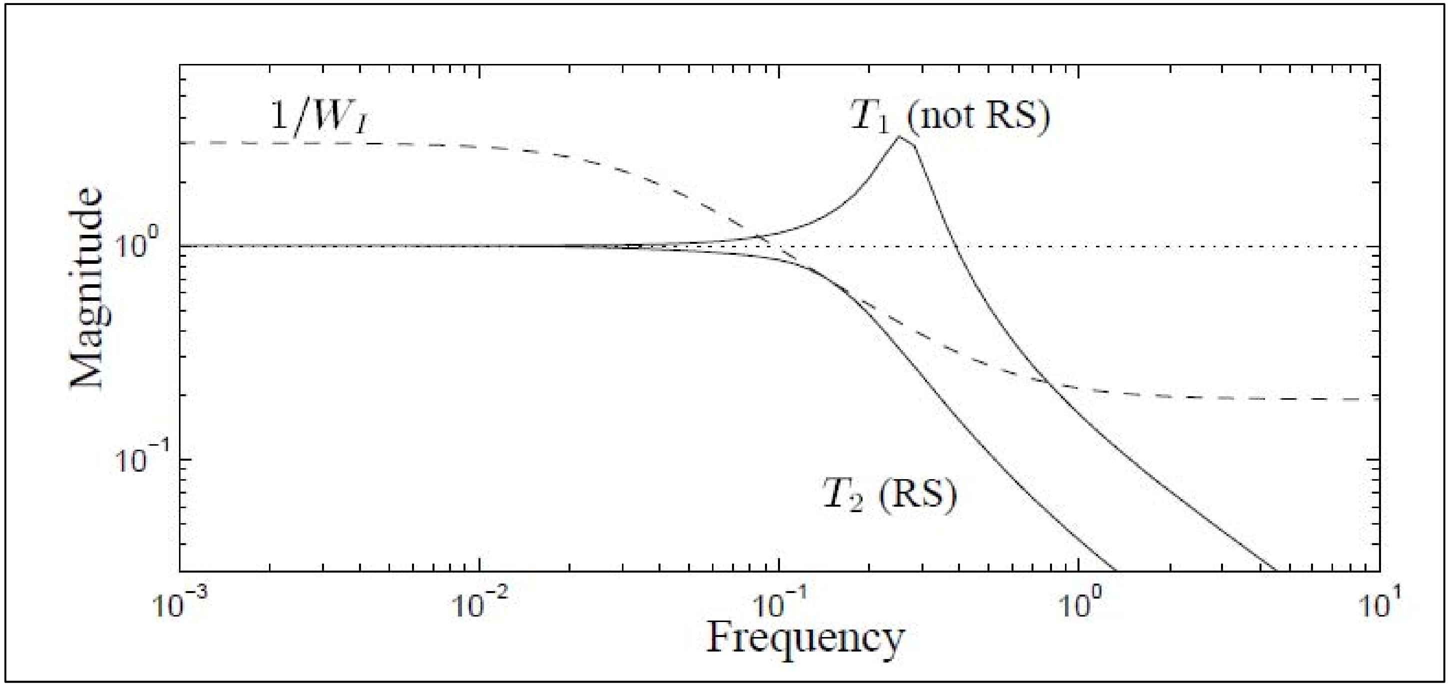




- Compute complementary sensitivity and verify robust stability with different controller gains

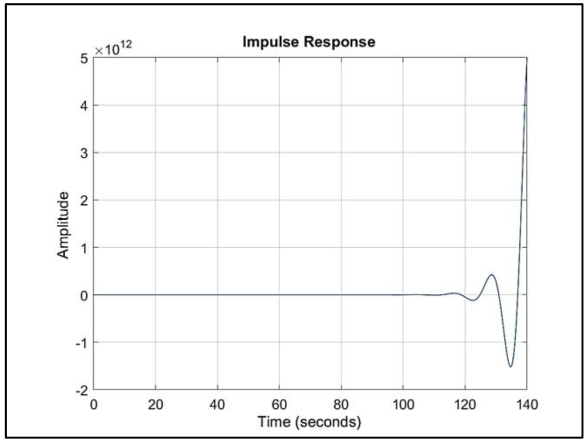
$$\boxed{\text{RS} \Leftrightarrow |T| < 1/|w_I|, \quad \forall \omega} \qquad T = G_P K \,/\, (1 + G_P K)$$

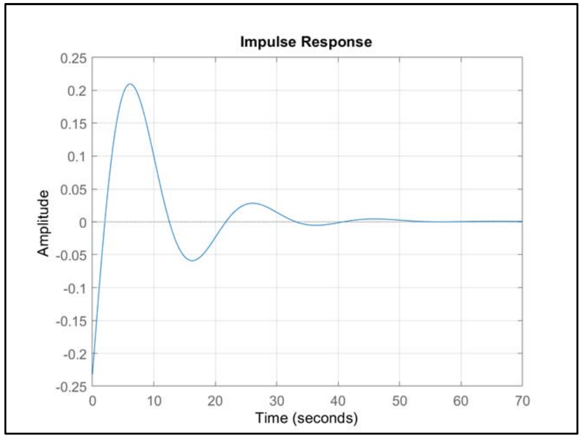
$K_{c1} = 1.13$ **Using ZN tuning** $\qquad$ $K_{c2} = 0.31$ **By trial and error**



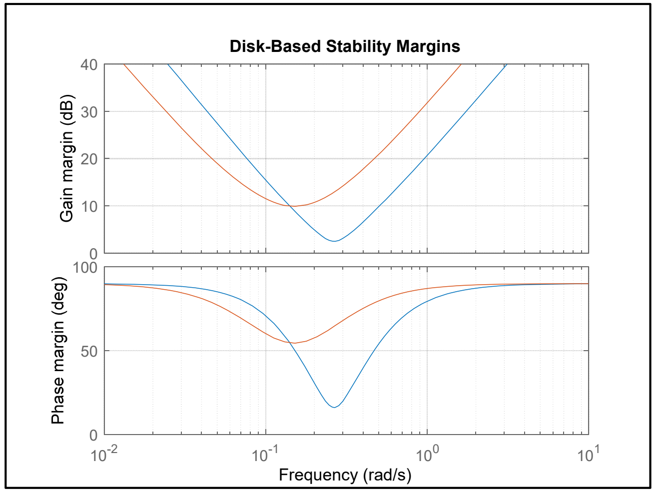
$$K_{c1} = 1.13$$

$$K_{c2} = 0.31$$

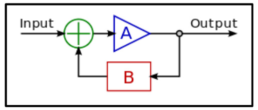




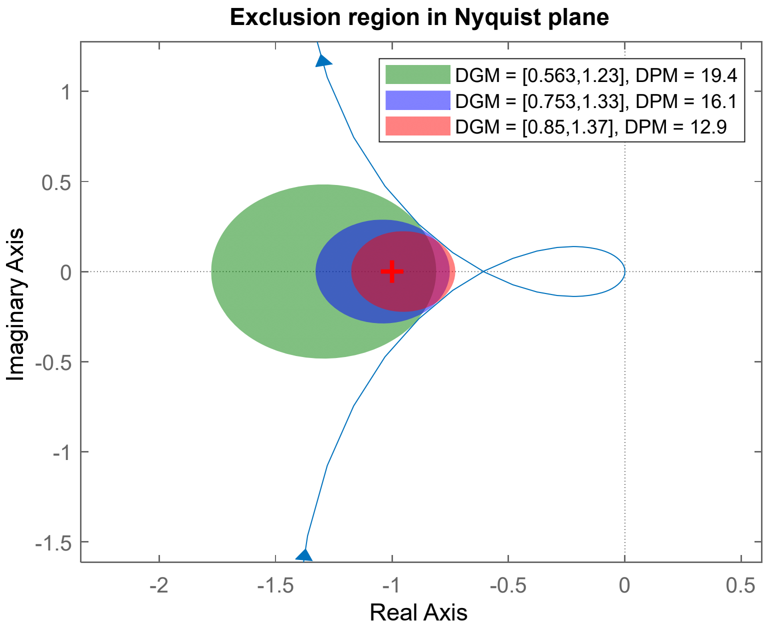

```
  GainMargin: [0.7526 1.3288]
 PhaseMargin: [-16.0714 16.0714]
  DiskMargin: 0.2824
  LowerBound: 0.2824
  UpperBound: 0.2824
   Frequency: 0.2654
```

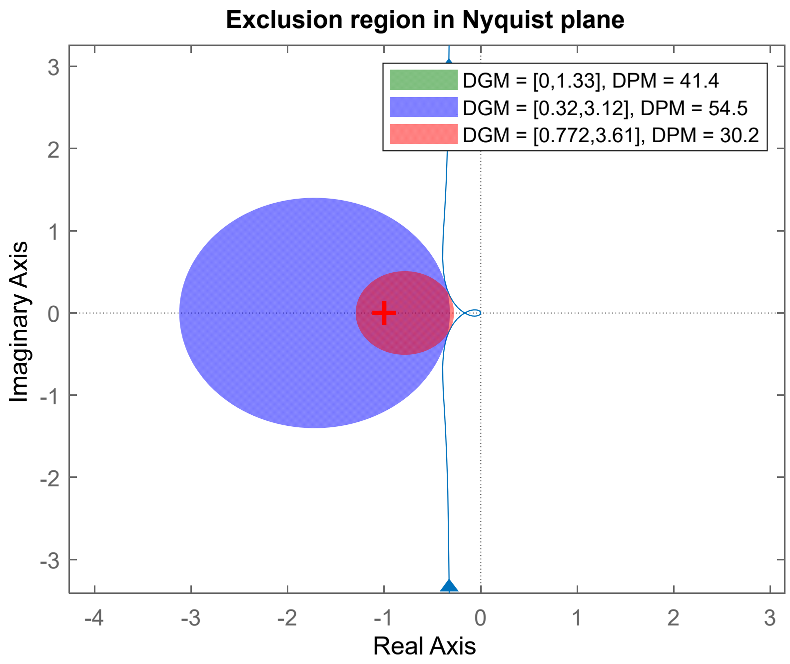



```
  GainMargin: [0.3202 3.1229]
 PhaseMargin: [-54.4881 54.4881]
  DiskMargin: 1.0298
  LowerBound: 1.0298
  UpperBound: 1.0298
   Frequency: 0.1503
```



$$K_{c1} = 1.13 \qquad\qquad K_{c2} = 0.31$$



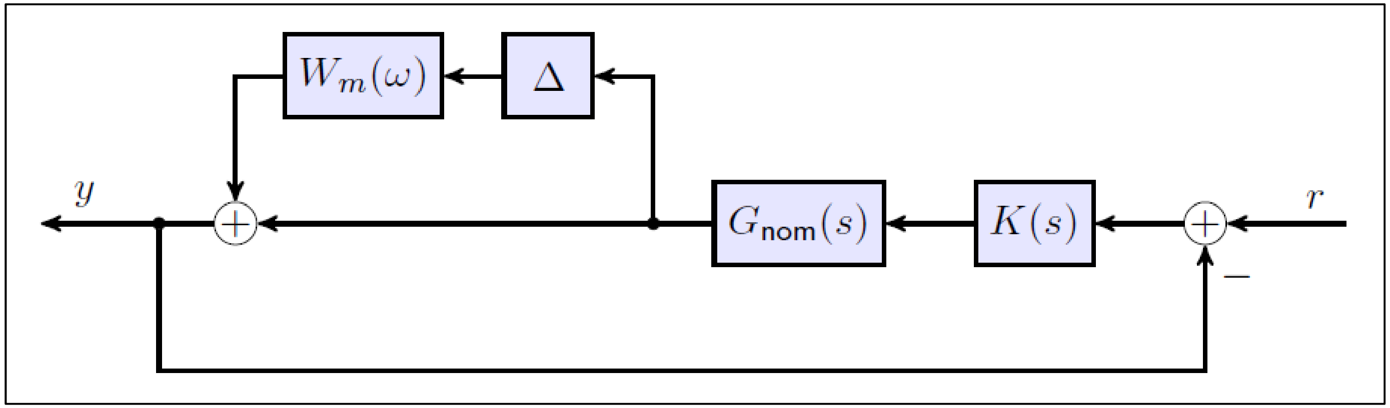


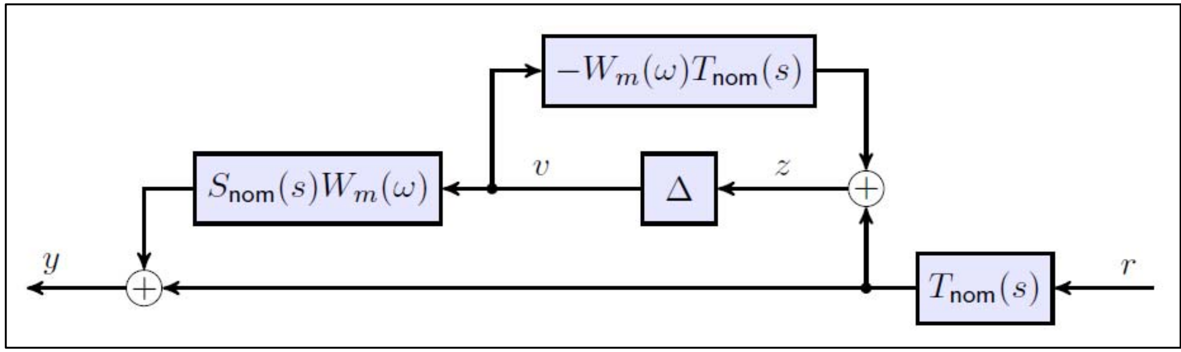
- **Home Problem**: for the example on the previous page (Skogestad 2005, Example 7.6):
    - Verify, the results,
    - Plot the step input responses of the nominal system for both cases
    - Plot the step responses of the perturbed system for both cases

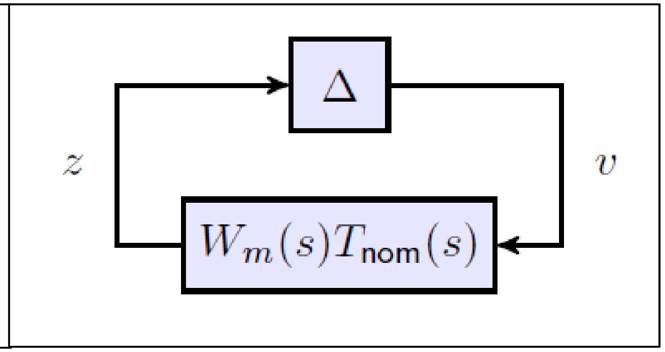
❑ M – Δ structure derivation of RS-condition. This derivation is a preview of a general analysis presented later.

$$|\Delta_I(j\omega)| \le 1, \forall\omega$$

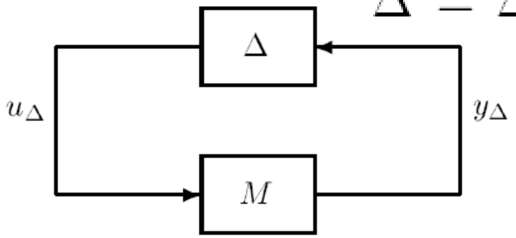
$$\text{RS} \Leftrightarrow |T| < 1/|w_I|, \quad \forall\omega$$







$$\Delta = \Delta_I$$



$M = w_I K(1+GK)^{-1}G = w_I T$ is the transfer function from the output of Δ_I to the input of Δ_I

We assume that Δ and $M = w_I T$ are stable; the former implies that G and G_p must have the same unstable poles, the latter is equivalent to $\boxed{\text{assuming nominal stability}}$ of the closed-loop system. The Nyquist stability condition then determines RS if and only if the "loop transfer function" $M\Delta$ does not encircle -1 for all Δ. Thus,

- Apply Nyquist Criterion to the loop transfer function $M\Delta$ instead of L_P.

$$\text{RS} \quad \Leftrightarrow \quad |1 + M\Delta| > 0, \quad \forall\omega, \forall|\Delta| \leq 1$$

- Consider the worst case:

$$\boxed{\begin{aligned} \text{RS} \quad &\Leftrightarrow \quad 1 - |M(j\omega)| > 0, \quad \forall\omega \\ &\Leftrightarrow \quad |M(j\omega)| < 1, \quad \forall\omega \end{aligned}}$$

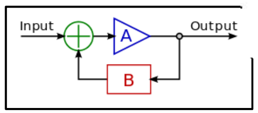
- The **M – Δ** structure provides a very general way of handling robust stability and is essentially a clever application of the small gain theorem.

❑ **Small Gain Theorem (Zhou, p. 212):** Let a nominal system $M(s)$ be internally stable. Moreover, let $\gamma > 0$ (also in the presence of stable weights); then :

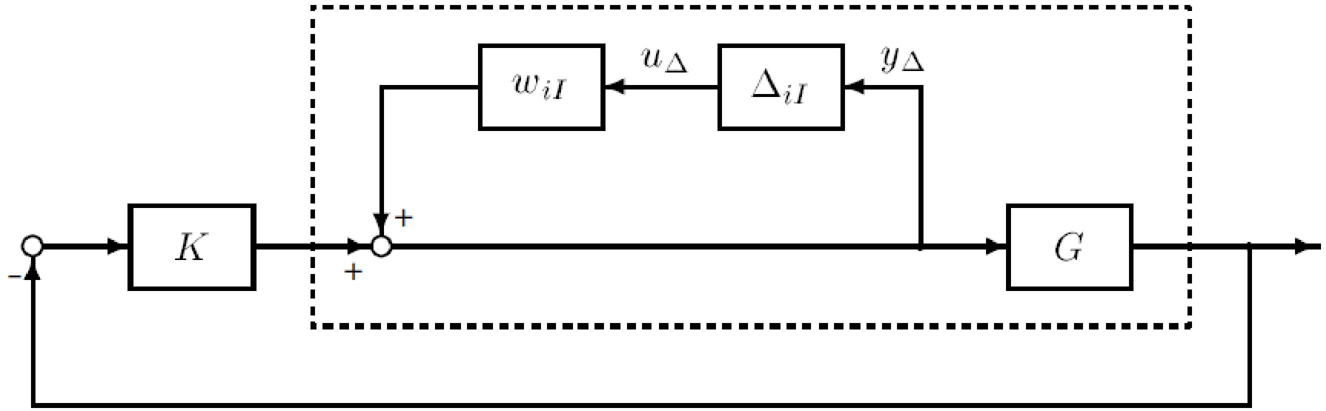
- The perturbed system is well-posed and Internally stable for all stable perturbations for which:

$$(a)\ \|\Delta\|_\infty \leq 1/\gamma \textit{ if and only if } \|M(s)\|_\infty < \gamma;$$

$$(b)\ \|\Delta\|_\infty < 1/\gamma \textit{ if and only if } \|M(s)\|_\infty \leq \gamma.$$



- Consider a system with an **inverse input multiplicative** perturbation from nominal



$$G_p = G(1 + w_{iI}(s)\Delta_{iI})^{-1}$$

- We can derive a corresponding RS-condition for a feedback system with inverse multiplicative uncertainty using the sensitivity function

Chapter 4: Summary

- Assume for simplicity that the loop transfer function $\boldsymbol{L_p}$ is stable, and assume stability of the nominal closed-loop system. Robust stability is then guaranteed if encirclements by $\boldsymbol{L_p}(j\omega)$ of the point -1 are avoided, and since $\boldsymbol{L_p}$ is in a norm-bounded set we have

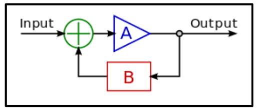
$$\begin{aligned} \text{RS} \quad & \Leftrightarrow \quad |1+L_p| > 0, \quad \forall L_p, \forall\omega \\ & \Leftrightarrow \quad |1+L(1+w_{iI}\Delta_{iI})^{-1}| > 0, \quad \forall|\Delta_{iI}| \leq 1, \forall\omega \\ & \Leftrightarrow \quad |1+w_{iI}\Delta_{iI}+L| > 0, \quad \forall|\Delta_{iI}| \leq 1, \forall\omega \end{aligned}$$

- Consider the worst case with Δ = 1 and in opposition phase:

$$\begin{aligned} \text{RS} \quad & \Leftrightarrow \quad |1+L| - |w_{iI}| > 0, \quad \forall\omega \\ & \Leftrightarrow \quad |w_{iI}S| < 1, \quad \forall\omega \end{aligned}$$

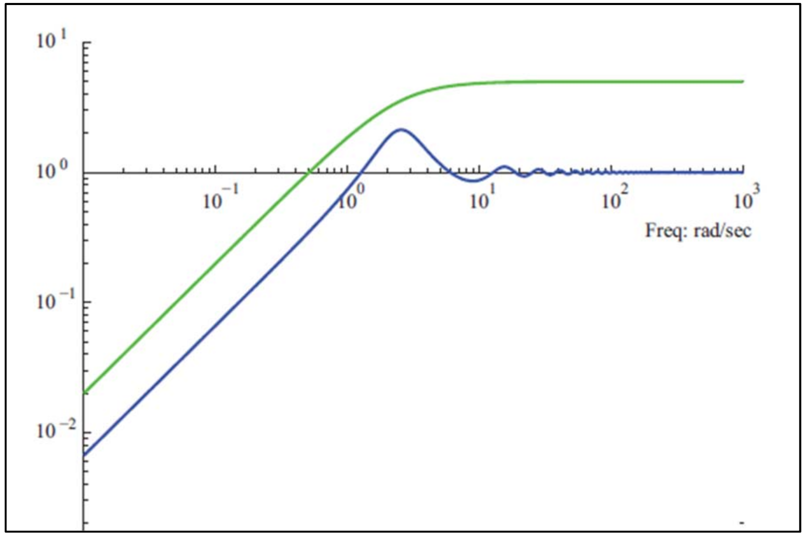
$$\boxed{\text{RS} \quad \Leftrightarrow \quad |S| < 1/|w_{iI}|, \quad \forall\omega}$$

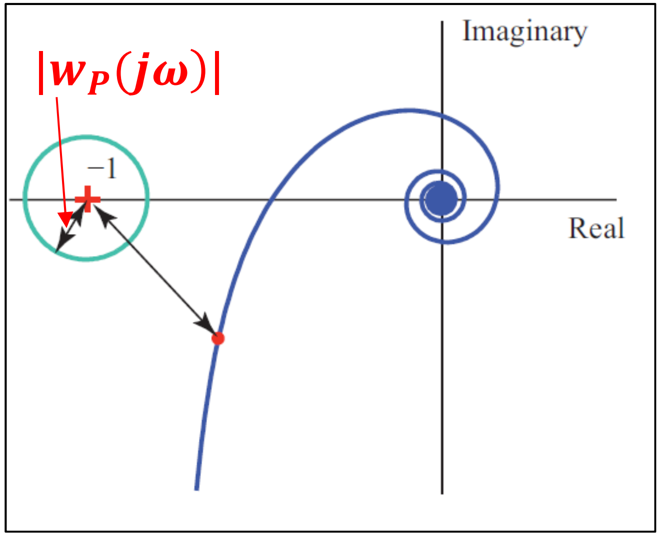
We see that we need tight control and have to make S small at frequencies where the uncertainty is large and $|w_{iI}|$ exceeds 1 in magnitude. This may be somewhat surprising since we intuitively expect to have to detune the system (and make $S \approx 1$) when we have uncertainty, while this condition tells us to do the opposite. The reason is that this uncertainty represents pole uncertainty, and at frequencies where $|w_{iI}|$ exceeds 1 we allow for poles crossing from the left to the right-half plane (G_p becoming unstable), and we then know that we need feedback ($|S| < 1$) in order to stabilize the system.



SISO Robust Performance

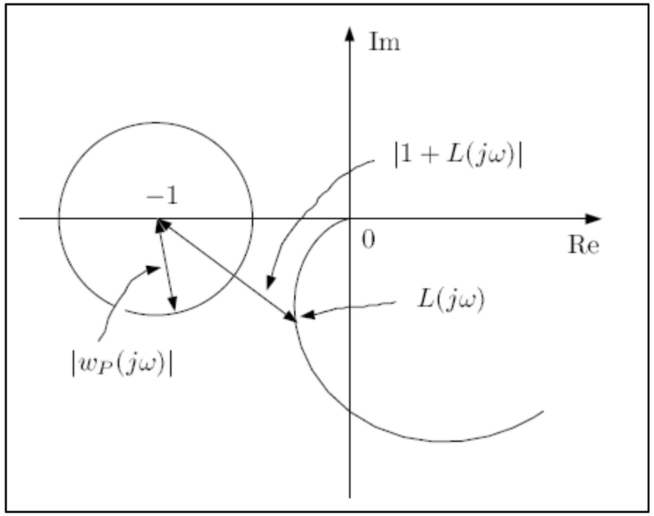
- From nominal performance analysis, we know that the sensitivity function S is a very good indicator both for SISO and MIMO systems.
- The main advantage of considering S is that because we ideally want S small, it is sufficient to consider just its magnitude.





$$\text{NP} \quad \Leftrightarrow \quad |w_P S| < 1 \quad \forall\omega \quad \Leftrightarrow \quad |w_P| < |1 + L| \quad \forall\omega$$

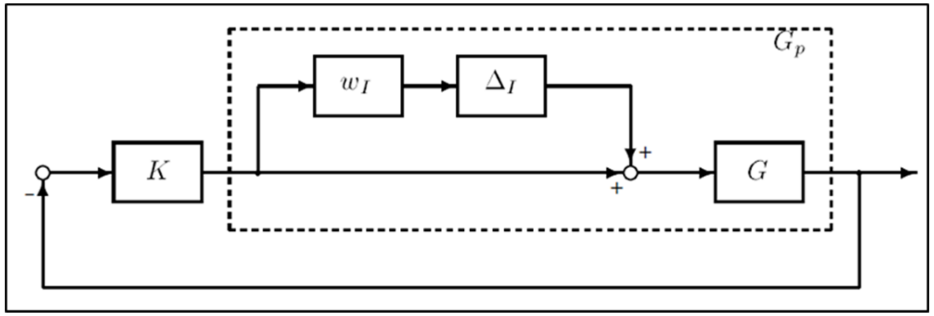
Now $|1 + L|$ represents at each frequency the distance of $L(j\omega)$ from the point -1 in the Nyquist plot, so $L(j\omega)$ must be at least a distance of $|w_P(j\omega)|$ from -1. This is illustrated graphically in Figure where we see that for NP, $L(j\omega)$ must stay outside a disc of radius $|w_P(j\omega)|$ centred on -1.



- For **robust performance** we require that the nominal performance condition to be satisfied for all possible plants, that is, including the worst-case uncertainty.

$$\text{RP} \overset{\text{def}}{\Longleftrightarrow} \quad |w_P S_p| < 1 \quad \forall S_p, \forall \omega$$
$$\Leftrightarrow \quad |w_P| < |1 + L_p| \quad \forall L_p, \forall \omega$$

$$L_p = G_p K = L(1 + w_I \Delta_I) = L + w_I L \Delta_I$$



$$\left|w_P(j\omega)S_P(j\omega)\right| < 1, S_P = \frac{1}{1+L_P} = \frac{1}{1+G_P K}$$

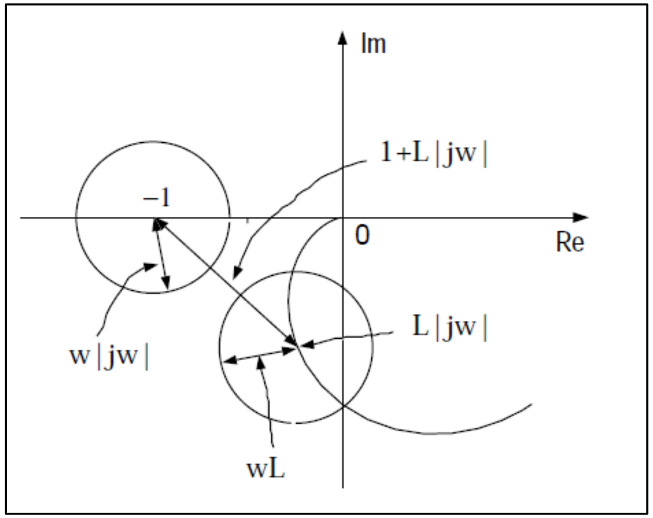
$$\left|w_P\right| < \left|1 + L_P\right| = \left|1 + [1 + w_I(j\omega)\Delta_I]GK\right|, \left|\Delta_I\right| \le 1$$

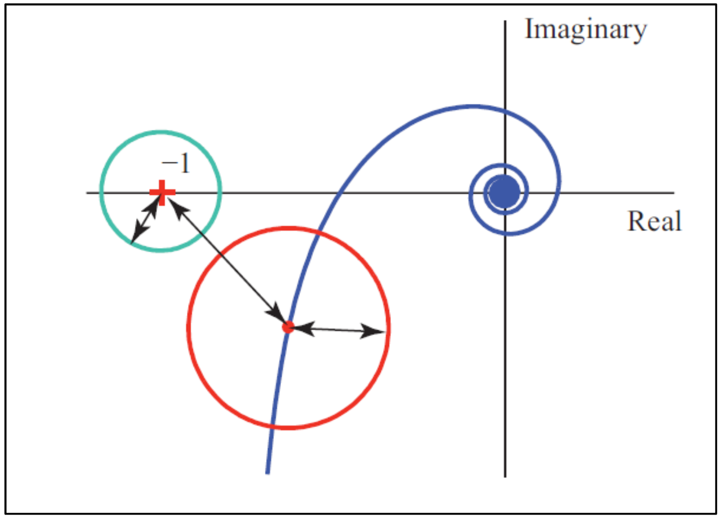
$$\left|w_P\right| < \left|1 + L + w_I \Delta_I L\right| \Rightarrow \left|1 + L\right| - \left|w_I L\right|$$

- the worst-case (maximum) is obtained at each frequency by selecting $|\Delta_I|=1$ such that the terms $(1 + L)$ and $w_I(j\omega)$ (which are complex numbers) point in opposite directions.

$$|w_P| + |w_I L| < |1 + L|$$

$$\boxed{RP \Rightarrow \max_{\omega} \left\{ |w_P S| + |w_I T| \right\} < 1}$$



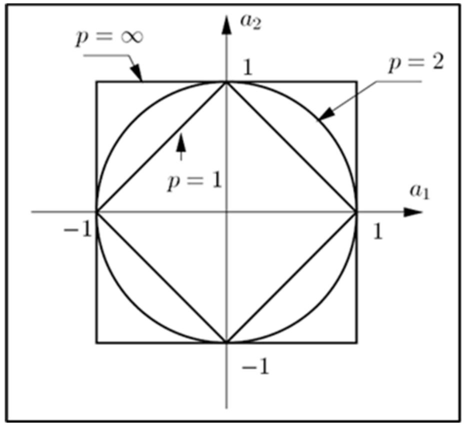


For RP we must require that all possible $L_p(j\omega)$ stay outside a disc of radius $|w_P(j\omega)|$ centred on -1. Since L_p at each frequency stays within a disc of radius $w_I L$ centred on L, we see from Figure that the condition for RP is that the two discs, with radii $|w_P|$ and $|w_I L|$, do not overlap. Since their centres are located a distance $|1 + L|$ apart, the RP-condition becomes

1. The Robust Performance requirement is similar to vector $\mathcal{H}_\infty$ computation:

$$RP \Rightarrow \max_{\omega} \left\{ \left| w_P S \right| + \left| w_I T \right| \right\} < 1 \quad \Longleftrightarrow \quad \left\| \begin{matrix} w_P S \\ w_I T \end{matrix} \right\|_\infty = \max_{\omega} \sqrt{|w_P S|^2 + |w_I T|^2} < 1$$

$$|w_P S| + |w_I T| \leq 2 \max\{|w_P S|, |w_I T|\} < 2$$



Contours for the vector p-norm,

$$\|a\|_p = 1 \text{ for } p = 1, 2, \infty$$

2. The Robust Performance condition can be used to derive bounds on the loop transfer function shape $|\boldsymbol{L}|$

$$RP \Rightarrow \max_{\omega}\left\{\left|w_P S\right| + \left|w_I T\right|\right\} < 1 \quad \Longleftrightarrow \quad \begin{array}{ll} |L| > \dfrac{1+|w_P|}{1-|w_I|}, & \text{(at frequencies where } |w_I| < 1) \\ |L| < \dfrac{1-|w_P|}{1+|w_I|}, & \text{(at frequencies where } |w_P| < 1) \end{array}$$

- The first condition is most useful at low frequencies where generally $|w_I|<1$ and $|w_P|>1$, and $|L|$ needs to be large for tight performance.
- Conversely, the second condition is most useful at high frequencies where generally $|w_I|>1$ (more than 100% uncertainty)and $|w_P|<1$, and $|L|$ needs to be small.

3. The Robust Performance condition is related to the <u>structured singular value</u> (SSV) μ. For a given matrix $N_{\mathbf{RP}}$ (see later), we can define:

$$\mu(N_{\mathrm{RP}}) = |w_P S| + |w_I T|$$

- **SSV for the present problem**

$$\mu^S = \max_{S_P} |w_P S_P| = \frac{|w_P S|}{1 - |w_I T|}$$

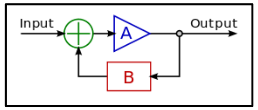
- **Skewed-SSV for the present problem, worst-case weighted sensitivity**

Note that μ and μ^s are closely related since $\mu \leq 1$ if and only if $\mu^s \leq 1$.

❑ Relationship between NP, RS, and RP

- Consider a nominally stable (NS) system with multiplicative uncertainty:

NP	$\Leftrightarrow$	$\lvert w_P S \rvert < 1, \forall \omega$
RS	$\Leftrightarrow$	$\lvert w_I T \rvert < 1, \forall \omega$
RP	$\Leftrightarrow$	$\lvert w_P S \rvert + \lvert w_I T \rvert < 1, \forall \omega$

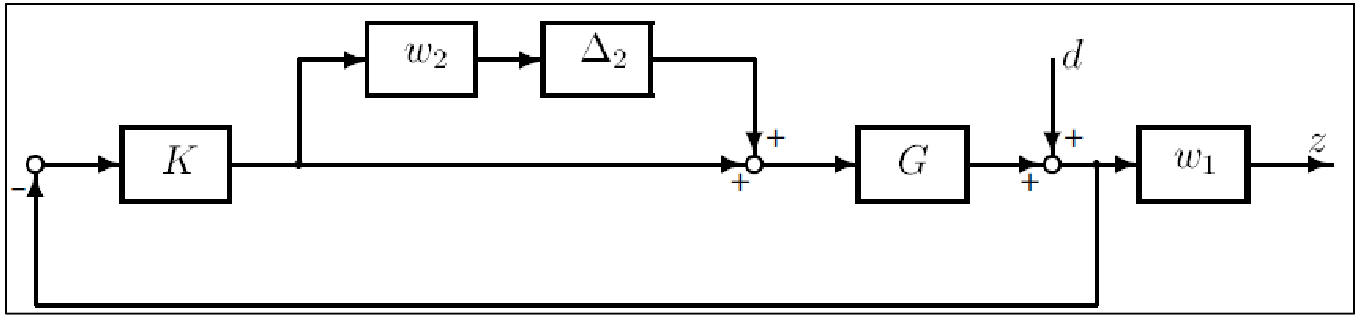


- From this we see that a prerequisite for RP is that we satisfy NP and RS

- **Robust performance may be viewed as a special case of robust stability (with multiple perturbations).**

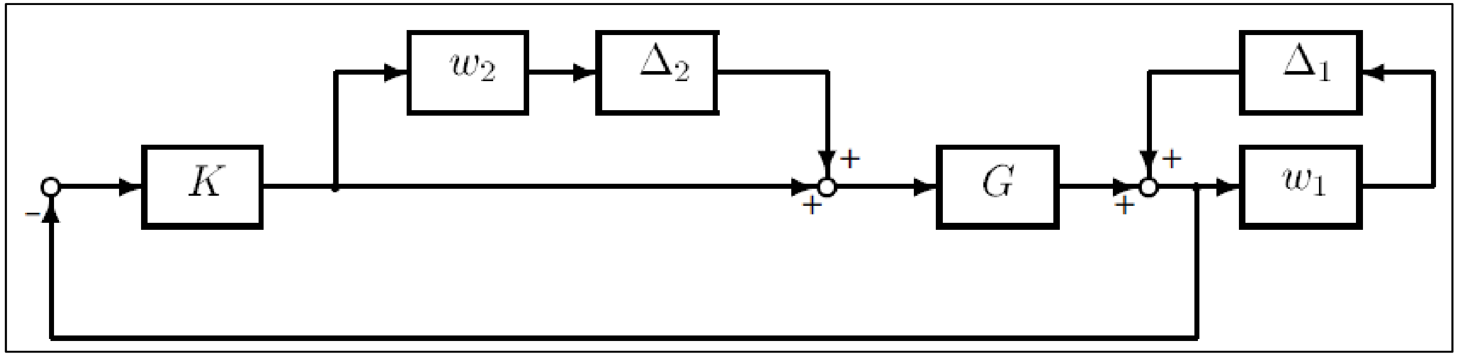
- Consider the problem of RP in the presence of an input multiplicative uncertainty (w_2 replaces w_I and w_1 replaces w_P).

- Assumed uncertainty $w_2(j\omega)\Delta_2, |\Delta_2| \le 1$
- Assumed performance req. $w_1(j\omega)\Delta_1, |\Delta_1| \le 1$



- Now consider the RS with combined multiplicative and inverse multiplicative uncertainty:

$$w_2(j\omega)\Delta_2, |\Delta_2| \leq 1 \qquad w_1(j\omega)\Delta_1, |\Delta_1| \leq 1$$



- Since we use the $\mathcal{H}_\infty$ norm to define both uncertainty and performance and since the weights are the same in both cases, the test for RP for the first system is the same as the test for RS for the second system.

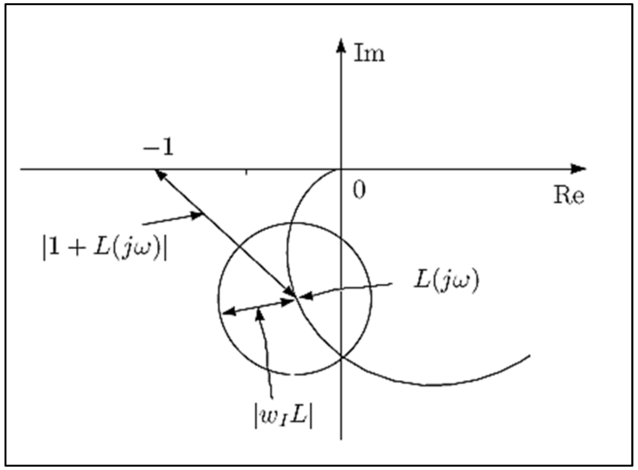
- Recall the RP condition

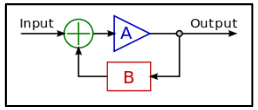
$$\mathrm{RP} \quad \Leftrightarrow \quad |w_1 S| + |w_2 T| < 1, \quad \forall \omega$$

with w_1 replaced by w_P and with w_2 replaced by w_I.

- Now derive the RS condition for the case where L_P is stable (second diagram). RS is equivalent to avoiding encirclements of the critical point -1 by the Nyquist plot of L_P.

$$\mathrm{RS} \quad \Leftrightarrow \quad |1 + L_p| > 0 \quad \forall L_p, \forall \omega$$





$$
\begin{aligned}
&\Leftrightarrow \quad |1 + L(1 + w_2\Delta_2)(1 - w_1\Delta_1)^{-1}| > 0,\ \forall\Delta_1, \forall\Delta_2, \forall\omega \\
&\Leftrightarrow \quad |1 + L + Lw_2\Delta_2 - w_1\Delta_1| > 0,\ \forall\Delta_1, \forall\Delta_2, \forall\omega
\end{aligned}
$$

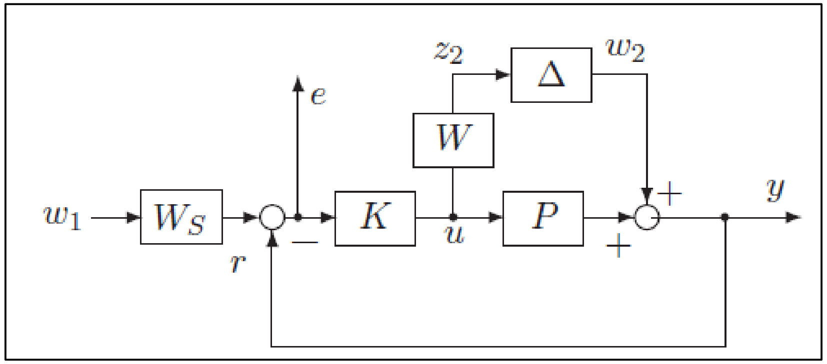
- Here the worst case is obtained when we choose Δ_1 and Δ_2 with magnitudes 1 such that the terms $Lw_2\Delta_2$ and $w_1\Delta_1$ are in the opposite direction of the term $1 + L$. We get

$$
\begin{aligned}
\text{RS} \quad &\Leftrightarrow \quad |1 + L| - |Lw_2| - |w_1| > 0, \quad \forall\omega \\
&\Leftrightarrow \quad |w_1 S| + |w_2 T| < 1, \quad \forall\omega
\end{aligned}
$$

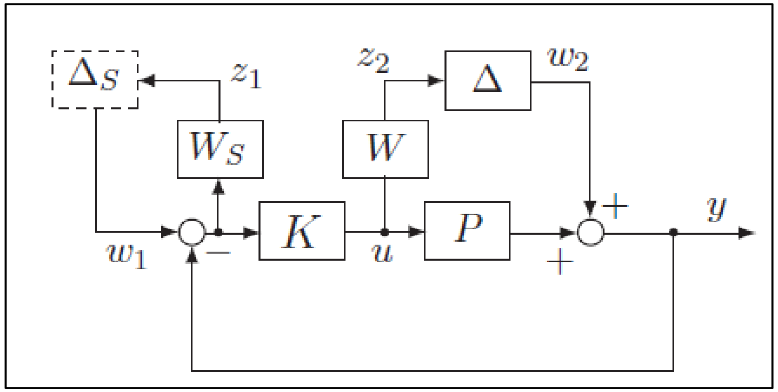
- **Conclusion:** The RP of a system with input multiplicative uncertainty $w_P\Delta_P$ can be tested by adding a fictitious uncertainty $w_I\Delta_I$ and evaluating the RS of the resulting system

- Graphically 1:

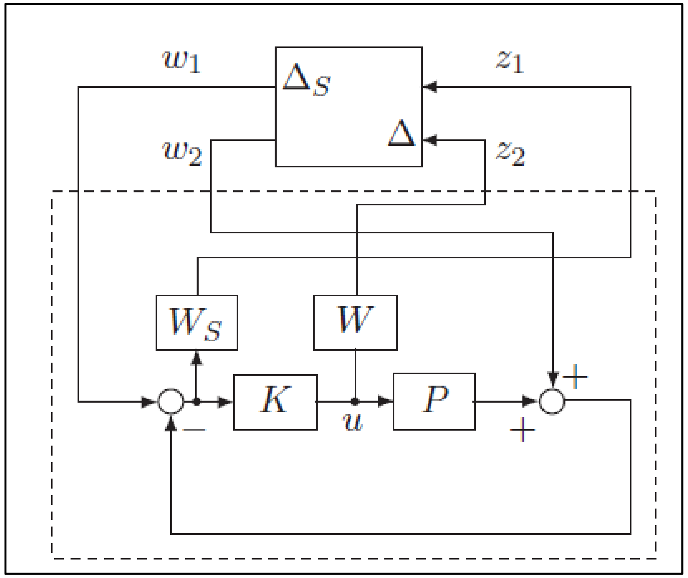
Want RP



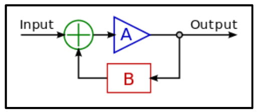
Introduce fictitious unstructured Δ_S



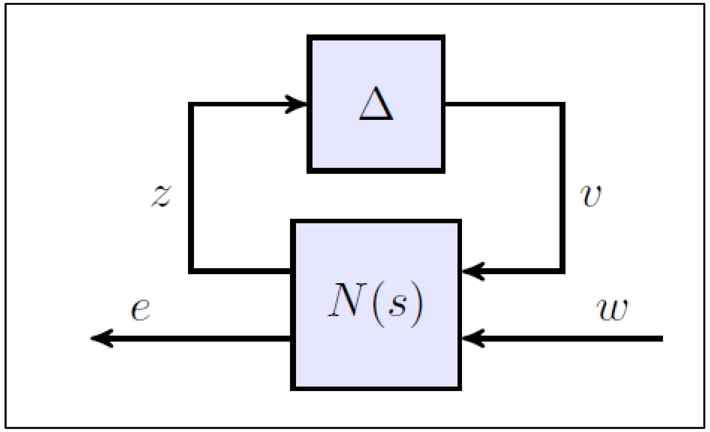
Check RS



$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \Delta_S & \\ & \Delta \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$



- Graphically 2:

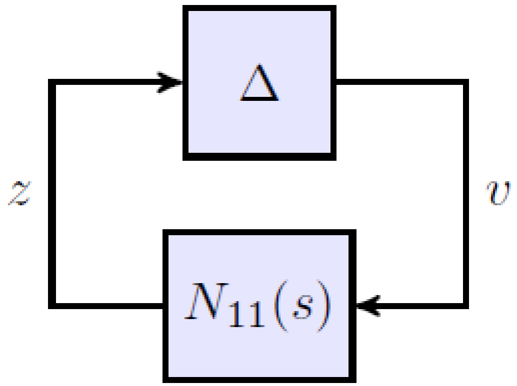


$$N(s) = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix}$$

$$\Delta = \begin{bmatrix} \Delta_1 & & 0 \\ & \ddots & \\ 0 & & \Delta_m \end{bmatrix}$$

$$\begin{aligned} e &= \left[N_{21}(s) \left(I - N_{11}(s) \Delta \right)^{-1} \Delta N_{12}(s) + N_{22}(s) \right] w \\ &= \mathcal{F}_u \left(N(s), \Delta \right) \, w \end{aligned}$$

❑ Robust Stability



Is the interconnection stable for given Δ?

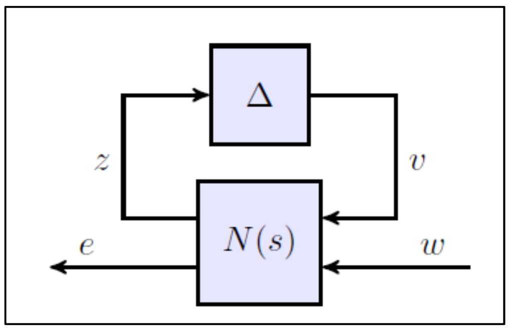
Δ is destabilizing $\iff$ $\det(I - N_{11}(j\omega)\Delta(j\omega))$ encircles 0.

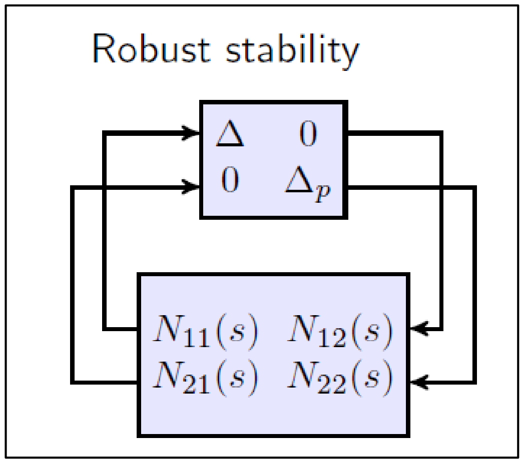
What is the smallest destabilising Δ?

$$\min_{\Delta \in \boldsymbol{\Delta}} \overline{\sigma}(\Delta) \text{ such that } \det(I - N_{11}(j\omega))\Delta(j\omega)) = 0$$

$$\boxed{\mu_{\boldsymbol{\Delta}}(N_{11}(j\omega)) < 1 \quad \text{for all } \omega.}$$

❑ Robust Performance

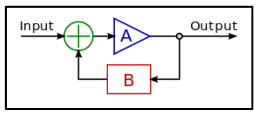




$$\boxed{\mu_{\tilde{\boldsymbol{\Delta}}}(N(j\omega)) < 1}$$

$$\tilde{\boldsymbol{\Delta}} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_p \end{bmatrix}$$

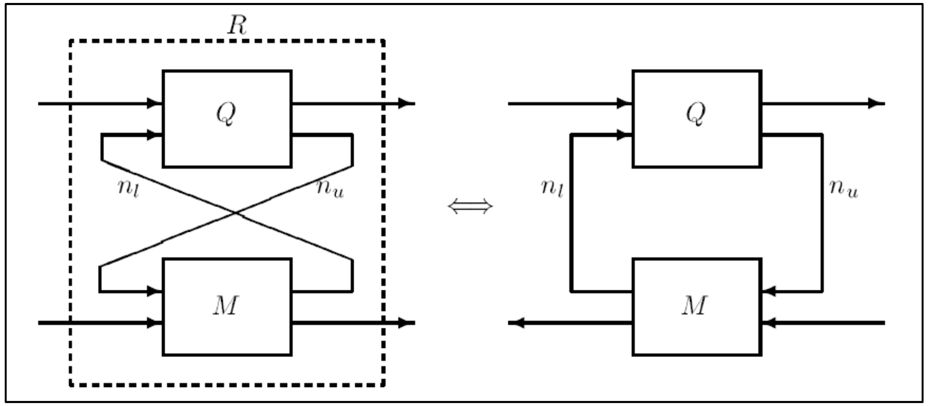
$$\Delta \in \boldsymbol{\Delta}, \Delta_p \in \mathcal{C}^{\dim(w)\times\dim(e)}$$



- Implementation of 2 block format uses RedHeffer star product

Q and M are interconnected such that the last n_u outputs from Q are the first n_u inputs of M, and the first n_l outputs from M are the last n_l inputs of Q. The corresponding partitioned matrices are

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \quad M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$



$$R = \mathcal{S}(Q, M) =$$

$$\begin{bmatrix} Q_{11} + Q_{12}M_{11}(I - Q_{22}M_{11})^{-1}Q_{21} & Q_{12}(I - M_{11}Q_{22})^{-1}M_{12} \\ M_{21}(I - Q_{22}M_{11})^{-1}Q_{21} & M_{22} + M_{21}Q_{22}(I - M_{11}Q_{22})^{-1}M_{12} \end{bmatrix}$$

- Additional References

ROBUST CONTROL
THEORY AND APPLICATIONS
Kang-Zhi Liu
Professor, Chiba University, Japan
Yu Yao
Professor, Harbin Institute of Technology, China

Chapters 12 and 17