Some critical Issues in LQR Control Implementation:

1. **Complete State** $x(t)$ availability is necessary for full state feedback. This requires a number of independent sensors equal to the system's dimensions (complexity). If the number of sensors is lower, then the remaining states must be reconstructed (estimated) by an additional dynamic system, requiring observability and/or detectability.
2. The loop may have signals/inputs that are not deterministic but stochastic representing a uncertainty in the process or inherent noise due to sensor dynamics.

- **Question:** Can we design a **dynamic compensator**, which is optimal (in a quadratic sense)?

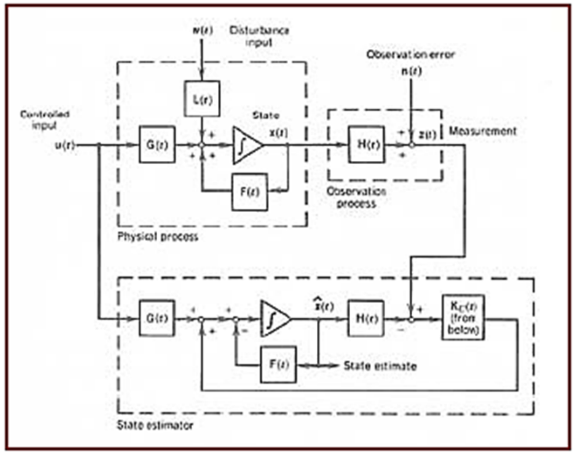
- **Answer 1:** From a controller standpoint the answer is **YES**, since we could design a LQR (constant feedback gain matrix, guaranteed stability, good stability margins,...)

- **Answer 2:** From the state estimation standpoint the answer is **NO**, since we only have a Luenberger observer, which is guaranteed to be stable but not optimal

LQG Control

- The answer to the second question is the development of an **optimal state estimator** (under some conditions) called **Kalman Filter**, and the use of the separation principle to connect it with an optimal LQR controller leading to the so-called **Linear Quadratic Gaussian (LQG)** compensator.

http://www.stengel.mycpanel.princeton.edu/MAE546Seminars.html



- **Although derived for a specific estimation problem, the Kalman Filter and its derivatives, is probably the most commonly used estimator in industry**
- **The Kalman filter, the linear-quadratic regulator, and the linear–quadratic–Gaussian controller are solutions to what arguably are among the most fundamental problems in control theory.**

Kalman Filter and Linear Quadratic Gaussian Optimal Controller – LQG

Kalman and Bucy (1977)

Observers for Multivariable Systems

D. G. LUENBERGER, MEMBER, IEEE

Abstract—Often in control design it is necessary to construct estimates of state variables which are not available by direct measurement. If a system is linear, its state vector can be approximately reconstructed by building an observer which is itself a linear system driven by the available outputs and inputs of the original system. The state vector of an nth order system with m independent outputs can be reconstructed with an observer of order $n-m$.

In this paper it is shown that the design of an observer for a system with M outputs can be reduced to the design of m separate observers for single-output subsystems. This result is a consequence of a special canonical form developed in the paper for multiple-output systems.

In the special case of reconstruction of a single linear functional of the unknown state vector, it is shown that a great reduction in observer complexity is often possible.

Finally, the application of observers to control design is investigated. It is shown that an observer's estimate of the system state vector can be used in place of the actual state vector in linear or nonlinear feedback designs without loss of stability.

R. E. KALMAN
Research Institute for Advanced Study,[2]
Baltimore, Md.

A New Approach to Linear Filtering and Prediction Problems[1]

The classical filtering and prediction problem is re-examined using the Bode-Shannon representation of random processes and the "state transition" method of analysis of dynamic systems. New results are:

(1) The formulation and methods of solution of the problem apply without modification to stationary and nonstationary statistics and to growing-memory and infinite-memory filters.

(2) A nonlinear difference (or differential) equation is derived for the covariance matrix of the optimal estimation error. From the solution of this equation the coefficients of the difference (or differential) equation of the optimal linear filter are obtained without further calculations.

(3) The filtering problem is shown to be the dual of the noise-free regulator problem.

The new method developed here is applied to two well-known problems, confirming and extending earlier results.

The discussion is largely self-contained and proceeds from first principles; basic concepts of the theory of random processes are reviewed in the Appendix.

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LQG Control

- Optimal State Estimation in a Quadratic Sense.

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} + \boldsymbol{w} \\ \boldsymbol{y} = C\boldsymbol{x} + \boldsymbol{v} \end{cases} \qquad \begin{cases} \boldsymbol{w}(t) \Rightarrow [0, W] \\ \boldsymbol{v}(t) \Rightarrow [0, V] \\ \boldsymbol{x}_0(t) \Rightarrow [\boldsymbol{\eta}_0, Q_0] \end{cases}$$

Markov Process

- $\boldsymbol{w}(t),\ \boldsymbol{v}(t)$ are wide sense stationary, white noise, uncorrelated, ergodic processes

$$W \geq 0, V > 0$$

- are symmetric constant matrices, called the *intensity matrices* of the two white noise processes (indicating the variance). The initial state is assumed to be a random vector.

- We wish to find:

$$\dot{\hat{\boldsymbol{x}}} = A\hat{\boldsymbol{x}} + B\boldsymbol{u} + K_F(\boldsymbol{y} - C\hat{\boldsymbol{x}})$$

$$\hat{\boldsymbol{x}}(t_0) = \hat{\boldsymbol{x}}_0$$

$$\min_{K_F, x_0} J = \min_{K_F, x_0} E\left\{\boldsymbol{e}^T Z \boldsymbol{e}\right\} \qquad \begin{cases} \boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}} \\ Z > 0 \end{cases}$$

Find: $\boxed{K_F, \hat{\boldsymbol{x}}_0}$

- **Note: Properties of E [.]**

$$\begin{cases} E\left\{\boldsymbol{x}^T(t)Q(t)\boldsymbol{x}(t)\right\} = tr\left[Q(t)C_x(t,t)\right] > 0 \\ \boldsymbol{x}(t) \Rightarrow stochastic\ process \\ Q(t) > 0 \end{cases}$$

$$C_x(t,t) = E\left\{\boldsymbol{x}^T(t)\boldsymbol{x}(t)\right\}$$

$$E\left\{\boldsymbol{x}^T Q \boldsymbol{x}\right\} = tr\left[QR_x(0)\right]$$

$$E\left\{\boldsymbol{x}^T A \boldsymbol{x}\right\} = tr\left[A\int_{-\infty}^{+\infty} \Sigma_x(\omega) df\right]$$

White Noise: a random process $x(t)$ with the same intensity at different frequency and a constant power spectral density
Wide Sense Stationary: a random process with mean independent of time $\mathrm{E}[x(t)] = \mu$ and autocorrelation function of time difference only $\mathrm{E}[x(t_1)\ x^T(t_2)] = \mathrm{E}[x(t)\ x^T(t)]$, $t = t_2 - t_1$
Ergodic: a random process is ergodic if its statistics are completely determined by a sufficiently long data sample

- **Markov Process**

Linear system driven by white noise Consider the stable linear system $\dot{x}(t) = Ax(t) + v(t)$, driven by white noise with intensity V, that is, $Ev(t)v^{\mathrm{T}}(s) = V\delta(t-s)$. The state of the system

$$x(t) = \int_{-\infty}^{t} \mathrm{e}^{A(t-s)} v(s)\, ds, \qquad t \in \mathbb{R}, \tag{4.196}$$

is a stationary stochastic process with covariance matrix

$$\begin{aligned} Y &= Ex(t)x^{\mathrm{T}}(t) = \int_{-\infty}^{t}\int_{-\infty}^{t} \mathrm{e}^{A(t-s_1)}\left(Ev(s_1)v^{\mathrm{T}}(s_2)\right)\mathrm{e}^{A^{\mathrm{T}}(t-s_2)}\, ds_1 ds_2 \\ &= \int_{-\infty}^{t} \mathrm{e}^{A(t-s)} V \mathrm{e}^{A^{\mathrm{T}}(t-s)}\, ds = \int_{0}^{\infty} \mathrm{e}^{A\tau} V \mathrm{e}^{A^{\mathrm{T}}\tau}\, d\tau. \end{aligned} \tag{4.197}$$

It follows that

$$\begin{aligned} AY + YA^{\mathrm{T}} &= \int_0^{\infty} \left(A\mathrm{e}^{A\tau} V \mathrm{e}^{A^{\mathrm{T}}\tau} + \mathrm{e}^{A\tau} V \mathrm{e}^{A^{\mathrm{T}}\tau} A^{\mathrm{T}}\right) d\tau \\ &= \int_0^{\infty} \frac{d}{d\tau}\left(\mathrm{e}^{A\tau} V \mathrm{e}^{A^{\mathrm{T}}\tau}\right) d\tau = \mathrm{e}^{A\tau} V \mathrm{e}^{A^{\mathrm{T}}\tau}\Big|_0^{\infty} = -V. \end{aligned} \tag{4.198}$$

Hence, the covariance matrix Y is the unique solution of the Lyapunov equation

$$AY + YA^{\mathrm{T}} + V = 0. \tag{4.199}$$

- **NOTE: The above result requires that A be Hurwitz. [A < 0]**

LQG Control

- Insert the Error Dynamics to modify the Performance Index to be minimized.

$$\begin{cases} \dot{\boldsymbol{e}} = (A - K_F C)\boldsymbol{e} + \boldsymbol{w} - K_F \boldsymbol{v} \\ \boldsymbol{e}(t_0) = \boldsymbol{x}_0 - \hat{\boldsymbol{x}}_0 = \boldsymbol{e}_0 \end{cases}$$

$$\boldsymbol{e}(t) = (\boldsymbol{\eta}_e(t), \tilde{Q}(t)) \quad E\left\{\boldsymbol{e}(t)\right\} = \boldsymbol{\eta}_e(t)$$

$$\tilde{Q}(t) = E\left\{\left[\boldsymbol{e} - \boldsymbol{\eta}_e\right] \left[\boldsymbol{e} - \boldsymbol{\eta}_e\right]^T\right\}$$

$$\downarrow$$

$$E\left\{\boldsymbol{e}\boldsymbol{e}^T\right\} = \tilde{Q}(t) + \boldsymbol{\eta}_e \boldsymbol{\eta}_e^{\ T}$$

$$J = E\left\{\boldsymbol{e}^T Z \boldsymbol{e}\right\} = \boldsymbol{\eta}_e^T Z \boldsymbol{\eta}_e + tr\left[\tilde{Q}(t) Z\right]$$

- Minimize each term separately

- Minimization of $\boldsymbol{\eta}_e^T Z \boldsymbol{\eta}_e$

$$\min\left(\boldsymbol{\eta}_e^T Z \boldsymbol{\eta}_e\right) \Rightarrow \boldsymbol{\eta}_e(t) = 0$$

$$\dot{\boldsymbol{\eta}}_e = (A - K_F C)\boldsymbol{\eta}_e$$

$$\boldsymbol{\eta}_e(t) = 0 \Rightarrow \boldsymbol{\eta}_e(t_0) = 0$$

$$E\left\{\hat{\boldsymbol{x}}_0 - \boldsymbol{x}_0\right\} = 0 \quad E\left\{\hat{\boldsymbol{x}}_0\right\} = E\left\{\boldsymbol{x}_0\right\}$$

- Minimization of $tr\left[\tilde{Q} Z\right]$

$$\tilde{Q}(t) \geq Q(t)$$

$$\begin{cases} \dot{Q} = AQ + QA^T + W - QC^T V^{-1} C Q \\ Q(t_0) = Q_0 \end{cases}$$

$$\tilde{Q}(t) \equiv Q(t) \Leftrightarrow K_F = Q C^T V^{-1}$$

LQG Control

- **Lemma 3.1.** 'Linear Optimal Control', Kwakernaak& Sivan, p. 218. Given the matrix differential equation:

$$\begin{cases} -\dot{\tilde{Q}} = -R_1 + K^T R_2 K + \tilde{Q}(A - BK) + (A - BK)^T \tilde{Q} \\ \tilde{Q}(t_1) = Q_1 \geq 0; R_1 \geq 0; R_2 > 0 \end{cases}$$

Given an arbitrary continuous matrix $K(t)$. Then for $t_0 \leq t \leq t_1$,

$$\tilde{Q}(t) \geq Q(t)$$

With $Q(t)$ solution of the matrix differential equation:

$$\begin{cases} \dot{Q} = AQ + QA^T + W - QC^T V^{-1} CQ \\ Q(t_0) = Q_0 \end{cases}$$

The equality holds if : $K = QC^T V^{-1}$

- For a controllable system, and infinite horizon optimization

$$Q(t) \Rightarrow Q > 0\,(const)$$

$$AQ + QA^T + W - QC^T V^{-1} CQ = 0$$

- KF Riccati equation (ARE)

LQG Control

- Proof of $Q(t) \le \tilde{Q}(t) \quad \tilde{Q}(t) \equiv Q(t) \Leftrightarrow K_F = QC^T V^{-1}$

Theorem 3.4. *The optimal input for the deterministic optimal linear regulator is generated by the linear control law*

$$u^0(t) = -F^0(t)x^0(t), \qquad \text{3-128}$$

where

$$F^0(t) = R_2^{-1}(t)B^T(t)P(t). \qquad \text{3-129}$$

Here the symmetric nonnegative-definite matrix $P(t)$ satisfies the matrix Riccati equation

$$-\dot{P}(t) = R_1(t) - P(t)B(t)R_2^{-1}(t)B^T(t)P(t) + P(t)A(t) + A^T(t)P(t), \qquad \text{3-130}$$

with the terminal condition

$$P(t_1) = P_1, \qquad \text{3-131}$$

and where

$$R_1(t) = D^T(t)R_3(t)D(t).$$

For the optimal solution we have

$$\int_t^{t_1} [x^{0T}(\tau)R_1(\tau)x^0(\tau) + u^{0T}(\tau)R_2(\tau)u^0(\tau)]\, d\tau + x^{0T}(t_1)P_1x^0(t_1)$$
$$= x^{0T}(t)P(t)x^0(t), \qquad t \le t_1. \qquad \text{3-132}$$

We see that the matrix $P(t)$ not only gives us the optimal feedback law but also allows us to evaluate the value of the criterion for any given initial state and initial time.

From the derivation of this section, we extract the following result (Wonham, 1968a), which will be useful when we consider the stochastic linear optimal regulator problem and the optimal observer problem.

Lemma 3.1. *Consider the matrix differential equation*

$$-\dot{\bar{P}}(t) = R_1(t) + F^T(t)R_2(t)F(t) + \bar{P}(t)[A(t) - B(t)F(t)]$$
$$+ [A(t) - B(t)F(t)]^T\bar{P}(t), \qquad \text{3-133}$$

with the terminal condition

$$\bar{P}(t_1) = P_1, \qquad \text{3-134}$$

where $R_1(t)$, $R_2(t)$, $A(t)$ and $B(t)$ are given time-varying matrices of appropriate dimensions, with $R_1(t)$ nonnegative-definite and $R_2(t)$ positive-definite for $t_0 \le t \le t_1$, and P_1 nonnegative-definite. Let $F(t)$ be an arbitrary continuous matrix function for $t_0 \le t \le t_1$. Then for $t_0 \le t \le t_1$

$$\bar{P}(t) \ge P(t), \qquad \text{3-135}$$

where $P(t)$ is the solution of the matrix Riccati equation

$$-\dot{P}(t) = R_1(t) - P(t)B(t)R_2^{-1}(t)B^T(t)P(t) + P(t)A(t) + A^T(t)P(t), \qquad \text{3-136}$$

$$P(t_1) = P_1. \qquad \text{3-137}$$

The inequality **3-135** *converts into an equality if*

$$F(\tau) = R_2^{-1}(\tau)B^T(\tau)P(\tau) \qquad \text{for } t \le \tau \le t_1. \qquad \text{3-138}$$

The lemma asserts that $\bar{P}(t)$ is "minimized" in the sense stated in **3-135** by choosing F as indicated in **3-138**. The proof is simple. The quantity

$$x^T(t)\bar{P}(t)x(t) \qquad \text{3-139}$$

is the value of the criterion **3-121** if the system is controlled with the arbitrary linear control law

$$u(\tau) = -F(\tau)x(\tau), \qquad t \le \tau \le t_1. \qquad \text{3-140}$$

The optimal control law, which happens to be linear and is therefore also the best linear control law, yields $x^T(t)P(t)x(t)$ for the criterion (Theorem 3.4), so that

$$x^T(t)\bar{P}(t)x(t) \ge x^T(t)P(t)x(t) \qquad \text{for all } x(t). \qquad \text{3-141}$$

This proves **3-135**.

...f the solution

Summary

Given the system:

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} + \boldsymbol{w} \\ \boldsymbol{y} = C\boldsymbol{x} + \boldsymbol{v} \end{cases}$$

$w = [0, W]$ uncorrelated, wide sense stationary White Noise

$v = [0, V]$ uncorrelated, wide sense stationary White Noise

$\boldsymbol{x}(t_0) = [\boldsymbol{\eta}(t_0), Q(t_0)]$ uncorrelated, wide sense stationary

for $Q \geq 0$ the following holds:

$$\begin{cases} \dot{Q} = AQ + QA^T + W - QC^TV^{-1}CQ \\ Q(t_0) = Q_0 \end{cases}$$

- a. Let: $Q_0 = 0$, for $t \to -\infty$

  the above solution tends to a constant matrix

  $$\bar{Q} > 0$$

  If and only if there are no poles that are simultaneously unstable, unobservable, and controllable;

- b. If the system is stabilizable and detectable, the solution tends to:

  $\bar{Q}$ for $t \to -\infty$ $\forall Q_0 \geq 0$

- c. If $\bar{Q}$ exists, it is $\geq$ 0 and solution of

  $$\boxed{AQ + QA^T + W - QC^TV^{-1}CQ = 0}$$

  (unique solution in case b.)

- d. If and only if the system is controllable. Then:

  $$\bar{Q} > 0$$

- e. If $\bar{Q}$ exists, the optimal estimator is:

  $$\boxed{\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + K_F(y - C\hat{x}) \\ K_F &= \bar{Q}C^TV^{-1} \end{aligned}}$$

  If the system is stabilizable and detectable, the estimator is asymptotically stable.

- f. The asymptotic estimator minimizes:

  $$\lim_{t \to -\infty} E\left\{e^T Z e\right\} = tr\left\{\bar{Q}Z\right\}$$

  The Point f. Is true iff $E\left\{\hat{x}_0\right\} = E\left\{x_0\right\}$

Observer error covariance matrix. We consider the system $\dot{x}(t) = Ax(t) + Bu(t) + v(t)$, $y(t) = Cx(t) + w(t)$, where v is white noise with intensity V and w white noise with intensity W. The estimation error $e(t) = \hat{x}(t) - x(t)$ of the observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)]$$

satisfies the differential equation

$$\dot{e}(t) = (A - KC)e(t) - Gv(t) + Kw(t).$$

The noise process $-Gv(t) + Kw(t)$ is white noise with intensity $G^{\mathrm{T}}VG + K^{\mathrm{T}}WK$. Hence, if the error system is stable then the error covariance matrix $Y = Ee(t)e^{\mathrm{T}}(t)$ is the unique solution of the Lyapunov equation

$$(A - KC)Y + Y(A - KC)^{\mathrm{T}} + G^{\mathrm{T}}VG + K^{\mathrm{T}}WK = 0. \tag{4.202}$$

The Kalman filter. We discuss how to choose the observer gain K to minimize the error covariance matrix Y. To this end we complete the square (in K) and rewrite the Lyapunov equation (4.202) as

$$(K - YC^{\mathrm{T}}W^{-1})W(K - YC^{\mathrm{T}}W^{-1})^{\mathrm{T}} + AY + YA^{\mathrm{T}} + GVG^{\mathrm{T}} - YC^{\mathrm{T}}W^{-1}CY = 0. \tag{4.203}$$

Suppose that there exists a gain K that stabilizes the error system and minimizes the error variance matrix Y. Then changing the gain to $K + \varepsilon\tilde{K}$, with ε a small scalar and $\tilde{K}$ an arbitrary matrix of the same dimensions as K, should only affect Y quadratically in ε. Inspection of (4.203) shows that this implies

$$K = YC^{\mathrm{T}}W^{-1}.$$

With this gain the observer reduces to the Kalman filter. The minimal error variance matrix Y satisfies the Riccati equation

$$AY + YA^{\mathrm{T}} + GVG^{\mathrm{T}} - YC^{\mathrm{T}}W^{-1}CY = 0.$$

Summary

Assumptions:

- The system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Gv(t),\\ y(t) &= Cx(t),\end{aligned} \qquad t \in \mathbb{R}, \tag{4.68}$$

  is stabilizable and detectable.

- The noise intensity matrices V and W are positive-definite.

The following facts follow from Summary 4.2.1 (p. 137) by duality:

1. The algebraic Riccati equation

$$AY + YA^{\mathrm{T}} + GVG^{\mathrm{T}} - YC^{\mathrm{T}}W^{-1}CY = 0 \tag{4.69}$$

   has a unique nonnegative-definite symmetric solution Y. If the system (4.68) is controllable rather than just stabilizable then Y is positive-definite.

2. The minimal value of the steady-state weighted mean square state reconstruction error $\lim_{t\to\infty} Ee^{\mathrm{T}}(t)W_e e(t)$ is[7] tr YW_e.

3. The minimal value of the mean square reconstruction error is achieved by the observer gain matrix $K = YC^{\mathrm{T}}W^{-1}$.

4. The error system

$$\dot{e}(t) = (A - KC)e(t), \qquad t \in \mathbb{R}, \tag{4.70}$$

   is stable, that is, all the eigenvalues of the matrix $A - KC$ have strictly negative real parts. As a consequence also the observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)], \qquad t \in \mathbb{R}, \tag{4.71}$$

   is stable.

□

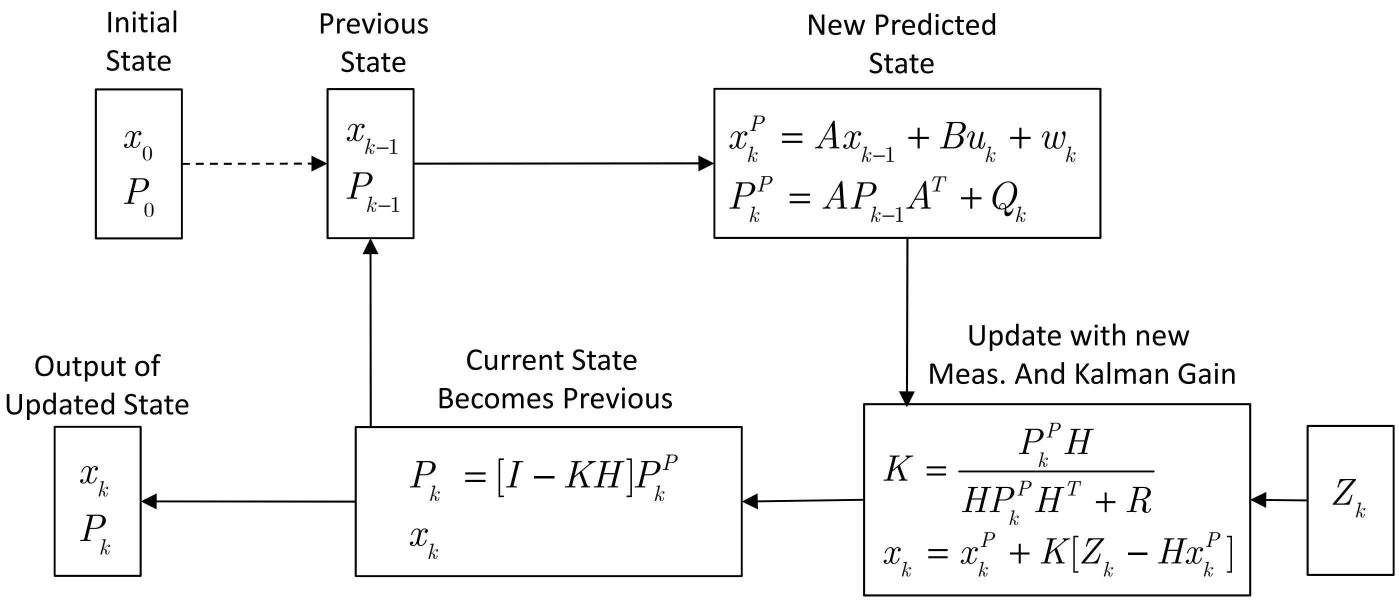
The reasons for the assumptions may be explained as follows. If the system (4.68) is not detectable then no observer with a stable error system exists. If the system is not stabilizable (with v as input) then there exist observers that are not stable but are immune to the state noise v. Hence, stability of the error system is not guaranteed. W needs to be positive-definite to prevent the Kalman filter from having infinite gain. If V is not positive-definite then there may be unstable modes that are not excited by the state noise and, hence, are not stabilized in the error system.

❑ Kalman Filter Implementation: the discrete case

$$\mathbf{x}_k = \mathbf{A}_{k-1}\mathbf{x}_{k-1} + \mathbf{B}_{k-1}\mathbf{u}_{k-1} + \mathbf{w}_{k-1}$$
$$\mathbf{z}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k$$

$$\mu_0 = E[\mathbf{x}_0]$$
$$\mathbf{P}_0 = E[(\mathbf{x}_0 - \mu_0)(\mathbf{x}_0 - \mu_0)^T]$$

$$E[\mathbf{w}_k] = 0 \quad E[\mathbf{w}_k\mathbf{w}_k^T] = \mathbf{Q}_k \quad E[\mathbf{w}_k\mathbf{w}_j^T] = 0 \ \text{ for } \ k \neq j \quad E[\mathbf{w}_k\mathbf{x}_0^T] = 0 \ \text{ for all } \ k$$
$$E[\mathbf{v}_k] = 0 \quad E[\mathbf{v}_k\mathbf{v}_k^T] = \mathbf{R}_k \quad E[\mathbf{v}_k\mathbf{v}_j^T] = 0 \ \text{ for } \ k \neq j \quad E[\mathbf{v}_k\mathbf{x}_0^T] = 0 \ \text{ for all } \ k$$



LQG Control

LQG Control Synthesis

- Given the LTI stochastic system

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} + D\boldsymbol{w} \\ \boldsymbol{y} = C\boldsymbol{x} + \boldsymbol{v} \end{cases} \qquad \begin{cases} \boldsymbol{w}(t) \Rightarrow \left[0, W\right] \\ \boldsymbol{v}(t) \Rightarrow \left[0, V\right] \\ \boldsymbol{x}_0(t) \Rightarrow \left[\boldsymbol{\eta}_0, Q_0\right] \end{cases}$$

- **Find the optimal controller $u(t)$ that minimizes:**

$$J = E\Big\{\boldsymbol{x}^T(t_f)S(t_f)\boldsymbol{x}(t_f) + \\ + \int_{t_0}^{t_f} \left(\boldsymbol{x}^T(\tau)Q(\tau)\boldsymbol{x}(\tau) + u^T(\tau)R(\tau)u(\tau)\right)d\tau\Big\}$$

$$Q \geq 0, R > 0, S(t_f) = S_f \geq 0$$

DWD^T

Main Result

<u>Theorem:</u>

The optimal control law is:

$\bar{u} = -K_c\hat{x}$ where:

- $K_c \quad -R^{-1}B^TS$ with S solution of $\begin{cases} \dot{S} = A^TS + SA + Q - SBR^{-1}B^TS \\ S(t_f) = S_f \end{cases}$

- $\hat{x}$ given by:

$\dot{\hat{x}} = A\hat{x} + B\bar{u} + K_F(y - C\bar{x})$ where

$K_F = QC^TV^{-1}$ with Q solution of: $\begin{cases} \dot{Q} = AQ + QA^T + W - QC^TV^{-1}CQ \\ Q(t_0) = Q_0 \end{cases}$

- **Outline of Proof:**

From KF

$$E\left\{\boldsymbol{x}^T Q \boldsymbol{x}\right\} = E\left\{\left(\boldsymbol{x}-\hat{\boldsymbol{x}}+\hat{\boldsymbol{x}}\right)^T Q\left(\boldsymbol{x}-\hat{\boldsymbol{x}}+\hat{\boldsymbol{x}}\right)\right\} = E\left\{\left(\boldsymbol{x}-\hat{\boldsymbol{x}}\right)^T Q\left(\boldsymbol{x}-\hat{\boldsymbol{x}}\right)\right\} + 2E\left\{\left(\boldsymbol{x}-\hat{\boldsymbol{x}}\right)^T Q\hat{\boldsymbol{x}}\right\} + E\left\{\hat{\boldsymbol{x}}^T Q\hat{\boldsymbol{x}}\right\}$$

  - Since noises are uncorrelated: $E\left\{\boldsymbol{e}^T Q\hat{\boldsymbol{x}}\right\} = 0$

- We know that:

$$E\left\{\left(\boldsymbol{x}-\hat{\boldsymbol{x}}\right)^T Q\left(\boldsymbol{x}-\hat{\boldsymbol{x}}\right)\right\} = E\left\{\boldsymbol{e}^T Q\boldsymbol{e}\right\} = tr\left[Q\Sigma\right]$$

- We obtain:

$$E\left\{\boldsymbol{x}^T Q\boldsymbol{x}\right\} = tr[Q\Sigma] + E\left\{\hat{\boldsymbol{x}}^T Q\hat{\boldsymbol{x}}\right\}$$

- Similarly:

$$E\left\{\boldsymbol{x}_f^T S_f \boldsymbol{x}_f\right\} = tr[S_f\Sigma] + E\left\{\hat{\boldsymbol{x}}_f^T S_f \hat{\boldsymbol{x}}_f\right\}$$

$$\boxed{J = E\left\{\int_{t_0}^{t_f}(\hat{\boldsymbol{x}}^T Q\hat{\boldsymbol{x}} + \boldsymbol{u}^T R\boldsymbol{u})dt + \hat{\boldsymbol{x}}_f^T S_f \hat{\boldsymbol{x}}_f\right\} + tr\left[\int_{t_0}^{t_f} Q\Sigma\, dt + S_f\Sigma_f\right]}$$

$$\tilde{J}(u) = E\left\{\int_{t_0}^{t_f}(\hat{\boldsymbol{x}}^T Q\hat{\boldsymbol{x}} + \boldsymbol{u}^T R\boldsymbol{u})dt + \hat{\boldsymbol{x}}_f^T S_f \hat{\boldsymbol{x}}_f\right\}$$

$$\boldsymbol{u} = -K_c\hat{\boldsymbol{x}} \qquad K_c = R^{-1}B^T S$$

$$\begin{cases}\dot{\hat{\boldsymbol{x}}} = A\hat{\boldsymbol{x}} + B\boldsymbol{u} + K_F\boldsymbol{\eta} \\ \boldsymbol{\eta} = \boldsymbol{y} - C\hat{\boldsymbol{x}}\end{cases}$$

$$\begin{cases}\dot{S} = A^T S + SA + Q - SBR^{-1}B^T S \\ S(t_f) = S_f\end{cases}$$

LQG Control

- The performance of the closed loop system is computed in a stochastic sense, since the state vector and the output are now stochastic processes

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{e} \\ \hat{\boldsymbol{x}} \end{bmatrix} \quad \boldsymbol{x}_0 = \begin{bmatrix} \boldsymbol{e}_0 \\ \hat{\boldsymbol{x}}_0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_0 - \boldsymbol{\eta}_0 \\ \boldsymbol{\eta}_0 \end{bmatrix}$$

$$\dot{\boldsymbol{x}} = \left[\begin{array}{c|c} A - K_F C & 0 \\ \hline K_F C & A - BK_c \end{array}\right] \boldsymbol{x} + \left[\begin{array}{c|c} D & -K_F \\ \hline 0 & K_F \end{array}\right] \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{v} \end{bmatrix}$$

Note: $\boldsymbol{x}$ is the augmented state vector of dimension **2n**

- **The closed loop dynamics are now a Linear System driven by white noise**

$$\dot{\boldsymbol{x}} = A_C \boldsymbol{x} + B_C \boldsymbol{\xi}$$

- **The statistics are given by the closed loop covariance matrix**

$$\bar{Q}^x(t) = E\left\{\left[\boldsymbol{x} - E(\boldsymbol{x})\right]^T \left[\left[\boldsymbol{x} - E(\boldsymbol{x})\right]\right]\right\} = \begin{bmatrix} Q_{11}^x & Q_{12}^x \\ Q_{21}^x & Q_{22}^x \end{bmatrix}$$

$$\begin{cases} \dot{Q}^x = A_C Q^x + Q^x A_C^T + B_C \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} B_C^T \\ Q^x(t_0) = Q_0^x \end{cases}$$

$$\begin{cases} \dot{Q}_{11}^x = [A - K_F C]Q_{11}^x + Q_{11}^x (A - K_F C)^T + DWD^T + K_F V K_F^T \\ \dot{Q}_{12}^x = Q_{11}^x C^T K_F^T + Q_{12}^x (A - BK_C)^T + (A - K_F C)Q_{12}^x - K_F V K_F^T \\ \dot{Q}_{22}^x = Q_{12}^{x\,T} C^T K_F^T + Q_{22}^x (A - BK_C)^T + K_F C Q_{12}^x + (A - BK_C)Q_{22}^x + K_F V K_F^T \end{cases}$$

$$Q_{11}^x(t_0) = Q_0$$

$$Q_{12}(t_0) = Q_{22}(t_0) = 0$$

LQG Control

- **Analysis of the Covariance Matrix differential equation solution:**

$$K_F = QC^T V^{-1} \quad \Rightarrow \quad Q_{11}^x C^T K_F^T - K_F V K_F^T = 0$$

- $Q_{11}^x(t)$ can be solved independently:

$$\begin{cases} \dot{Q}_{11}^x = [A - K_F C]Q_{11}^x + Q_{11}^x(A - K_F C)^T + DWD^T + K_F V K_F^T \\ Q_{11}^x(t_0) = Q_0 \end{cases}$$

- Error and state estimate are uncorrelated since:

$$Q_{12}^x(t_0) = 0 \Rightarrow Q_{12}^x(t) = 0 \quad \Rightarrow \quad E\left\{\left[\boldsymbol{e} - E(\boldsymbol{e})\right]\left[\hat{\boldsymbol{x}} - E(\hat{\boldsymbol{x}})\right]^T\right\} = 0$$

$$\begin{cases} \dot{Q}_{22}^x = Q_{22}^x(A - BK_C)^T + (A - BK_C)Q_{22}^x + K_F V K_F^T \\ Q_{22}^x(t_0) = 0 \end{cases}$$

$$\vdots$$

$$\boxed{\begin{bmatrix} Q_{11}^x & 0 \\ 0 & Q_{22}^x \end{bmatrix} = \begin{bmatrix} \text{Filter Riccati Solution} & 0 \\ 0 & \text{Lyapunov Equation Solution} \end{bmatrix}}$$

  - Once $Q^x(t)$ is computed all the statistics are known since:

$$\boldsymbol{x} = \boldsymbol{e} + \hat{\boldsymbol{x}}$$

NOTE: In the case of infinite horizon the equations are algebraic and not differential

- **LQG Statistics** (see Kwakernaak and Sivan, 'Linear Optimal Control' , page 95, Theorem 1.50)

- **State Vector Mean Square Value**

$$E\left\{\boldsymbol{x}^T Q \boldsymbol{x}\right\} = tr\left[QE\left\{\boldsymbol{x}\boldsymbol{x}^T\right\}\right] = tr\left[Q(\boldsymbol{\eta}\boldsymbol{\eta}^T + Q_{11}^x + Q_{22}^x)\right]$$

$$\boldsymbol{\eta} = E\left\{\hat{\boldsymbol{x}}\right\}$$

- **Output Vector Mean Square Value**

$$E\left\{\boldsymbol{y}^T \boldsymbol{y}\right\} = tr\left[C^T C(\boldsymbol{\eta}\boldsymbol{\eta}^T + Q_{11}^x + Q_{22}^x)\right]$$

- **Input Mean Square Value**

$$E\left\{\boldsymbol{u}^T R \boldsymbol{u}\right\} = E\left\{\hat{\boldsymbol{x}}^T K_C^T R K_C \hat{\boldsymbol{x}}\right\} =$$
$$= tr\left[K_C^T R K_C(\boldsymbol{\eta}\boldsymbol{\eta}^T + Q_{22}^x)\right]$$

- **State Vector Expected Value**

$$E\left\{\boldsymbol{x}(t)\right\} = e^{A_C t} E\left\{\boldsymbol{x}(t_0)\right\}$$

$$E\left\{\boldsymbol{x}(t_0)\right\} = \begin{bmatrix} E\left\{\boldsymbol{e}_0\right\} \\ E\left\{\hat{\boldsymbol{x}}_0\right\} \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{\eta}_0 = E\left\{\boldsymbol{x}_0\right\} \end{bmatrix}$$

- **Relationship between Mean Square Value and Spectral Density**

Given a vector stochastic process $v(t)$ and a positive Semidefinite symmetric Matrix $W(t)$, then:

$$E\left\{v^T(t)W(t)v(t)\right\} = tr\left\{W(t)C_v(t,t)\right\}$$
$$C_v(t_1,t_2) = E\left\{\overline{v}(t_1)\overline{v}^T(t_2)\right\}$$

If $v(t)$ is wide-sense stationary with zero Mean, and Covariance $R_v(t_1 - t_2)$

And $W(t) = W$ = constant, then:

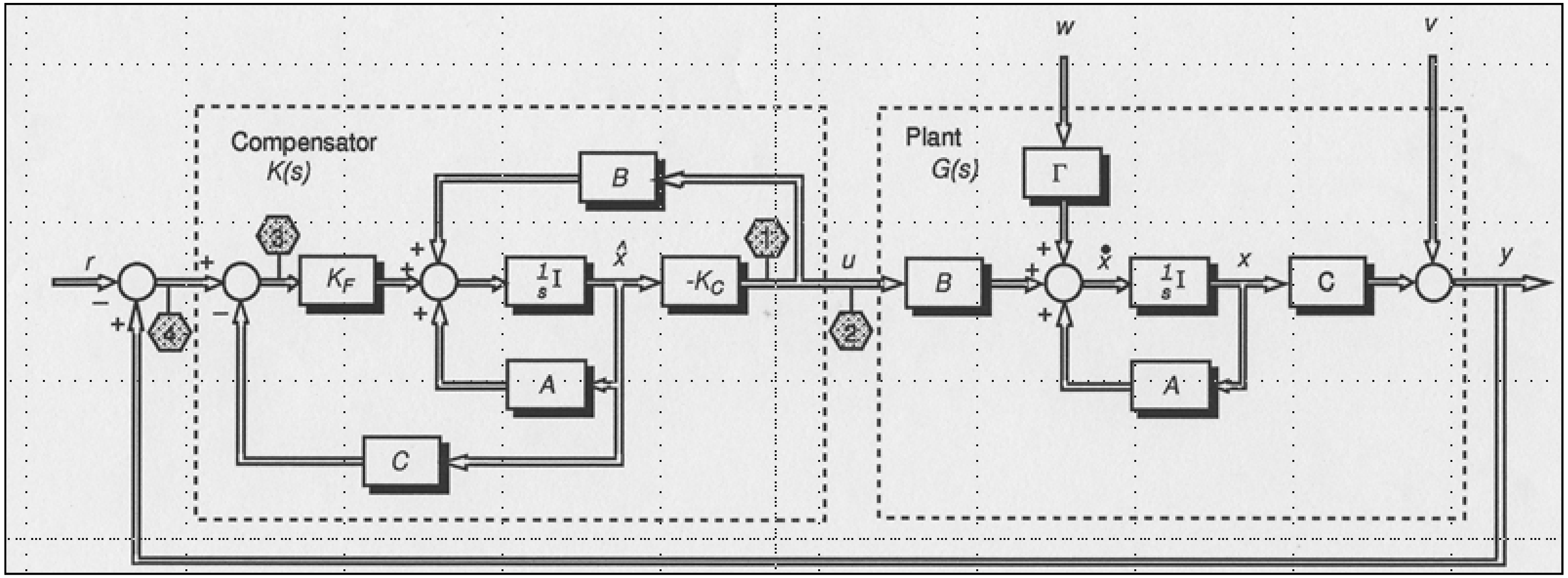
$$E\left\{v^T(t)Wv(t)\right\} = tr\left\{WR_v(0)\right\}$$

If $v(t)$ has zero Mean, and PSD $\sum_v(\omega)$

$$E\left\{v^T(t)Wv(t)\right\} = tr\left\{\int_{-\infty}^{+\infty} W \Sigma_v(\omega) df\right\},$$

$$f = \frac{\omega}{2\pi}, R_v(0) = \int_{-\infty}^{+\infty} \Sigma_v(\omega) df$$

LQG Control



$$\boxed{K_{LQG}(s) = -K_C\left(sI - A + BK_C + K_FC\right)^{-1}K_F}$$

$$\dot{\boldsymbol{x}} = \left[\begin{array}{c:c} A-K_FC & 0 \\ \hdashline K_FC & A-BK_c \end{array}\right]\boldsymbol{x} + \left[\begin{array}{c:c} D & -K_F \\ \hdashline 0 & K_F \end{array}\right]\begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{v} \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{e} \\ \hat{\boldsymbol{x}} \end{bmatrix} \qquad \boldsymbol{x}_0 = \begin{bmatrix} \boldsymbol{e}_0 \\ \hat{\boldsymbol{x}}_0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_0 - \boldsymbol{\eta}_0 \\ \boldsymbol{\eta}_0 \end{bmatrix}$$

8. LQG Control Examples

LQG Control Commands

lqg

R2013b

Linear-Quadratic-Gaussian (LQG) design

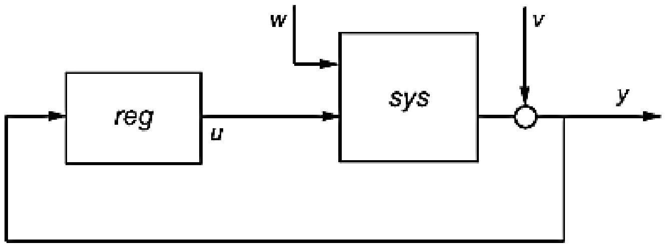
expand all in page

Syntax

```
reg = lqg(sys,QXU,QWV)
reg = lqg(sys,QXU,QWV,QI)
reg = lqg(sys,QXU,QWV,QI,'1dof')
reg = lqg(sys,QXU,QWV,QI,'2dof')
```

Description

`reg = lqg(sys,QXU,QWV)` computes an optimal linear-quadratic-Gaussian (LQG) regulator `reg` given a state-space model `sys` of the plant and weighting matrices `QXU` and `QWV`. The dynamic regulator `sys` uses the measurements *y* to generate a control signal *u* that regulates *y* around the zero value. Use positive feedback to connect this regulator to the plant output *y*.



The LQG regulator minimizes the cost function

$$J = E\left\{ \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} \left[x^T, u^T \right] Q_{xu} \begin{bmatrix} x \\ u \end{bmatrix} dt \right\}$$

subject to the plant equations

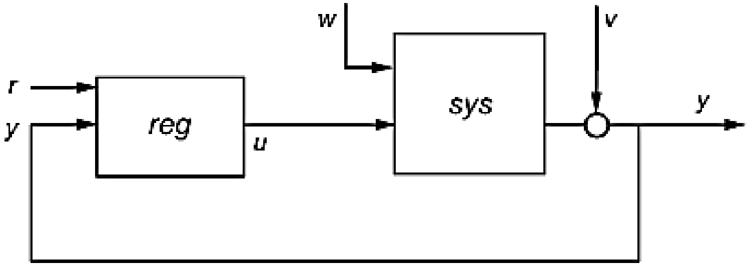
```
dx/dt = Ax + Bu + w
y     = Cx + Du + v
```

where the process noise *w* and measurement noise *v* are Gaussian white noises with covariance:

```
E([w;v] * [w',v']) = QWV
```

LQG Control Commands

`reg = lqg(sys,QXU,QWV,QI)` uses the setpoint command r and measurements y to generate the control signal u. `reg` has integral action to ensure that y tracks the command r.



The LQG servo-controller minimizes the cost function

$$J = E\left\{ \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} \left(\left[x^T, u^T\right] Q_{xu} \begin{bmatrix} x \\ u \end{bmatrix} + x_i^T Q_i x_i \right) dt \right\}$$

where x_i is the integral of the tracking error $r - y$. For MIMO systems, r, y, and x_i must have the same length.

`reg = lqg(sys,QXU,QWV,QI,'1dof')` computes a one-degree-of-freedom servo controller that takes $e = r - y$ rather than $[r ; y]$ as input.

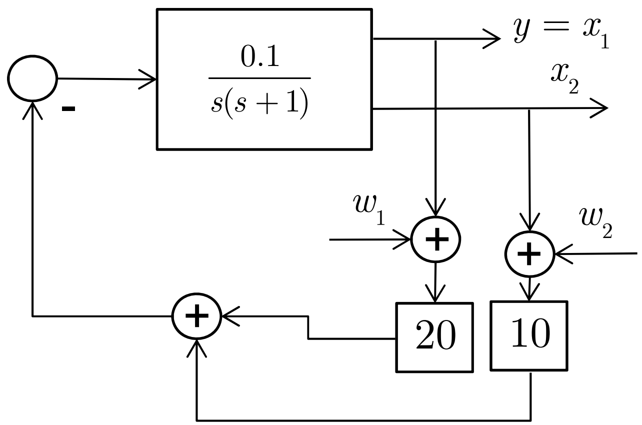
`reg = lqg(sys,QXU,QWV,QI,'2dof')` is equivalent to `LQG(sys,QXU,QWV,QI)` and produces the two-degree-of-freedom servo-controller shown previously.

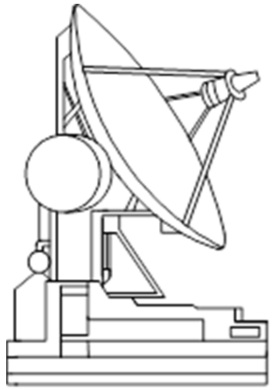
LQG Control Examples

- **Example No. 1:** Single Channel Rigid Antenna with noisy controller

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

$$u(t) = -\begin{bmatrix} 20 & 10 \end{bmatrix} \left\{ \boldsymbol{x}(t) + \boldsymbol{w}(t) \right\}$$





$$\boldsymbol{w}(t) := [0, S_w] \Rightarrow S_w = \begin{bmatrix} .0025 rad^2 / Hz & 0 \\ 0 & 0.05 rad^2 / \sec^2 / Hz \end{bmatrix}$$

- Compute the solution to the associated Lyapunov equation, to get the state vector statistics

$$AQ + QA^T + DWD^T = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}, W = \begin{bmatrix} .0025 & 0 \\ 0 & .05 \end{bmatrix}$$

```
>> x=lyap([0 1;-2 -2],[0 0;0 0.6])

x =

   0.075000000000000   0.000000000000000
   0.000000000000000   0.150000000000000
```

LQG Control Examples

- The solution is found to be: $$Q_\infty = diag\left\{.0075 \quad .015\right\}$$

- The Standard Deviation of the output is 0.087 rad = 5 degrees. This means that the controller is keeping the antenna position within 5 degrees from desired position for 67 % of the time.

❑ **Example No. 2:** Consider now the same controller, but we add a disturbance torque τ_d, (white noise with intensity V_d). The output is the angular position with measurement noise v_m.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} x + \begin{bmatrix} 0 \\ k \end{bmatrix} \mu + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} \tau_d \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + v_m ; v_m = [0, V_m]$$

- Kalman Filter finite time Riccati equation:

$$\dot{Q}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} Q(t) + Q(t) \begin{bmatrix} 0 & 0 \\ 1 & -\alpha \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 V_d \end{bmatrix} - Q(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{V_m} \begin{bmatrix} 1 & 0 \end{bmatrix} Q(t)$$

$$\begin{cases} \dot{q}_{11}(t) = 2q_{12}(t) - \dfrac{1}{V_m} q_{11}^2(t) \\ \dot{q}_{12}(t) = q_{22}(t) - \alpha q_{12}(t) - \dfrac{1}{V_m} q_{11}(t) q_{12}(t) \\ \dot{q}_{22}(t) = -2\alpha \; q_{22}(t) + \gamma^2 V_d - \dfrac{1}{V_m} q_{12}^2(t) \end{cases}$$

- Asymptotic Solution: $\dot{Q}(t) = 0 \Rightarrow \lim_{t \to \infty} Q(t) = Q_\infty$

$$Q_\infty = V_m \begin{bmatrix} -\alpha + \sqrt{\alpha^2 + 2\beta} & \alpha^2 + \beta - \alpha\sqrt{\alpha^2 + 2\beta} \\ \alpha^2 + \beta - \alpha\sqrt{\alpha^2 + 2\beta} & -\alpha^3 - 2\alpha\beta + \left(\alpha^2 + \beta\right)\sqrt{\alpha^2 + 2\beta} \end{bmatrix} \qquad \beta = \gamma\sqrt{\frac{V_d}{V_m}}$$

LQG Control Examples

- Kalman Filter Gain Matrix: $$K_F = \begin{bmatrix} -\alpha + \sqrt{\alpha^2 + 2\beta} \\ \alpha^2 + \beta - \alpha\sqrt{\alpha^2 + 2\beta} \end{bmatrix}$$

- Kalman Filter Poles: $$\det\left(sI - A + \bar{K}C\right) = s^2 + s\left(\sqrt{\alpha^2 + 2\beta}\right) + \beta \Rightarrow \frac{1}{2}\left(-\sqrt{\alpha^2 + 2\beta} \pm \sqrt{\alpha^2 - 2\beta}\right)$$

- **Numerical Example:**
- Disturbance torque RMS value of 31.6 Nm with constant spectral density between ± 50 Hz
- Measurement noise RMS value of 0.01 rad. With constant spectral density between ± 500 Hz

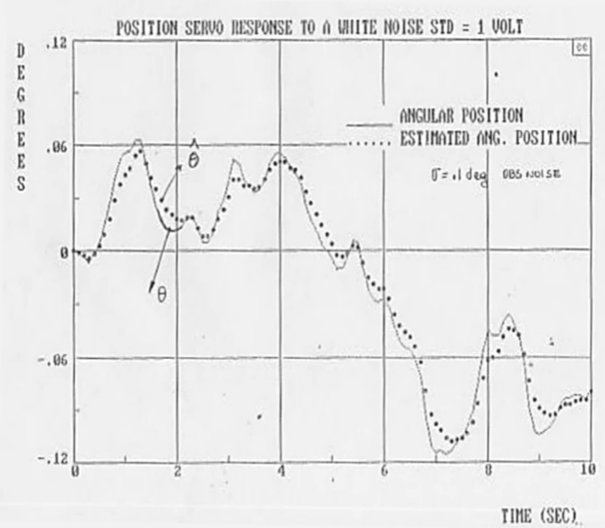
$$\begin{cases} k = 0.787 \ \ rad \,/\, (Vs^2) \\ \alpha = 4.6 \ \ s^{-1} \\ \gamma = 0.1 \ \ Kg^{-1}m^{-2} \\ V_d = 10 \ \ N^2m^2s \\ V_m = 10^{-7} \ \ rad^2 s \end{cases}$$

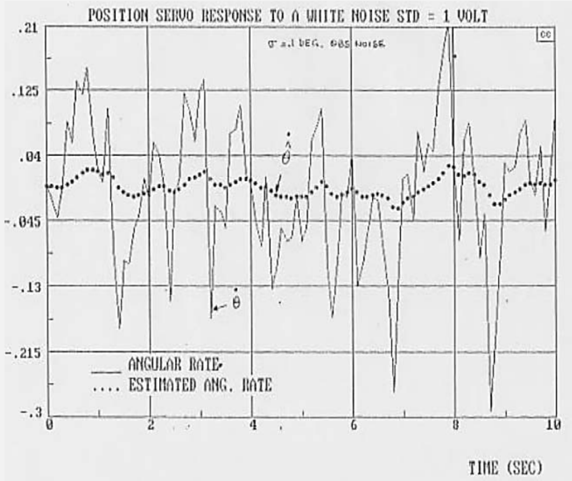
$$K_F = \begin{bmatrix} 40.36 \\ 814.3 \end{bmatrix} \qquad -22.48 \pm j22.24$$

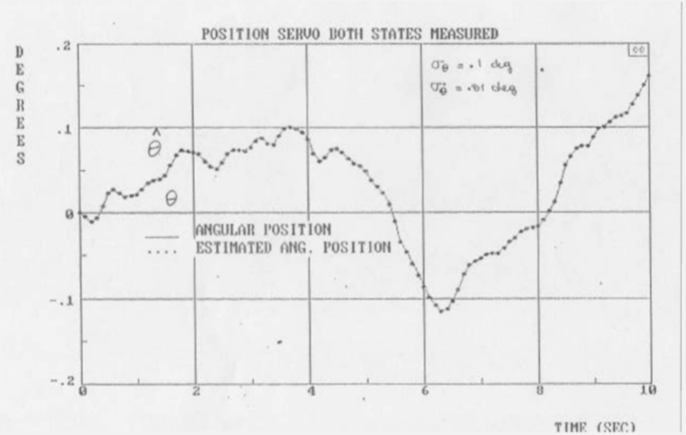
- From the square root of the diagonal terms of Q we can determine the statistics of the reconstruction error, which are 0.002 *rad* and 0.06 *rad/sec* in position and velocity respectively.

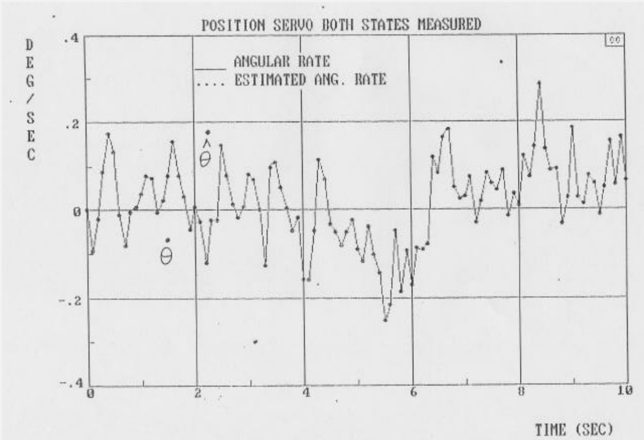
- **Use Matlab to simulate and plot position and rate**

LQG Control Examples



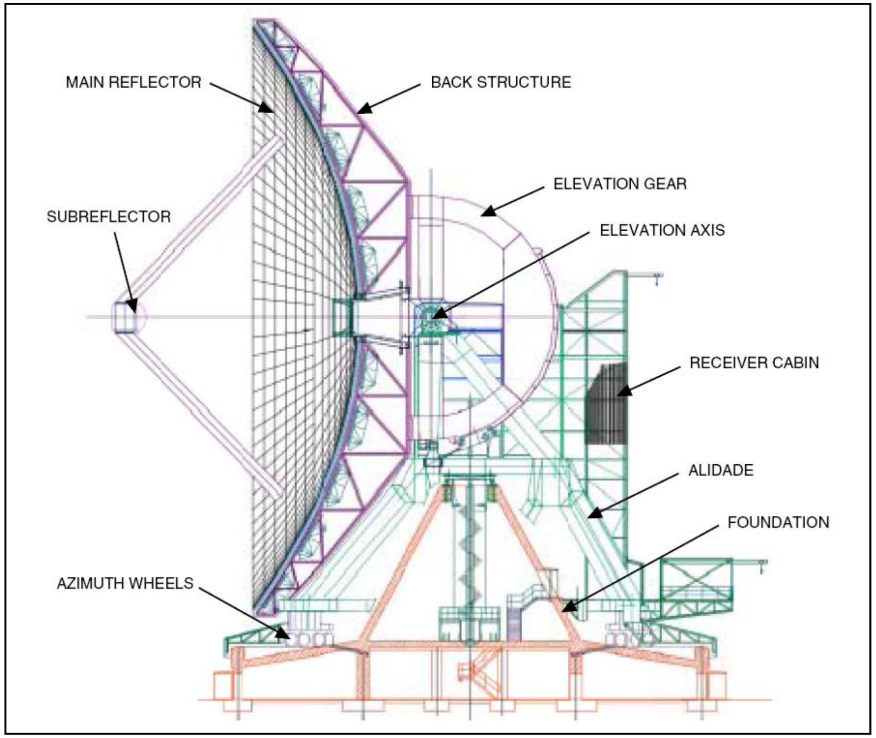


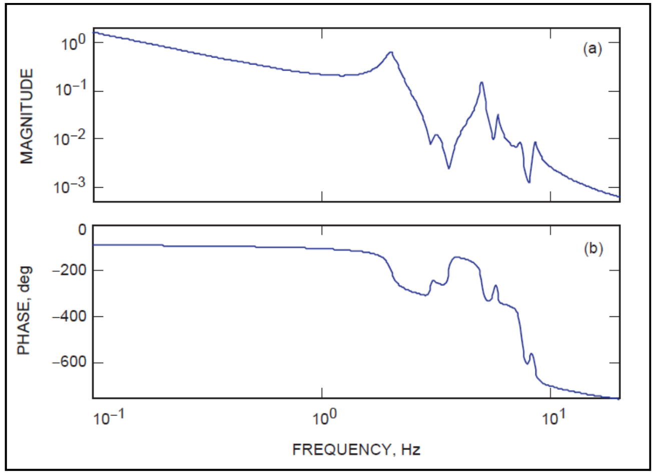




NASA JPL DSN




MAIN REFLECTOR
BACK STRUCTURE
SUBREFLECTOR
ELEVATION GEAR
ELEVATION AXIS
RECEIVER CABIN
ALIDADE
FOUNDATION
AZIMUTH WHEELS


MAGNITUDE
10^0
10^{-1}
10^{-2}
10^{-3}
(a)
PHASE, deg
0
−200
−400
−600
(b)
10^{-1}
10^0
10^1
FREQUENCY, Hz

Motivation / Considerations

- The performance of a controller improves with the gain increase. However, high gains excite structural vibrations. For a PI controller, structural vibrations cannot easily be controlled since the structural deformations are not directly measured by encoders.

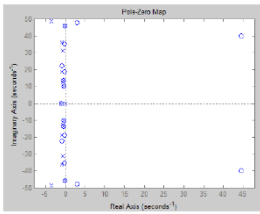
- An LQG controller uses a Kalman filter to estimate telescope vibrations, overcoming the difficulty of direct measurement. The Kalman filter consists of a telescope analytical model that needs to be accurate to produce an accurate estimate of the telescope structural dynamics.

- LQG controllers guarantee wide bandwidth and good wind-disturbance rejection properties; thus, they can be used for antennas with stringent pointing requirements in the presence of wind disturbances (modeled as random processes).

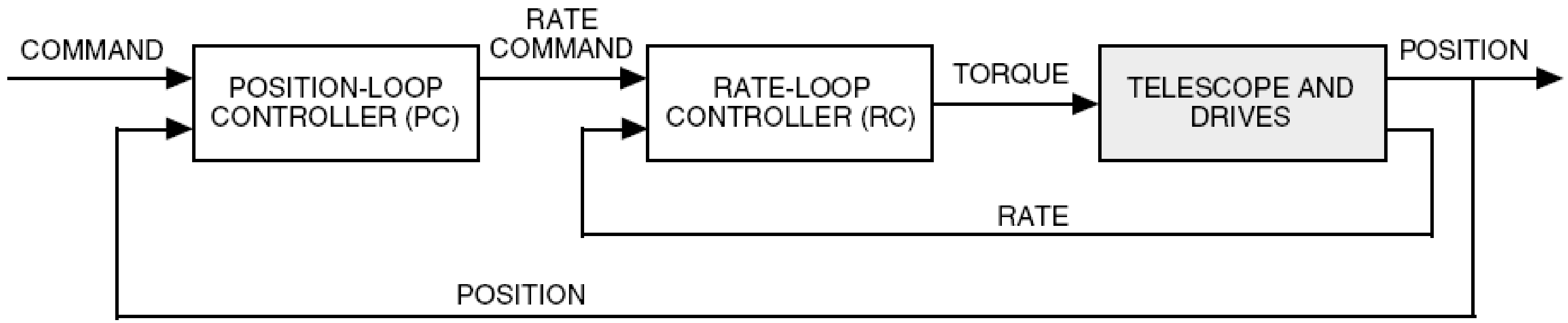
- For antennas and radio telescopes, the LQG control systems can be implemented in two different ways: at the telescope rate loop or at the position loop (as any other controller).

- The analysis is augmented with the performance of the proportional-and-integral (PI) controller. The latter analysis is necessary not only for comparison purposes (the PI controller is a standard antenna industry feature), but also because the implementation of the LQG controller requires preliminary installation of PI controllers in both rate and position loops.

- **Remainder:** The implementation of the LQG algorithm needs an accurate telescope model, and it can be obtained from the field test of the telescope.



General Controller Structure



LMT control system	Rate-loop controller	Position-loop controller
PP	PI	PI
PL	PI	LQG
LP	LQG	PID
LL	LQG	LQG

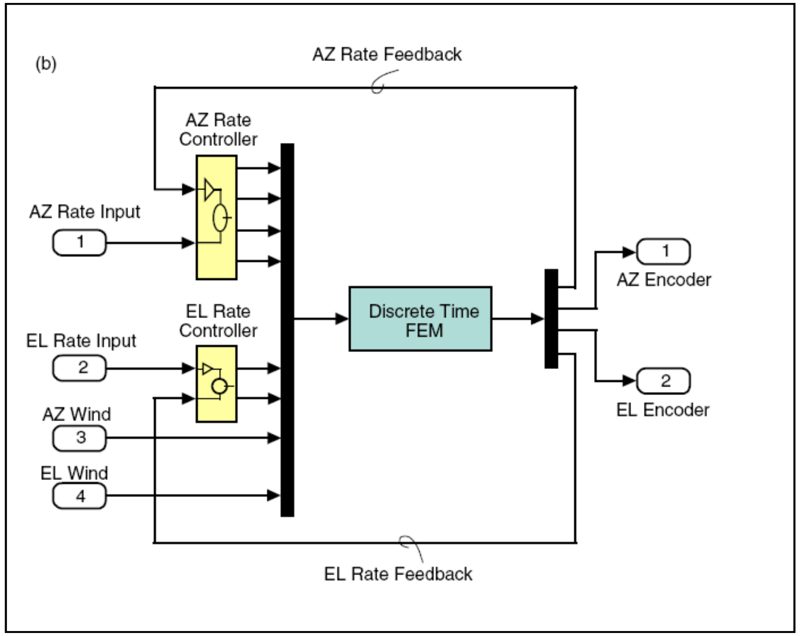
The PP control system is a typical telescope control system configuration.
The PL case is the configuration of the NASA Deep Space Network (DSN) antenna control system, and it was implemented at the 34-meter antennas in Goldstone, California.
The LQG controller has been considered by MAN Technologies. The LP and LL control systems were implemented in 2010, and 2012

- **A Command Preprocessor (CPP) deals with the rate and acceleration limits imposed at the drives. During tracking, the telescope motion is within the rate and acceleration limits. However, during slewing, the large position-offset commands exceed the acceleration limit or both the acceleration and rate limits. When limits are exceeded, the telescope dynamics are no longer linear, and the telescope becomes unstable, which is observed in the form of limit cycling. The CPP modifies the telescope commands such that they remain unaltered if they do not exceed the rate and acceleration limits, and it processes the command to the maximum acceleration and rate limits if the limits are exceeded by the command. The CPP algorithm represents an integrator, rate and acceleration limits, a variable-gain controller, and a feed-forward gain.**

PP Controller

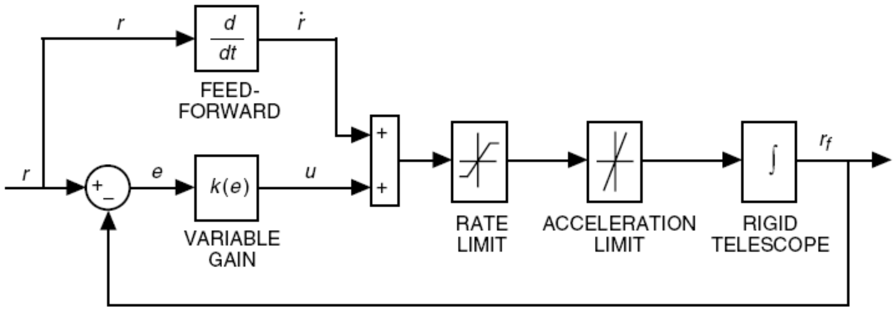
- The PP control system consists of a PI controller in the position loop and a PI controller in the rate loop.

The rate-loop model consists of the finite-element model (FEM) of the telescope structure, which includes the drives and the azimuth (AZ) and elevation (EL) rate-loop controllers. It is a discrete-time control system with a 0.001-s sampling time. The proportional-and-integral gains of the azimuth controller are 300; for the elevation controller, the proportional gain is also 300, and the integral gain is 400. The bandwidth of the rate-loop transfer function is 1.0 Hz, both in azimuth and in elevation.



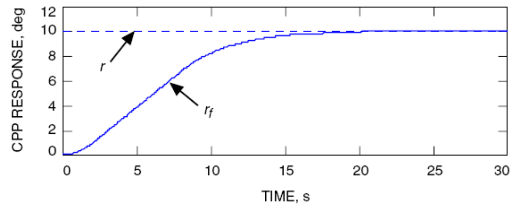
- **A Command Preprocessor (CPP) deals with the rate and acceleration limits imposed at the drives. During tracking, the telescope motion is within the rate and acceleration limits. However, during slewing, the large position-offset commands exceed the acceleration limit or both the acceleration and rate limits. When limits are exceeded, the telescope dynamics are no longer linear, and the telescope becomes unstable, which is observed in the form of limit cycling. The CPP modifies the telescope commands such that they remain unaltered if they do not exceed the rate and acceleration limits, and it processes the command to the maximum acceleration and rate limits if the limits are exceeded by the command. The CPP algorithm represents an integrator, rate and acceleration limits, a variable-gain controller, and a feed-forward gain.**

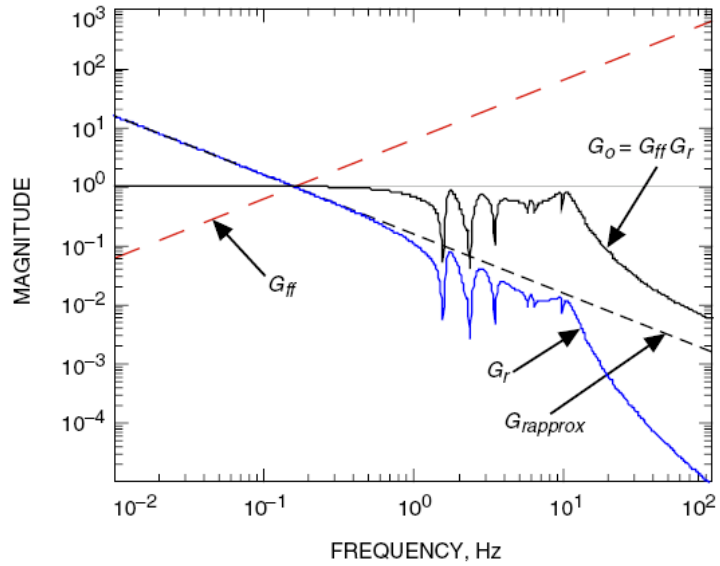
PP Controller



The variable gain k depends on the CPP tracking error e:

$$k(e) = k_o + k_v e^{-\beta|e|}$$

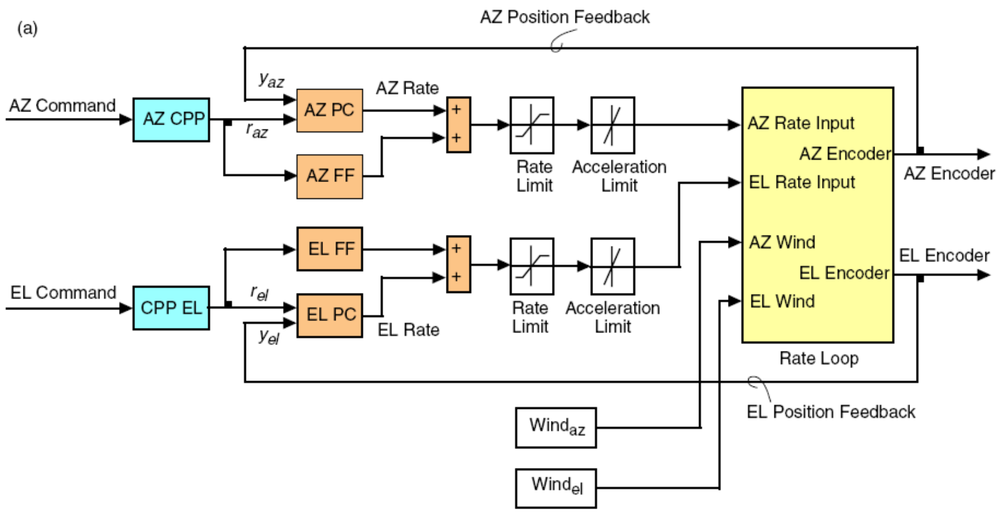




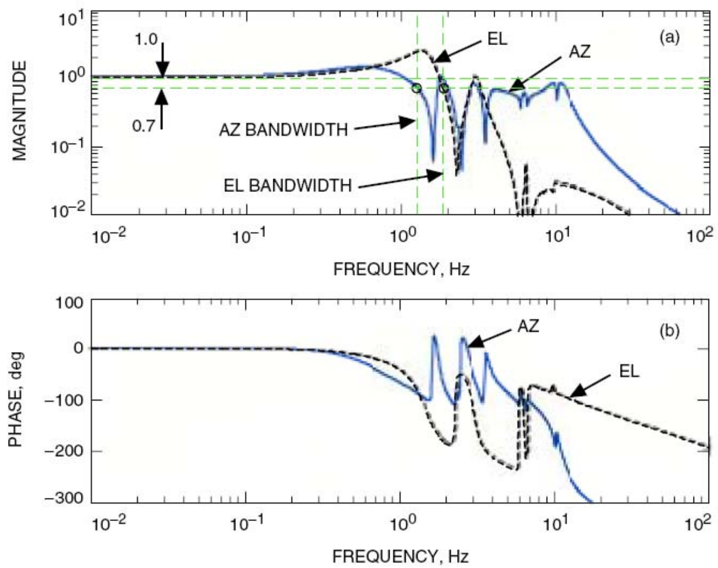
A feed-forward loop is added to improve tracking accuracy, especially at high rates. The feed-forward loop differentiates the command and forwards it to the rate-loop input. The derivative is the inversion of the rate-loop transfer function. <u>In this way, we obtain the open-loop transfer function from the command to the encoder, approximately equal to 1.</u> Indeed, the magnitude of the rate-loop transfer function *Gr* is shown in Fig. It can be approximated (up to 1 Hz) with an integrator *(Grapprox* = 1/s), which is shown in the figure (short-dashed line). The feed forward transfer function is a derivative *(Gff(s) = s)* shown as (long-dashed line), so that the overall open-loop transfer function is a series connection of the feed-forward and the rate loop *Go(s) = Gr(s)Gff(s),* which is approximately equal to 1 up to a frequency of 1 Hz. In this way, the transfer function of the system is equal to 1 (up to 1 Hz) without applying the position feedback. The position feedback is added to compensate disturbances and system imperfections.

PP Controller

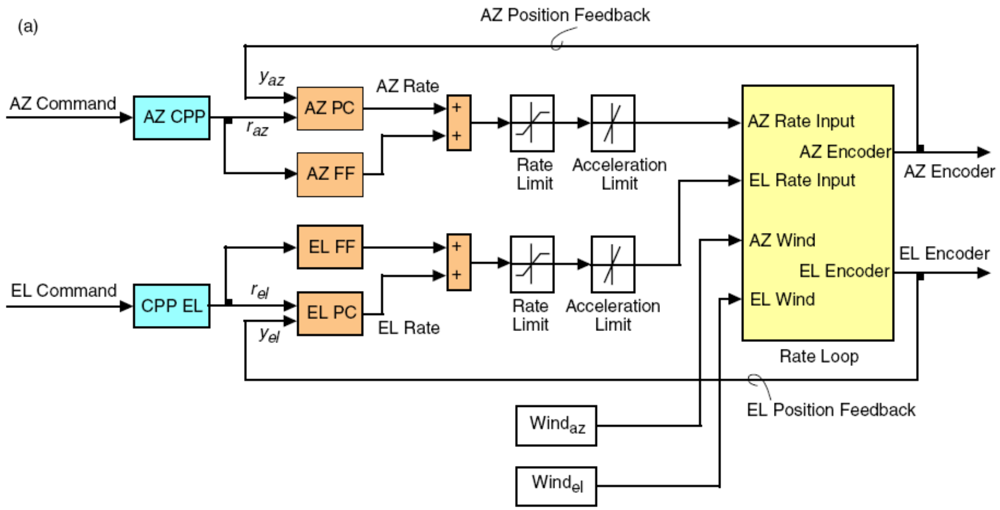
The position-loop model consists of the rate-loop model, PI and feed-forward controllers in azimuth and elevation, command preprocessors in azimuth and elevation, and rate and acceleration limiters in azimuth and elevation. The telescope rate limit is 1.0 deg/s, and the acceleration limit is 0.5 deg/s2, both in azimuth and elevation. The PI controller gains were selected to minimize settling time and servo error in wind gusts. They also guarantee zero steady-state error for constant rate tracking.

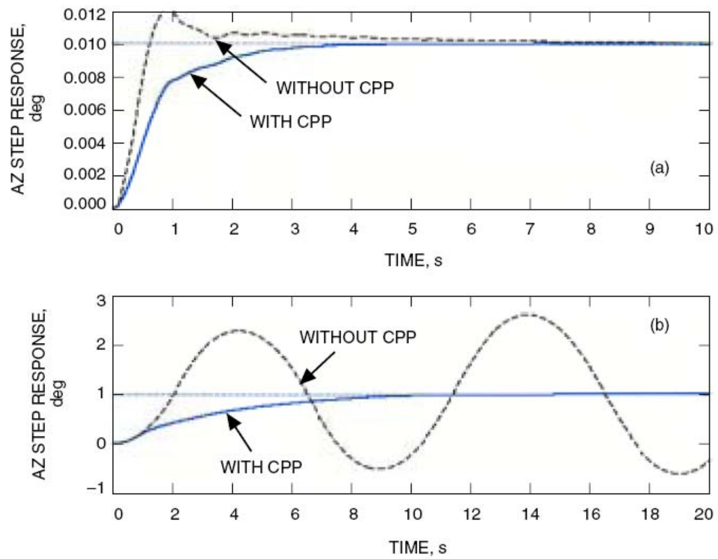


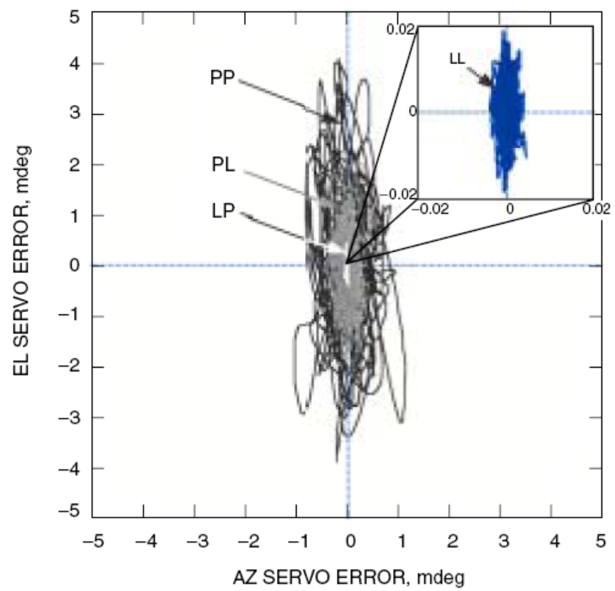
Note that the position-loop bandwidth is higher than the rate-loop bandwidth, both in azimuth and elevation. Note also that the elevation-axis bandwidth is higher than the azimuth-axis bandwidth, mainly because the elevation transfer function has only a few resonances and the resonances are well damped. Azimuth bandwidth is lower since the controller gains will be low in order to prevent excitation of multiple resonances.



PP Controller





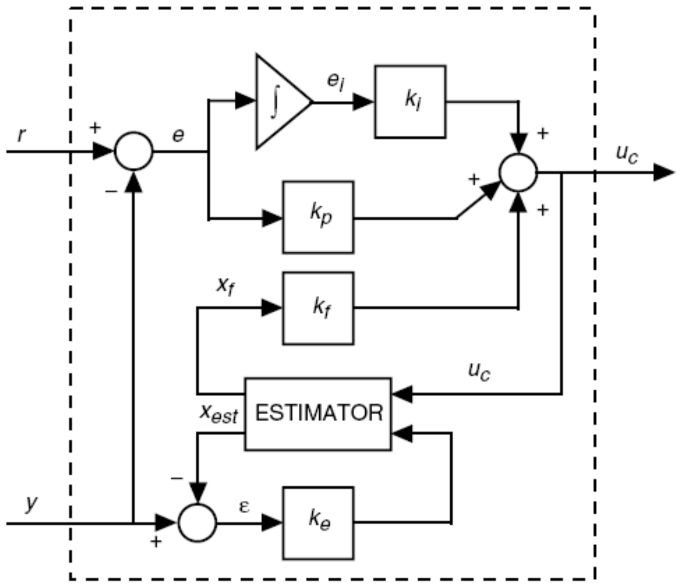


PL Controller

- **The PL control system consists of the LQG controller in the position loop and the PI Controller in the rate loop.**

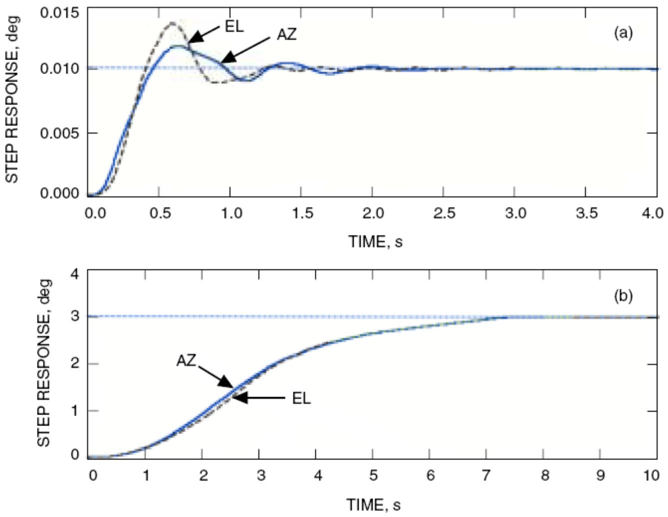
In this case, the rate-loop model of the telescope is as shown before in the PP control.

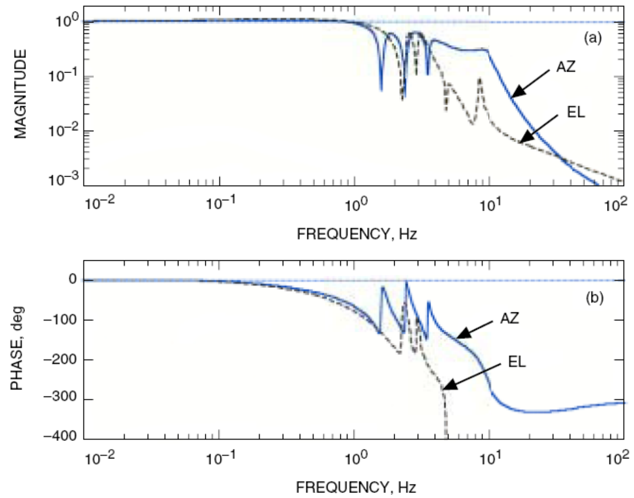
The position loop is similar to the previous one, but the PI controllers are replaced with the LQG controllers. The LQG controller structure is shown below. The controller includes the estimator, which is an analytical model of the telescope.



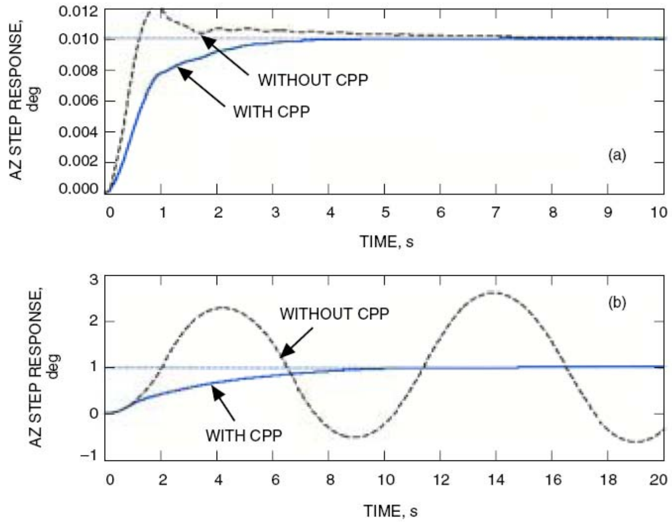
The estimator is driven by the same input *(uc)* as the telescope, but also by the estimation error *ε* (the difference between the actual rate *y* and the estimated rate *yest).* The error is amplified with the estimator gain *ke* to correct for transient dynamics. The estimator outputs are the estimated telescope states that consist of the telescope rate and the telescope flexible deformations *xf.* The latter are the missing vibration measurements, which allow for suppression of the telescope vibrations

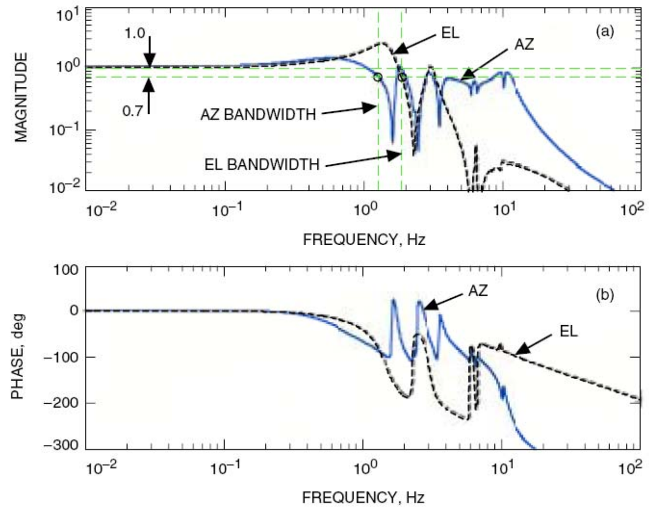
PL Controller





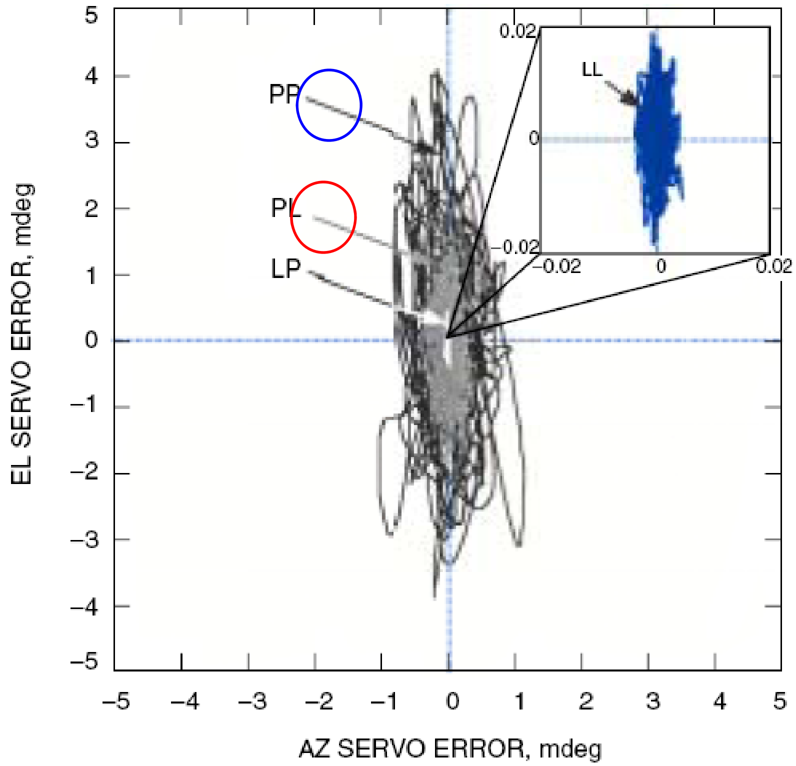
PL





PP

PL Controller

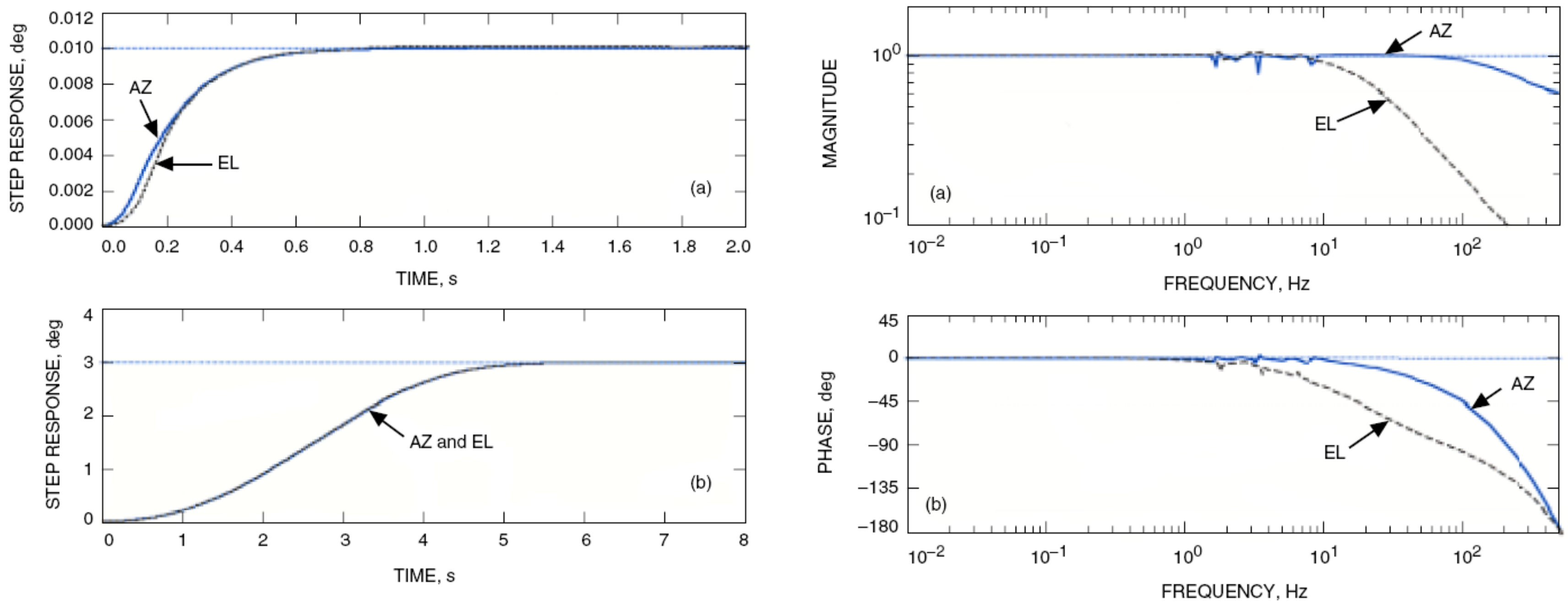


- The results obtained on the LQG position controller produce a settling time of 1.6 s, bandwidth of 1.3 Hz, and wind servo error of 0.76 mdeg or 2 times smaller than that of the PP control system.

- The wind-gust simulations show 0.15-mdeg rms servo error in azimuth and 0.74-mdeg rms servo error in elevation. These numbers are compared with the PP control system (0.35 mdeg in azimuth and 1.4 mdeg in elevation). This means that the LQG controller improves the servo error in wind over the PI controller by a factor of 2.3 in azimuth and a factor of 1.9 in elevation.

LP Controller

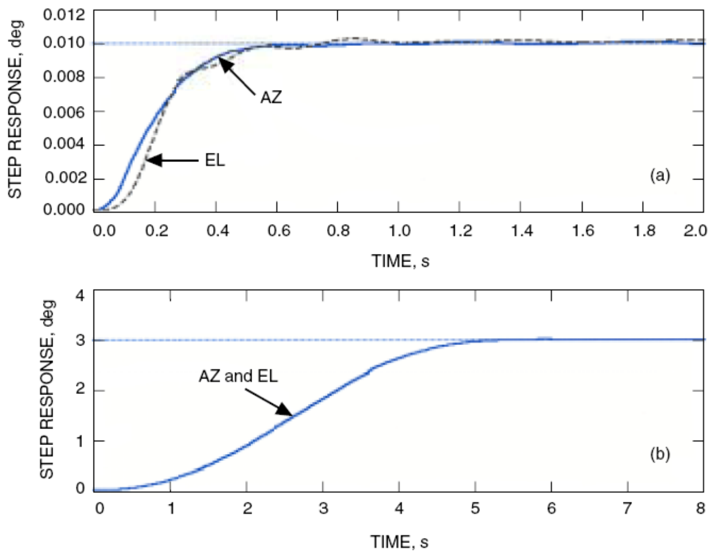
- **The LP control system consists of the PID (proportional-integral-derivative) controller in the position loop and the LQG controller in the rate loop.**

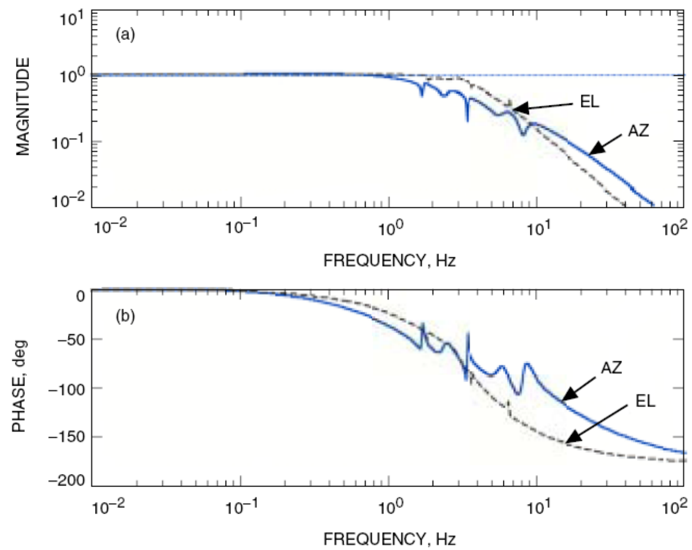


- The position-loop transfer functions for azimuth and elevation show a wide bandwidth of 200 Hz in azimuth and 20 Hz in elevation. The steady-state error due to rate offsets is zero. The wind-gust simulations for a 12-m/s wind show 0.012-mdeg rms servo error in azimuth and 0.150-mdeg rms servo error in elevation. **These small numbers show that, as compared with the PP control system, the LQG controller in the rate loop improves the servo error in wind by a factor of 30 in azimuth and a factor of 10 in elevation.**

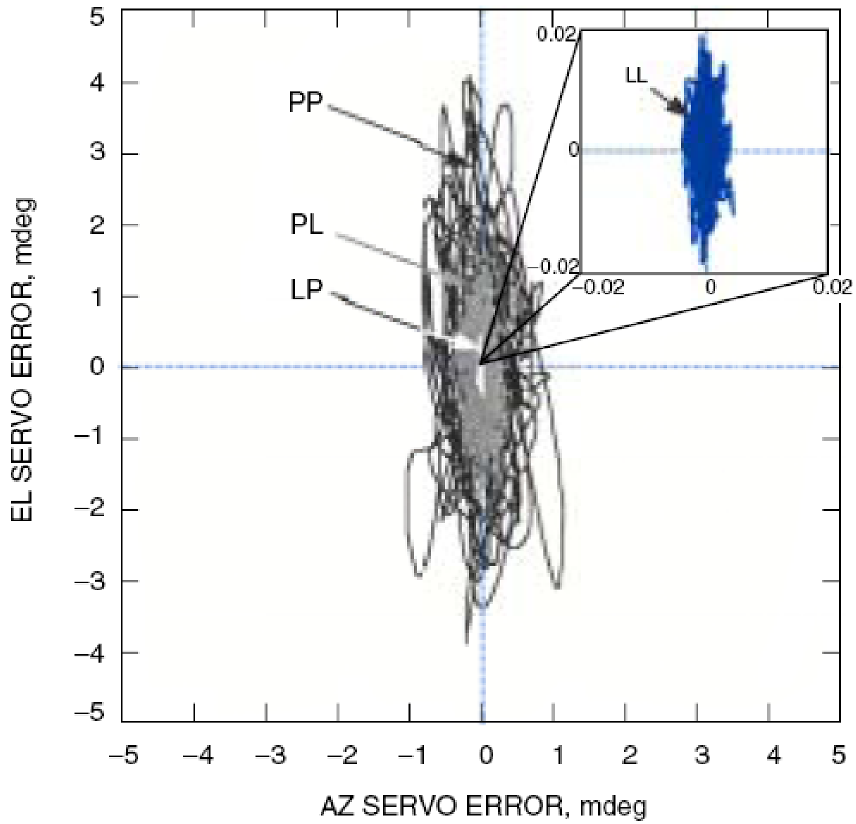
LL Controller (implemented in 2010)

- Finally, consider the complete telescope control system with a LQG controller in the rate and position loops. This is also a novel configuration in the antenna industry.
  - The rate loop is the same as for the LP control system. For the given rate loop, the position-loop controller was designed to minimize the servo error in the wind gusts.
  - The position-loop characteristics are plotted below, it follows that the system settling time is 0.5 s, and there is no overshoot in either azimuth or elevation. The bandwidth is 20 Hz in azimuth and 40 Hz in elevation.





LL Controller

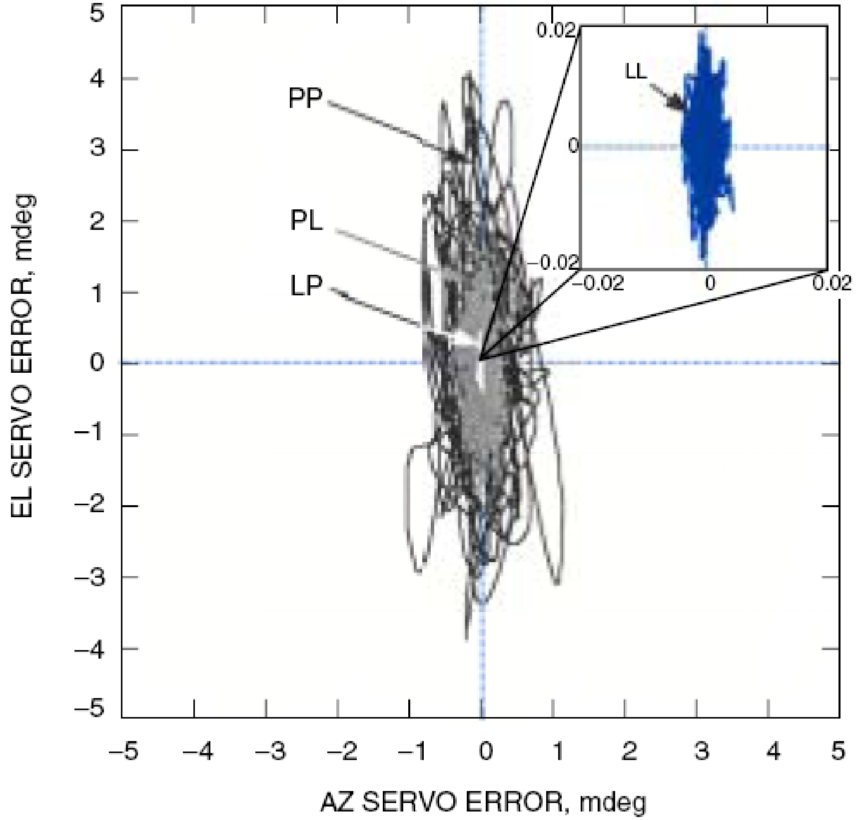


- Finally, the wind-gust simulations for a 12-m/s wind are plotted in the zoomed insert. The figure shows 0.0012-mdeg rms servo error in azimuth and 0.0057-mdeg rms servo error in elevation, which give a total rms error of 0.0058 mdeg. It is 250 times smaller than the error of the PP control system.

- Thus, the LL control system performance is the best of all the systems presented, although the system is the most complicated and will require careful tuning of both rate- and position-loop LQG controllers in order to obtain the predicted performance.

Control system	Settling time, s	Overshoot, percent	Bandwidth, Hz	Wind-gust servo error, mdeg
PP	3.0	20	1.2	1.48
LP	0.6	0	20	0.15
PL	1.4	20	1.4	0.76
LL	0.5	0	20	0.004

LL Controller



Control system	Settling time, s	Overshoot, percent	Bandwidth, Hz	Wind-gust servo error, mdeg
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References

[1] W. Gawronski, "Antenna Control Systems: From PI to H_∞," *IEEE Antennas and Propagation Magazine*, vol. 43, no. 1, pp. 52–60, 2001.

[2] W. Gawronski, C. Racho, and J. Mellstrom, "Application of the LQG and Feedforward Controllers for the DSN Antennas," *IEEE Transactions on Control Systems Technology*, vol. 3, pp. 417–421, December 1995.

[3] W. Gawronski and K. Souccar, "Control Systems of the Large Millimeter Telescope," *Proceedings of the SPIE Conference on Astronomical Telescopes and Instrumentation*, Glasgow, United Kingdom, in press, 2004.

[4] W. Gawronski, *Advanced Structural Dynamics, and Active Control of Structures*, New York: Springer, 2004.

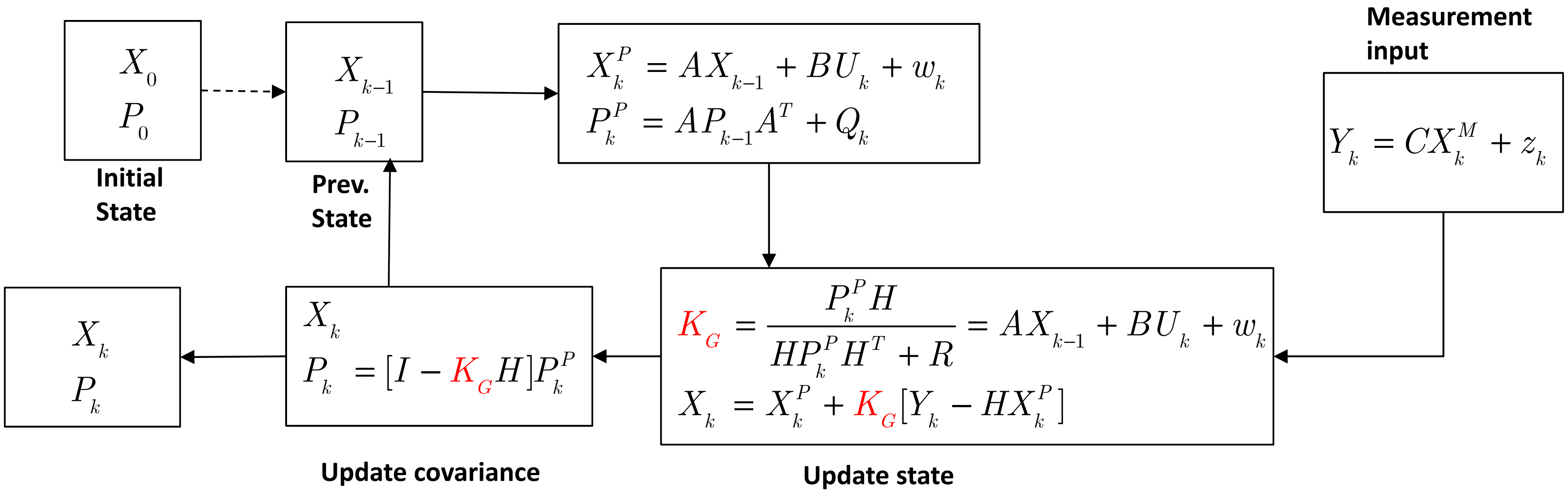
[5] C. D. Johnson, "Optimal Control of the Linear Regulator with Constant Disturbances," *IEEE Transactions on Automatic Control*, vol. 13, pp. 416–421, 1968.

[6] M. Athans, "On the Design of PID Controllers Using Optimal Linear Regulator Theory," *Automatica*, vol. 7, pp. 643–647, 1971.

[7] B. Porter, "Optimal Control of Multivariable Linear Systems Incorporating Linear Feedback," *Electronic Letters*, vol. 7, pp. 170–174, 1971.

[8] E. Maneri and W. Gawronski, "LQG Controller Design Using GUI: Application to Antennas and Radio-Telescopes," *ISA Transactions*, vol. 39, pp. 243–264, 2000.

Extra



X, U, Y= Process state, input, measurement

w, z = predicted process noise and measurement uncertainty

P = predicted process covariance or error in the process estimate
Q = error in the prediction of covariance matrix due to noise
R = sensor noise covariance matrix