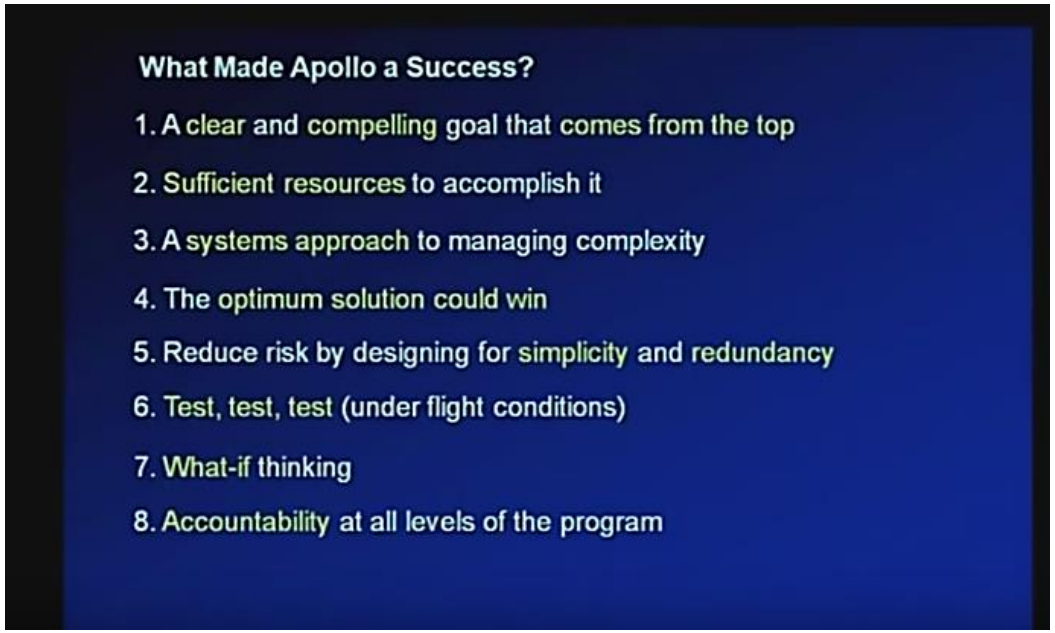
Orbit Transfer – Orbital Maneuvers

- **Two – Body Problem:** Relative forces acting on a spacecraft (orbiting body), geometry of orbits around the main body.
- **Kepler's Equation:** Relation between position at different instants and time intervals spanned by the motion.
- **Orbital Elements:** Definition of reference systems; relationship between orbital elements and position and velocity vectors in space (Ephemerides).

  - i = Inclination of the orbit
  - Ω = Right Ascension of ascending node
  - ω = Argument of periapsis
  - a = Semi major axis (or semilatus rectum)
  - e = Eccentricity of the orbit
  - f = True anomaly (or time to periapsis)

- **Time and Distance:** Measurement of time and distance computation.

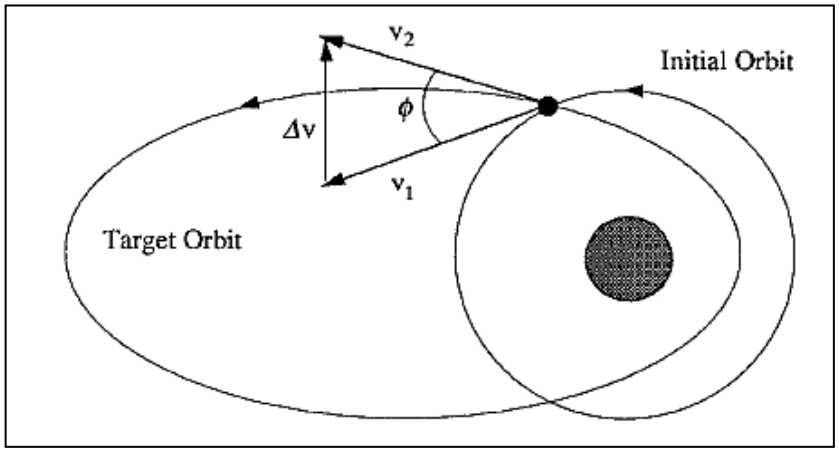
https://www.youtube.com/watch?v=B1J2RMorJXM&t=3641s



Orbit Transfer – Orbital Maneuvers

- **Orbit Transfer:** It is used in several segments of a mission.
  - To change from a parking orbit to the final orbit
  - As part of a more complex interplanetary flight

- **Main Assumption:** Orbit transfer is mostly based on propellant consumption, defined as Δv, that is the instantaneous change in velocity (acceleration) needed for transfer. Some Examples are:
  - Impulsive transfer (constant thrust, no position change at the impulse)
  - Continuous thrust transfer (Option for relative motion)
  - Single Impulse transfer
  - Coplanar orbit transfer
  - Minimum Fuel transfer (Hohmann)
  - Bielliptical orbit transfer
  - Out-of-plane orbit transfer
  - Planet Escape
  - Planet Capture
  - Swing by (Fly by or gravity assist)
  - …



❑ Single Impulse Maneuver

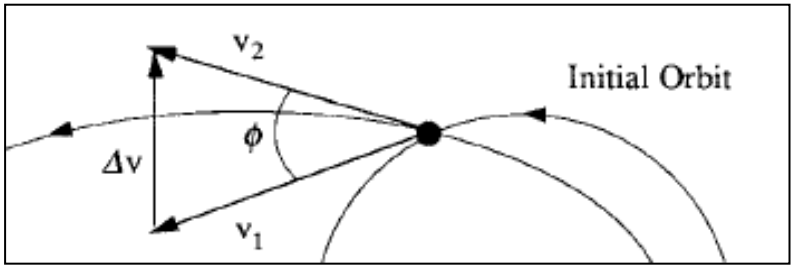
- Δv is typically provided by the thrust of a rocket engine, or other reaction engines (on-off thrusters). The time-rate of change of the velocity Δv is the magnitude of the acceleration caused by the engines, i.e., the thrust per total vehicle mass.

$$\Delta \boldsymbol{v}(t) = \int_{t0}^{t_1} \frac{\left|\boldsymbol{T}(\tau)\right|}{m(\tau)} d\tau = \int_{t0}^{t_1} \left|\boldsymbol{a}(\tau)\right| d\tau$$

$$\boldsymbol{T} = V_e \frac{dm}{dt}$$

$$V_e = g_0 I_{SP}$$

Exit velocity, specific Impulse



- Tsiolkovsky's rocket equation

$$\Delta v = -g_0 I_{SP} \ln\left(\frac{m_0 + \Delta m}{m_0}\right); \Delta m < 0$$

$$\Delta m = m_0 \left\{1 - e^{\left(-\frac{\Delta v}{g_0 I_{SP}}\right)}\right\}$$

- Kinetic Energy Change due to Δv:

$$\boldsymbol{v}_2 = \boldsymbol{v}_1 + \Delta \boldsymbol{v}$$

$$\left|\Delta \boldsymbol{v}\right| = \Delta v = \sqrt{v_1^2 + v_2^2 - 2 v_1 v_2 \cos\phi}$$

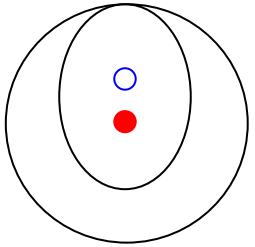
A Δv maneuver can:

- Raise/lower the apogee/perigee
- A change in inclination
- Escape
- Reduction/Increase in period
- Change in RAAN Ω
- Begin a 2+ maneuver sequence of burns.
  - Creates a **Transfer Orbit**.

Orbit Transfer – Orbital Maneuvers

❑ **Problem:** Single coplanar impulse maneuver from Elliptic to Circular Earth orbit (apogee)

- Known needed parameters: semi major axis a, eccentricity e.



At apogee, we have that

$$r_a = a(1+e)$$

From the vis-viva equation, we can calculate the velocity at apogee.

$$v_a = \sqrt{\mu\left(\frac{2}{r_a} - \frac{1}{a}\right)} = \sqrt{\frac{\mu}{a}\left(\frac{1-e}{(1+e)}\right)}$$

$\frac{d}{dt}(\frac{v^2}{2} - \frac{\mu}{r}) = 0 \Rightarrow E_{SPEC} = \mathbb{C}$

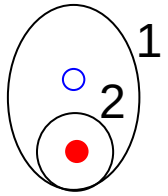
However, for a circular orbit at the same point, we calculate from vis-viva

$$v_c = \sqrt{\frac{\mu}{r_a}} = \sqrt{\frac{\mu}{a(1+e)}}$$

Therefore, the Δv required to circularize the orbit

$$\Delta v = v_c - v_a = \sqrt{\frac{\mu}{a(1+e)}} - \sqrt{\frac{\mu}{a}\left(\frac{1-e}{(1+e)}\right)}$$

- For a change to a smaller circular orbit, the transfer should start at perigee of the elliptical orbit



$$\begin{cases} a = 10,000\ Km \\ e = 0.5 \end{cases} \quad \begin{cases} r_a = 15,000\ Km \\ r_p = 5,000\ Km \end{cases} \quad \begin{cases} v_a = 3.6451\ Km/\sec \\ v_c = 5.1549\ Km/\sec \end{cases} \quad \Delta v = +1.5098\ Km/\sec$$

❑ Two – Impulse Maneuver, Hohmann Transfer

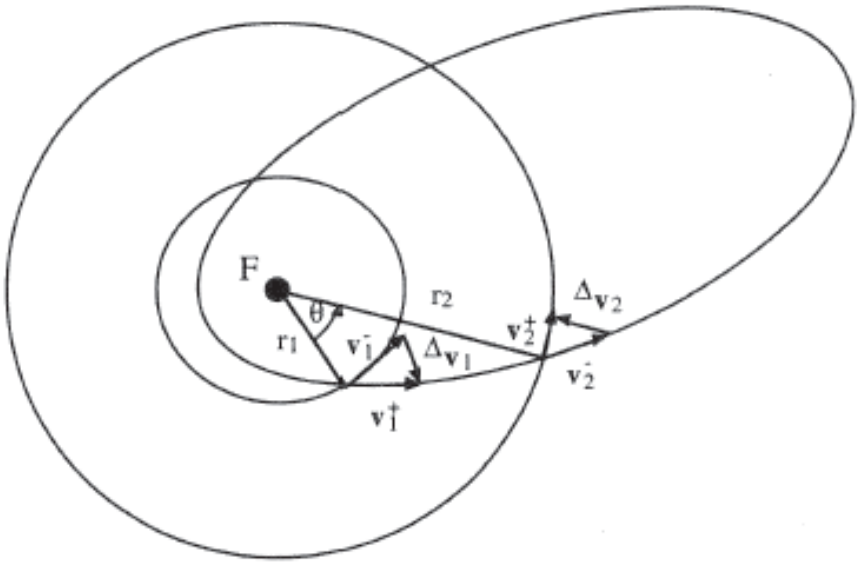
- Two (or more) impulsive maneuvers are used for complex orbit changes and also as segments for interplanetary flight mission planning. The simplest case consists of the initial orbit, the target orbit, and the transfer orbit (coplanar case for now).

Step 1: Design a transfer orbit (a,e,i, etc.).

Step 2: Calculate $\vec{v}_{tr,1}$ at the point of intersection with initial orbit.

Step 3: Calculate initial burn to maneuver into transfer orbit.

$$\Delta v_1 = \vec{v}_{tr,1} - \vec{v}_{init}$$



Step 4: Calculate $\vec{v}_{tr,2}$ at the point of intersection with target orbit.

Step 5: Calculate velocity of the target orbit, $\vec{v}_{fin}$, at the point of intersection with transfer orbit.

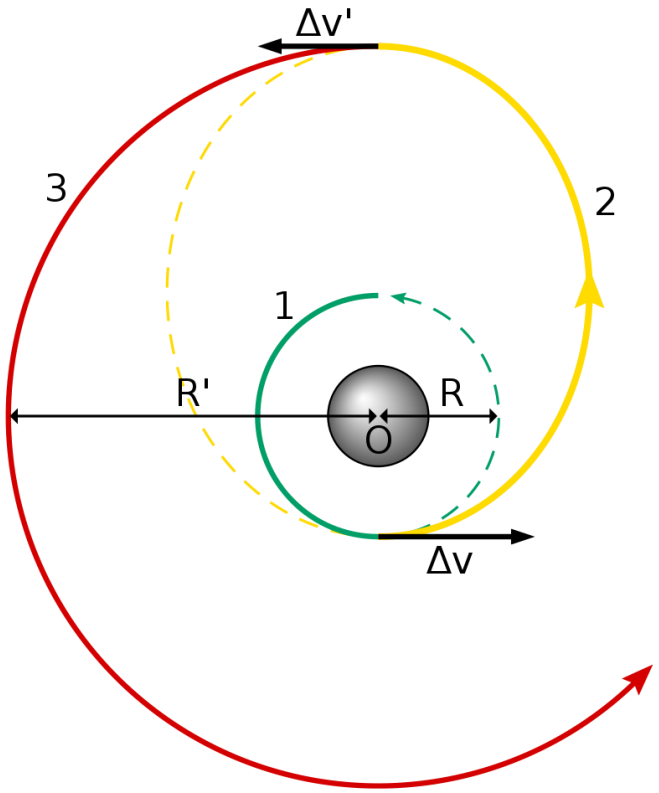
Step 6: Calculate the final burn to maneuver into target orbit.

$$\Delta v_2 = \vec{v}_{fin} - \vec{v}_{tr,2}$$

❑ Hohmann Transfer

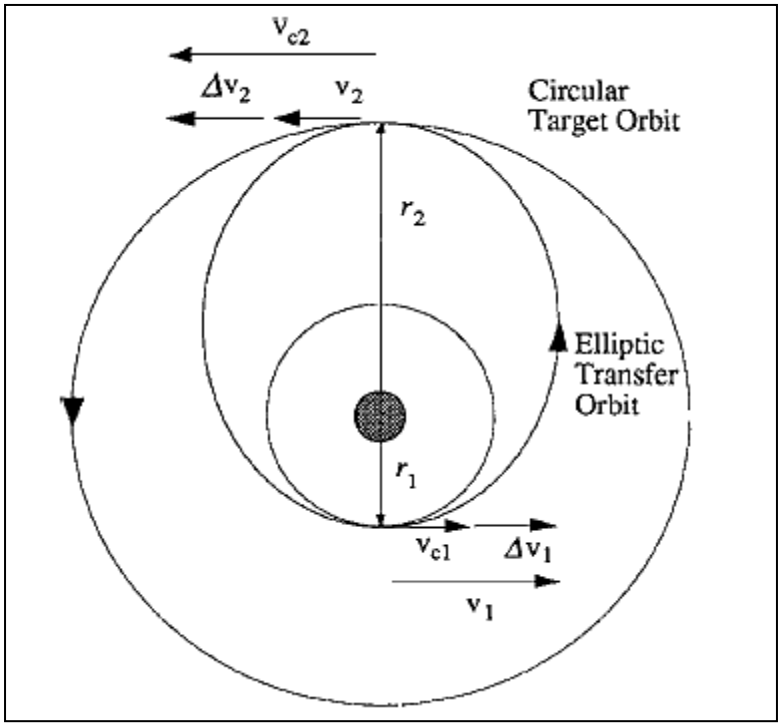
The Hohmann transfer is the energy-optimal two burn maneuver between any two coaxial elliptic orbits.

- Originally proposed by Hohmann (1925)
- Proved optimal for circular orbits (1952)
- Proved optimal for elliptic orbits (1986)



Orbit Transfer – Orbital Maneuvers

- **Example:** Consider an outward Hohmann transfer:



1. Speed increase from circular orbit to elliptical transfer at periapsis.
2. Speed increase at elliptical apoapsis to achieve circular target orbit

- Orbital elements of transfer orbit:

$$\begin{cases} a = \dfrac{r_p^e + r_a^e}{2} = r_1^c \\ e = \dfrac{r_p^e}{a} = \dfrac{r_2 - r_1}{r_2 + r_1} \end{cases}$$

$$v_{init} = \sqrt{\frac{\mu}{r_1}} \qquad v_{trans,p} = \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a}} = \sqrt{2\mu}\sqrt{\frac{1}{r_1} - \frac{1}{r_1 + r_2}} = \sqrt{2\mu\frac{r_2}{r_1(r_1 + r_2)}}$$

Orbit Transfer – Orbital Maneuvers

- Initial impulse Δv_1:

$$\Delta v_1 = v_{trans,p} - v_{init} = \sqrt{2\mu \frac{r_2}{r_1(r_1+r_2)}} - \sqrt{\frac{\mu}{r_1}} = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{(r_1+r_2)}} - 1\right)$$

- At apoapsis:

$$v_{trans,a} = \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{a}} = \sqrt{2\mu \frac{r_1}{r_2(r_1+r_2)}} \qquad v_{fin} = \sqrt{\frac{\mu}{r_2}}$$

- Final impulse Δv_2:

$$\Delta v_2 = v_{fin} - v_{trans,a} = \sqrt{\frac{\mu}{r_2}} - \sqrt{2\mu \frac{r_1}{r_2(r_1+r_2)}} = \sqrt{\frac{\mu}{r_2}}\left(1 - \sqrt{\frac{2r_1}{(r_1+r_2)}}\right)$$

The transfer time is simply half the period of the orbit. Hence

$$\Delta t = \frac{\tau}{2} = \pi\sqrt{\frac{a^3}{\mu}}$$

$$= \pi\sqrt{\frac{(r_1+r_2)^3}{8\mu}}$$

Orbit Transfer – Orbital Maneuvers

- **Problem:** Transfer a satellite from parking orbit at 200 Km, to a geostationary orbit with a period of 24 hours

$$\begin{cases} r_p^e = r_1 = 6378 + 200 = 6,578\,Km \Rightarrow e = 0.7305 \\ P^2 = (4\frac{\pi^2}{\mu}) r_2^3 \Rightarrow r_2 = 42,241\,Km \Rightarrow a = 24,409.55\,Km \end{cases}$$

$$\begin{cases} v_1 = 7.7843 \\ v_p = 10.240 \\ v_a = 1.5947 \\ v_2 = 3.0719 \end{cases}$$

$$\begin{cases} \Delta v_1 = 2.455\,Km/\sec \\ \Delta v_2 = 1.48\,Km/\sec \\ TOF = 5.27\,hours \end{cases}$$

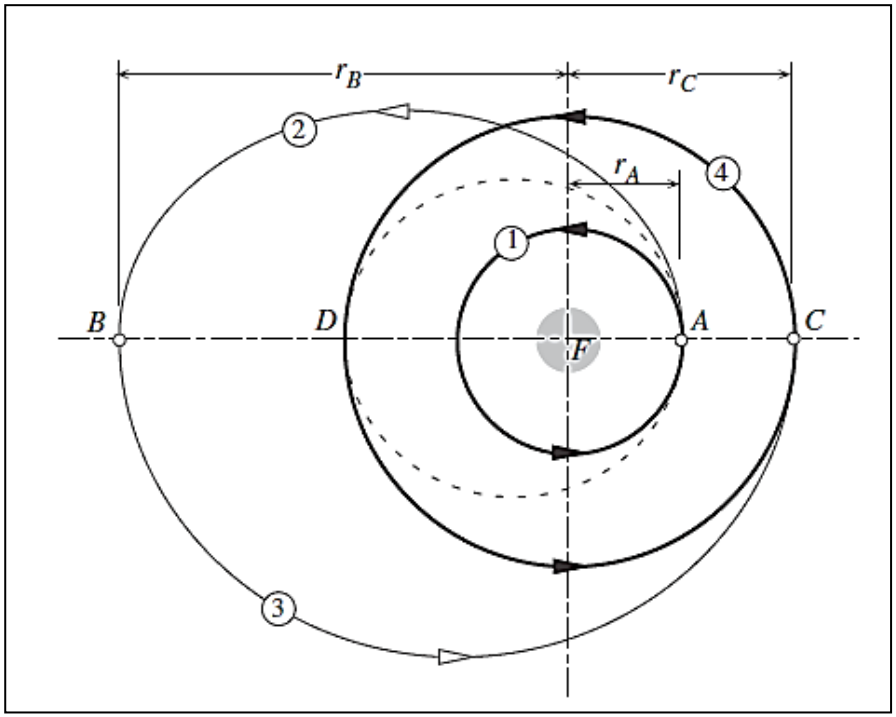
- **Properties of Hohmann Transfer:**

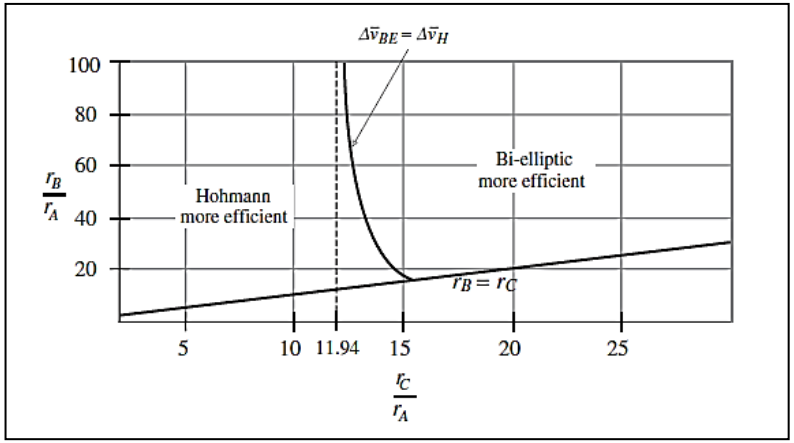
1. Optimal 2 – Impulse transfer in terms of fuel consumption for circular to circular and elliptic to elliptic
2. Has the longest time of flight
3. It can be suboptimal with respect to multiple impulse transfers

   - From Kinetic Energy Arguments $E_{COST} = \frac{\|\Delta v_1\|^2 + \|\Delta v_2\|^2}{2}$ $\Delta E_{MIN} = -\frac{\mu}{2a_2} + \frac{\mu}{2a_1}$

Orbit Transfer – Orbital Maneuvers

- **Bi-Elliptical Transfer:** There are situations in which addional fuel can be saved, at the expense of longer flight time.
  - One example is to raise the orbit of a satellite to a higher circular orbit





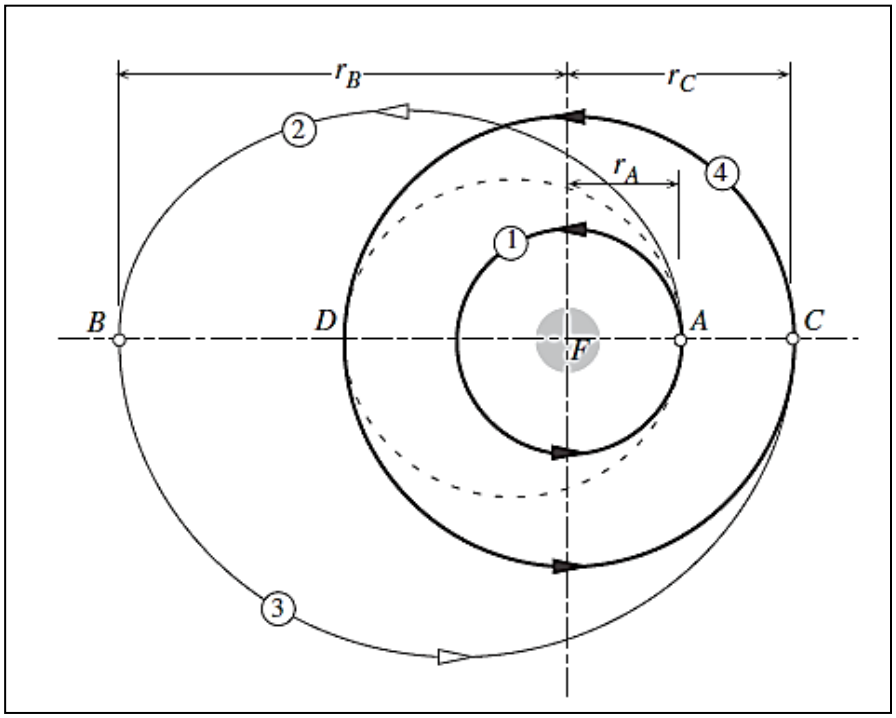
1. Original Parking Orbit (Circular)

$$v_{c1} = \sqrt{\frac{\mu}{r_A}}$$

2. First Elliptical Transfer (A – B)

$$\begin{cases} a_2 = \dfrac{r_A + r_B}{2} \\ e_2 = \dfrac{r_B - r_A}{r_A + r_B} \end{cases} \qquad \begin{cases} v_{P2} = \sqrt{2\mu \dfrac{r_B}{r_A \; r_A + r_B}} \\ v_{A2} = \sqrt{2\mu \dfrac{r_A}{r_B \; r_A + r_B}} \end{cases}$$

Orbit Transfer – Orbital Maneuvers



- **Problem:**

  Find the total delta-v requirement for a bi-elliptic Hohmann transfer from a geocentric circular orbit of 7000 km radius to one of 105,000 km radius. Let the apogee of the first ellipse be 210,000 km. Compare the delta-v, schedule and total flight time with that for an ordinary single Hohmann transfer ellipse.

3. Second Elliptical Transfer (B – C)

$$\begin{cases} a_3 = \dfrac{r_B + r_C}{2} \\ e_3 = \dfrac{r_B - r_C}{r_C + r_B} \end{cases} \qquad \begin{cases} v_{P3} = \sqrt{2\mu \dfrac{r_B}{r_C \; r_C + r_B}} \\ v_{A3} = \sqrt{2\mu \dfrac{r_C}{r_B \; r_C + r_B}} \end{cases}$$

4. Final Orbit (Circular)

$$v_{c2} = \sqrt{\frac{\mu}{r_C}}$$

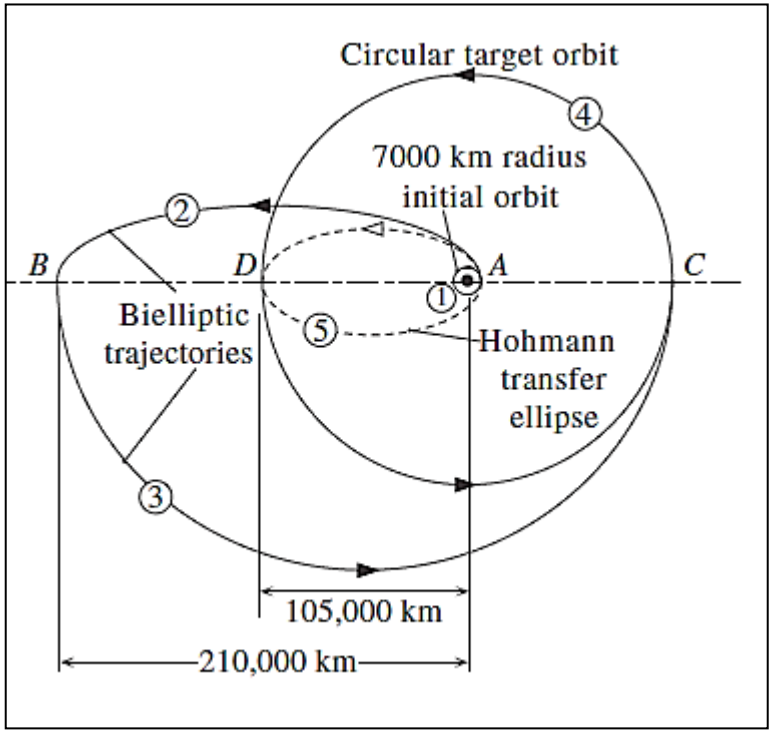
- Burn Sequence

$$\Delta v_1 = v_{P2} - v_{C1} \, (+)$$

$$\Delta v_2 = v_{A3} - v_{A2} \, (+)$$

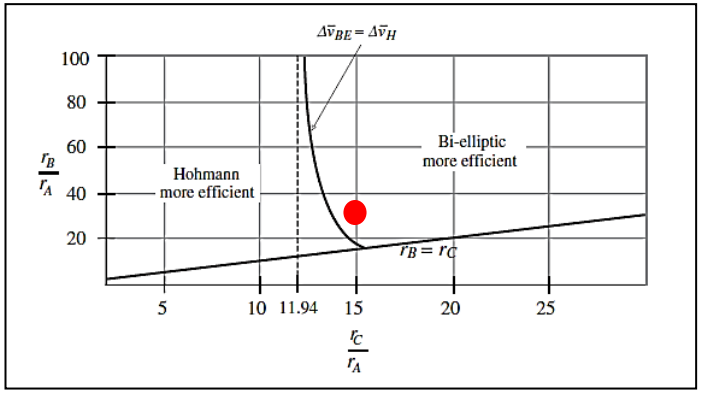
$$\Delta v_3 = v_{C2} - v_{A3} \, (-)$$

Orbit Transfer – Orbital Maneuvers



- Problem Data

$$r_{c1} = 7,000 Km \qquad \mu = 398,600 Km^3 / \sec^2$$
$$r_{c2} = 105,000 Km$$
$$r_{a1} = 210,000 Km$$
$$r_{p1} = r_{c1}$$
$$r_{a2} = r_{a1}$$
$$r_{p2} = r_{c2}$$



$$v_{c1} = \sqrt{\frac{\mu}{r_{c1}}} = 7.54 Km / \sec$$

$$\begin{cases} a_1 = \dfrac{r_{c1} + r_{a1}}{2} = 108,500\, Km \\ e_1 = \dfrac{r_{a1} - r_{c1}}{r_{c1} + r_{a1}} = 0.9355 \end{cases}$$

$$\begin{cases} h_1 = \sqrt{2\mu}\sqrt{\dfrac{r_{c1} r_{a1}}{r_{c1} + r_{a1}}} = 73,487\ Km^2 / \sec \\ E = -\dfrac{\mu}{2a} = -1.8369\ Km^4 / \sec^2 \end{cases}$$

$$\begin{cases} v_{P1} = \sqrt{2\mu \dfrac{r_B}{r_A\ r_A + r_B}} = \sqrt{2 \cdot 398,600 \dfrac{210,000}{7,000 \cdot 217,000}} = 10.4982 Km / \sec \\ v_{A1} = \sqrt{2\mu \dfrac{r_A}{r_B\ r_A + r_B}} = \sqrt{2 \cdot 398,600 \dfrac{7,000}{210,000 \cdot 217,000}} = 0.3499 Km / \sec \end{cases}$$

Orbit Transfer – Orbital Maneuvers

$$\begin{cases} a_2 = \dfrac{r_{c2} + r_{a1}}{2} = 157,500\ Km \\ e_2 = \dfrac{r_{a1} - r_{c2}}{r_{c2} + r_{a1}} = 0.333 \end{cases}$$

$$\begin{cases} h_2 = \sqrt{2\mu}\sqrt{\dfrac{r_{c2} r_{a1}}{r_{c2} + r_{a1}}} = 236,238\ Km^2 / \sec \\ E = -\dfrac{\mu}{2a} = -1.2654\ Km^4 / \sec^2 \end{cases}$$

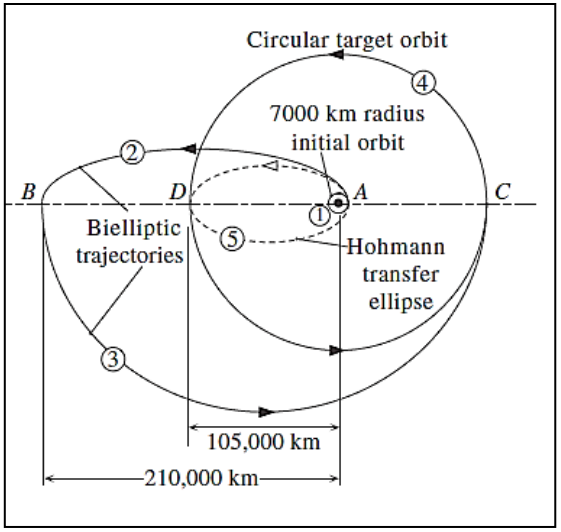
$$\begin{cases} v_{P2} = \dfrac{h_2}{r_{c2}} = \dfrac{236,238}{105,000} = 2.24992 Km / \sec \\ v_{A2} = \dfrac{h_2}{r_{a1}} = \dfrac{236,238}{210,000} = 1.1249 Km / \sec \end{cases}$$

$$v_{c2} = \sqrt{\frac{\mu}{r_{c1}}} = 1.9484 Km / \sec$$

$$\Delta v = \left|10.4982 - 7.54\right| + \left|1.1249 - 0.3489\right| + \left|1.9484 - 2.24992\right| = 3.4327\ Km / \sec$$

$$TOF = 5.66\ days$$

- Consider now a 2 impulse Hohmann



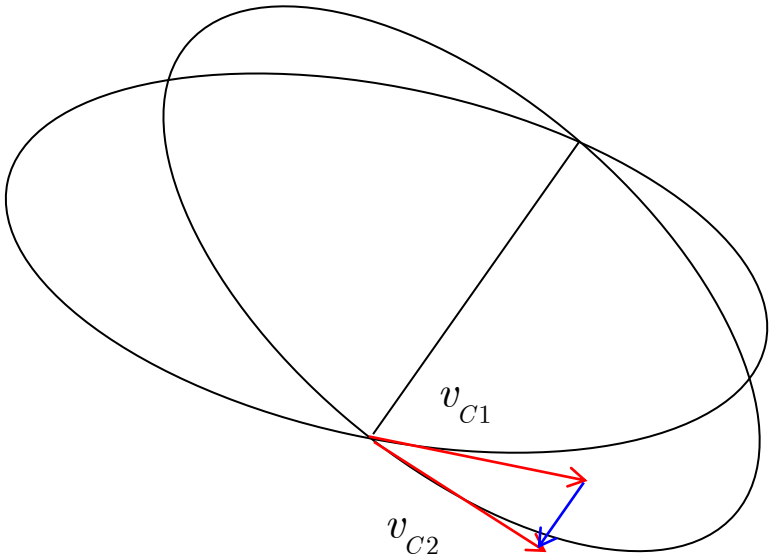
$$\begin{cases} a_H = \dfrac{r_p + r_a}{2} = 56,000\ Km \\ e_H = \dfrac{r_a - r_p}{r_a + r_p} = 0.875 \end{cases}$$

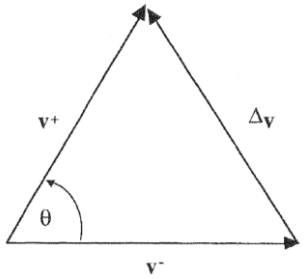
$$\begin{cases} h = 72,330\ Km^2 / \sec \\ v_p = 10.333\ Km / \sec \\ v_a = 0.68886\ Km / \sec \\ TOF = 0.763\ days \end{cases}$$

$$\Delta v = \left|10.333 - 7.54\right| + \left|1.9484 - 0.68886\right| = 4.0463\ Km / \sec$$

Orbit Transfer – Orbital Maneuvers

- Orbital inclination change is an orbital maneuver aimed at **changing the inclination** of an orbiting body's orbit. This maneuver is also known as an orbital plane change as the plane of the orbit is tipped. This maneuver requires a change in the orbital velocity vector at the orbital nodes.
- In general, inclination changes can require a great deal of delta-v to perform, and most mission planners try to avoid them whenever possible to conserve fuel.
- Maximum efficiency of inclination change is achieved at apoapsis, where orbital velocity is the lowest. In some cases, it may require less total delta-v to raise the satellite into a higher orbit, change the orbit plane at the higher apogee, and then lower the satellite to its original altitude.





$$\Delta v = 2v \sin\frac{\theta}{2} \quad \theta = \Delta i$$

The direction of trust is: $90^\circ + \frac{\theta}{2}$

- Out of Plane maneuvers can also be used to **change both inclination and Longitude of the Ascending Node Ω (RAAN)**

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