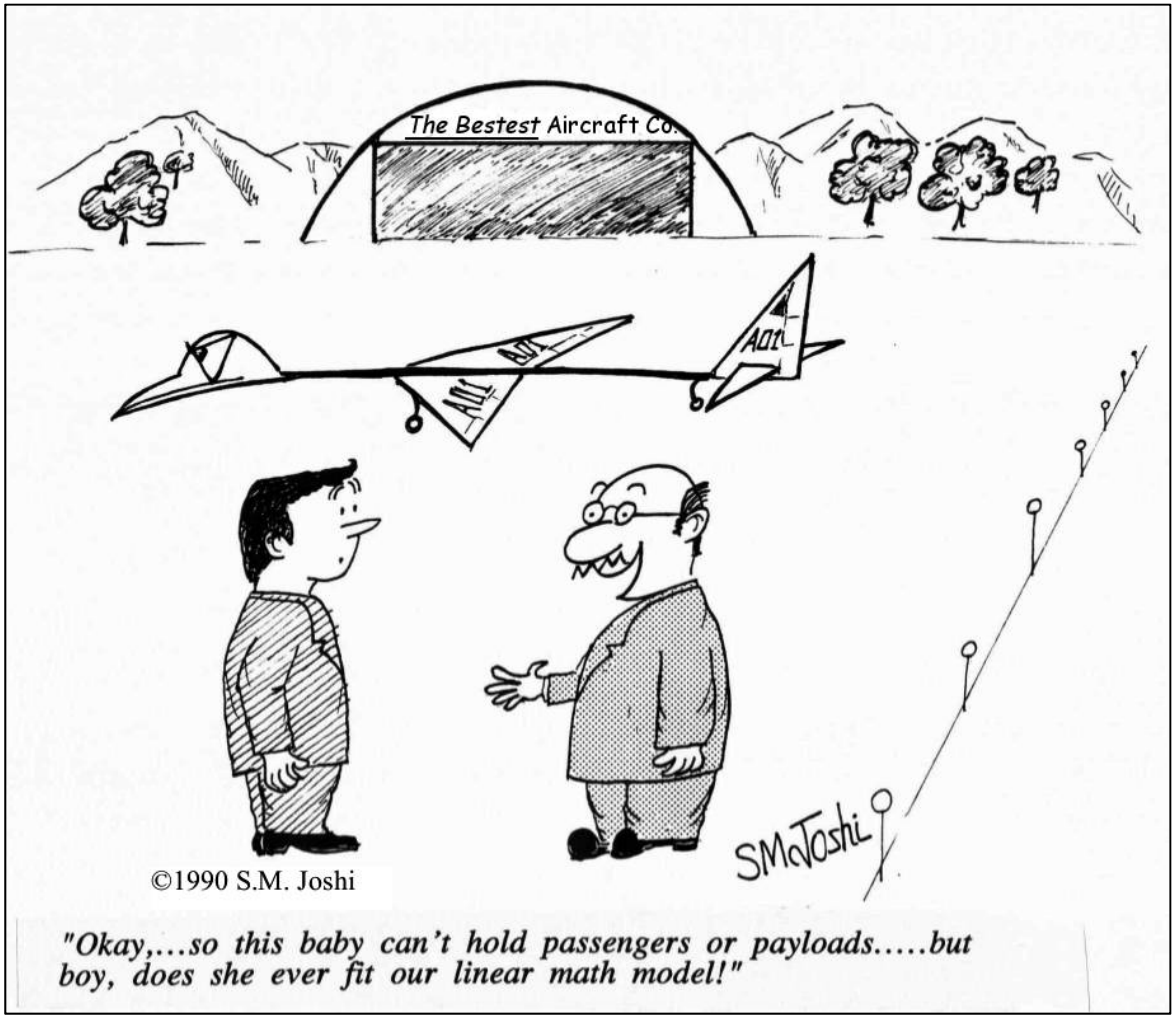
❑ Miracles of Linearization



❑ Linearization of the Equations of Motion

- The complete set Nonlinear Equations of motion are used for advanced simulation purposes. Control Analysis and Synthesis requires simplified models obtained from knowledge of operating conditions and linearization.

1. The first step is the extraction of the gravity contribution from the other forces and moments

$$\begin{cases} m\left[\dot{u} - vr + wq\right] = X - mg\sin\Theta \\ m\left[\dot{v} - wp + ur\right] = Y + mg\cos\Theta\sin\Phi \\ m\left[\dot{w} - uq + vp\right] = Z + mg\cos\Theta\cos\Phi \end{cases} \qquad \begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{z}_I \end{bmatrix} = \begin{bmatrix} c\Theta c\Psi & -c\Phi s\Psi + s\Phi s\Theta c\Psi & s\Phi s\Psi + c\Phi s\Theta c\Psi \\ c\Theta s\Psi & c\Phi c\Psi + s\Phi s\Theta s\Psi & -s\Phi c\Psi + c\Phi s\Theta s\Psi \\ -s\Theta & s\Phi c\Theta & c\Phi c\Theta \end{bmatrix} \cdot \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

2. The second step is the elimination of terms due to xz plane symmetry of the vehicle ($I_{yz} = I_{yx} = 0$)

$$\begin{cases} I_{xx}\dot{p} + (I_{zz} - I_{yy})qr - (\dot{r} + pq)I_{xz} = L \\ I_{yy}\dot{q} + (I_{xx} - I_{zz})rp + (p^2 - r^2)I_{xz} = M \\ I_{zz}\dot{r} + (I_{yy} - I_{xx})pq + (rq - \dot{p})I_{xz} = N \end{cases} \qquad \begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\Phi\tan\Theta & \cos\Phi\tan\Theta \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & \dfrac{\sin\Phi}{\cos\Theta} & \dfrac{\cos\Phi}{\cos\Theta} \end{bmatrix} \cdot \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

- **Note: Control forces and moments are the combined effect of 4 primary contributions:**
  - **Aerodynamic**: Elevator (pitch, δ_E), Ailerons (roll, δ_A), Rudder (yaw, δ_R)
  - **Propulsion**: engine (primary longitudinal, δ_T)

$$M_{RB}\dot{\boldsymbol{\nu}} + C_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \tau_{RB} \Leftrightarrow \dot{\boldsymbol{x}} = \boldsymbol{f}[\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}_d(t)]$$

3. The third step is the separation between longitudinal and latero-directional states, controls, and disturbances.

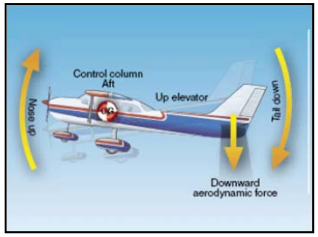
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \hline x_4 \\ x_5 \\ x_6 \\ \hline x_7 \\ x_8 \\ x_9 \\ \hline x_{10} \\ x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \\ \hline x \\ y \\ z \\ \hline p \\ q \\ r \\ \hline \phi \\ \theta \\ \psi \end{bmatrix}$$

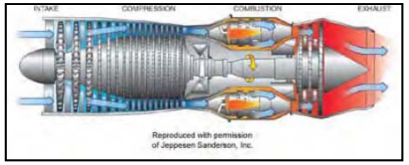
First six elements of the state are longitudinal variables

Second six elements of the state are lateral-directional variables

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ \hline x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{Lon} \\ \hline \mathbf{x}_{Lat-Dir} \end{bmatrix} = \begin{bmatrix} u \\ w \\ x \\ z \\ q \\ \theta \\ \hline v \\ y \\ p \\ r \\ \phi \\ \psi \end{bmatrix}$$

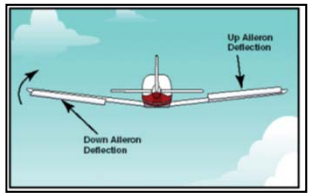
δ_E



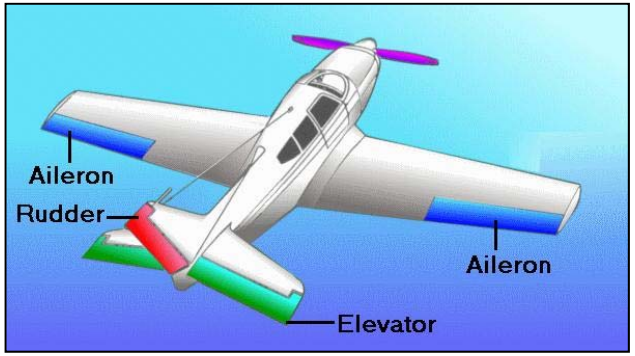


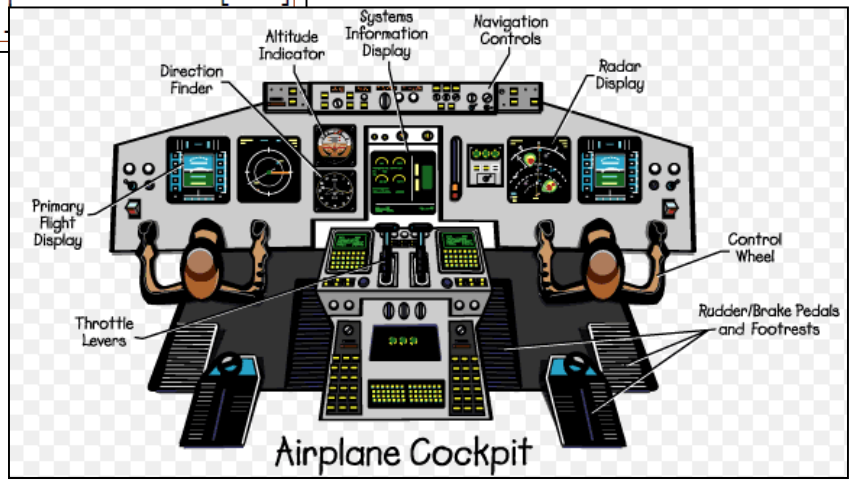
δ_T

δ_A



δ_R





- The fourth step is the linearization about some equilibrium condition (or trim condition) within the operational envelope of the vehicle.
  - **The standard procedure is to identify all the variables of motion, to expand the equations in a Taylor series, retaining the first order derivatives only, and to subtract the equilibrium condition.**

- Rewrite the equations of motion, neglecting the kinematic translation for now

$$\begin{cases} \dot{u} = \dfrac{1}{m}\left[X + vr - wq - mg\sin\Theta\right] \\ \dot{w} = \dfrac{1}{m}\left[Z + uq - vp + mg\cos\Theta\cos\Phi\right] \\ \dot{q} = \dfrac{1}{I_{yy}}\left[M - (I_{xx} - I_{zz})rp - (p^2 - r^2)I_{zx}\right] \end{cases}$$

$$\begin{cases} \dot{v} = \dfrac{1}{m}\left[Y + wp - ur + mg\cos\Theta\sin\Phi\right] \\ \dot{p} = \dfrac{1}{I_{xx}}\left[L - (I_{zz} - I_{yy})qr + (\dot{r} + pq)I_{xz}\right] \\ \dot{r} = \dfrac{1}{I_{zz}}\left[N - (I_{yy} - I_{xx})pq - (rq - \dot{p})I_{zx}\right] \end{cases}$$

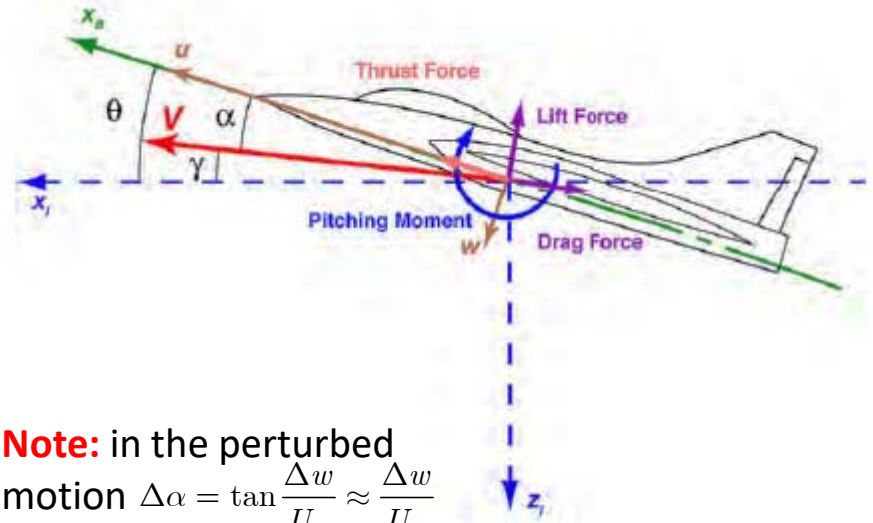
$$\dot{\Theta} = q\cos\Phi - r\sin\Phi$$

$$\dot{\Phi} = p + q\sin\Phi\tan\Theta + r\cos\Phi\tan\Theta$$

$$\dot{\Psi} = q\frac{\sin\Phi}{\cos\Theta} + r\frac{\cos\Phi}{\cos\Theta}$$

- The equilibrium condition about which to linearize is not unique. $\boxed{\dot{\boldsymbol{x}}_{\boldsymbol{E}} = \boldsymbol{f}[\boldsymbol{x}_{\boldsymbol{E}}(t), \boldsymbol{u}_{\boldsymbol{E}}(t)] = 0}$
  - Longitudinal steady flight (constant speed, climb, cruise, descend))
  - Longitudinal steady pull – up maneuver (constant centripetal acceleration)
  - Coordinated Turn (steady turn at zero sideslip)
  - Helical turn
  - …

- **Longitudinal Steady Flight Condition:** Linearization about steady, longitudinal flight (climbing, cruising, descending at constant speed), and small perturbations from equilibrium:



Note: in the perturbed motion $\Delta\alpha = \tan\frac{\Delta w}{U_0} \approx \frac{\Delta w}{U_0}$

$$\dot{\boldsymbol{x}}_0 + \Delta\dot{\boldsymbol{x}} = f[\boldsymbol{x}_0, \boldsymbol{u}_0] + \left.\frac{\Delta f}{\Delta \boldsymbol{x}}\right|_0 \Delta\boldsymbol{x} + \left.\frac{\Delta f}{\Delta \boldsymbol{u}}\right|_0 \Delta\boldsymbol{u} + \ldots$$

$$V_0 = 0 \quad R_0 = 0$$
$$P_0 = 0 \quad \Phi_0 = 0$$
$$Q_0 = 0 \quad \Psi_0 = 0$$

*$W_0 = 0$ if Wind axes are used

$$\begin{cases} u = U_0 + \Delta u \\ v = \Delta v \\ w^* = W_0 + \Delta w \end{cases}; \begin{cases} p = \Delta p \\ q = \Delta q \\ r = \Delta r \end{cases}; \begin{cases} \Phi = \Delta\varphi \\ \Theta = \Theta_0 + \Delta\theta \\ \Psi = \Delta\psi \end{cases}$$

- Taylor Series expansion of all applied forces and moments, with respect to the implicit and explicit variables of motion, yields a number of STABILITY DERIVATIVES that describe their perturbation from equilibrium (Bryan Hypothesis)

$$A = A_0 + \Delta A(u, v, w, p, q, r, \delta_E, \delta_A, \delta_R, \delta_T, \dot{u}, \dot{v}, \dot{w}, \dot{p}, \dot{q}, \dot{r}, \ldots)$$
$$\Delta A \approx A_u u + A_v v + \ldots + A_p p + A_q q + \ldots + A_{\delta_E}\delta_E + \ldots$$
$$A_u = \frac{\partial A}{\partial u}, \ldots$$

$$\begin{cases} m\left[\dot{u} - vr + wq\right] = X - mg\sin\Theta \\ m\left[\dot{v} - wp + ur\right] = Y + mg\cos\Theta\sin\Phi \\ m\left[\dot{w} - uq + vp\right] = Z + mg\cos\Theta\cos\Phi \end{cases} \qquad \begin{cases} I_{xx}\dot{p} + (I_{zz} - I_{yy})qr - (\dot{r} + pq)I_{xz} = L \\ I_{yy}\dot{q} + (I_{xx} - I_{zz})rp + (p^2 - r^2)I_{zx} = M \\ I_{zz}\dot{r} + (I_{yy} - I_{xx})pq + (rq - \dot{p})I_{zx} = N \end{cases}$$

- **Linearize the inertia part of the equations:** as an example consider the component in the Xbody direction

$$X - mg\sin\theta = m(\dot{u} + qw - rv)$$

$$X_0 + \Delta X - mg\sin(\theta_0 + \Delta\theta) = m\left[\frac{d}{dt}(u_0 + \Delta u) + (q_0 + \Delta q)(w_0 + \Delta w) - (r_0 + \Delta r)(v_0 + \Delta v)\right]$$

$$w_0 = v_0 = p_0 = q_0 = r_0 = \Phi_0 = \psi_0 = 0$$

$$X_0 + \Delta X - mg\sin(\theta_0 + \Delta\theta) = m\,\Delta\dot{u}$$

$$\sin(\theta_0 + \Delta\theta) = \sin\theta_0\cos\Delta\theta + \cos\theta_0\sin\Delta\theta = \sin\theta_0 + \Delta\theta\cos\theta_0$$

$$X_0 + \Delta X - mg(\sin\theta_0 + \Delta\theta\cos\theta_0) = m\,\Delta\dot{u}$$

$$X_0 - mg\sin\theta_0 = 0$$

$$\boxed{\Delta X - mg\,\Delta\theta\cos\theta_0 = m\,\Delta\dot{u}}$$

$$\begin{cases} \Delta\dot{u} = \dfrac{1}{m}\left[\Delta X - mg\Delta\theta\cos\Theta_0\right] \\ \Delta\dot{w} - \dfrac{U_0}{m}\Delta q = \dfrac{1}{m}\left[\Delta Z - mg\Delta\theta\sin\Theta_0\right] \\ \Delta\dot{q} = \dfrac{1}{I_{yy}}\Delta M \\ \Delta\dot{\theta} = \Delta q \end{cases}$$

$$\begin{cases} \Delta\dot{v} + U_0\Delta r = \dfrac{1}{m}\left[\Delta Y + mg\Delta\phi\cos\Theta_0\right] \\ I_{xx}\Delta\dot{p} - I_{zx}\Delta\dot{r} = \Delta L \\ I_{zz}\Delta\dot{r} - I_{xz}\Delta\dot{p} = \Delta N \\ \Delta\dot{\phi} = p + r\tan\Theta_0 \\ \Delta\dot{\psi} = r\sec\Theta_0 \end{cases}$$

- **Linearize the Forces and Moments part of the equations, using Bryan Hypotesis:**

$$\Delta X = \frac{\partial X}{\partial u}\Delta u + \frac{\partial X}{\partial w}\Delta w + \frac{\partial X}{\partial \delta_e}\Delta\delta_e + \frac{\partial X}{\partial \delta_T}\Delta\delta_T$$

$$\Delta\alpha = \tan\frac{\Delta w}{U_0} \approx \frac{\Delta w}{U_0}$$

$$\Delta\beta \approx \frac{\Delta v}{U_0}$$

- **For all six equations:**

$$\Delta X = \frac{\partial X}{\partial u}u + \frac{\partial X}{\partial w}w + \frac{\partial X}{\partial \delta_e}\delta_e + \frac{\partial X}{\partial \delta_T}\delta_T$$
$$\Delta Z = \frac{\partial Z}{\partial u}u + \frac{\partial Z}{\partial w}w + \frac{\partial Z}{\partial \dot{w}}\dot{w} + \frac{\partial Z}{\partial q}q + \frac{\partial Z}{\partial \delta_e}\delta_e + \frac{\partial Z}{\partial \delta_T}\delta_T$$
$$\Delta M = \frac{\partial M}{\partial u}u + \frac{\partial M}{\partial w}w + \frac{\partial M}{\partial \dot{w}}\dot{w} + \frac{\partial M}{\partial q}q + \frac{\partial M}{\partial \delta_e}\delta_e + \frac{\partial M}{\partial \delta_T}\delta_T$$

$$X_u \equiv \frac{1}{m}\frac{\partial X}{\partial u}, \quad X_w \equiv \frac{1}{m}\frac{\partial X}{\partial w}, \quad \dots \quad X_{\delta_T} \equiv \frac{1}{m}\frac{\partial X}{\partial \delta_T};$$
$$Y_v \equiv \frac{1}{m}\frac{\partial Y}{\partial v}, \quad Y_p \equiv \frac{1}{m}\frac{\partial Y}{\partial p}, \quad \dots \quad Y_{\delta_r} \equiv \frac{1}{m}\frac{\partial Y}{\partial \delta_r};$$
$$Z_u \equiv \frac{1}{m}\frac{\partial Z}{\partial u}, \quad Z_w \equiv \frac{1}{m}\frac{\partial Z}{\partial w}, \quad \dots \quad Z_{\delta_T} \equiv \frac{1}{m}\frac{\partial Z}{\partial \delta_T};$$

$$\Delta Y = \frac{\partial Y}{\partial v}v + \frac{\partial Y}{\partial p}p + \frac{\partial Y}{\partial r}r + \frac{\partial Y}{\partial \delta_r}\delta_r$$
$$\Delta L = \frac{\partial L}{\partial v}v + \frac{\partial L}{\partial p}p + \frac{\partial L}{\partial r}r + \frac{\partial L}{\partial \delta_r}\delta_r + \frac{\partial L}{\partial \delta_a}\delta_a$$
$$\Delta N = \frac{\partial N}{\partial v}v + \frac{\partial N}{\partial p}p + \frac{\partial N}{\partial r}r + \frac{\partial N}{\partial \delta_r}\delta_r + \frac{\partial N}{\partial \delta_a}\delta_a$$

$$L_v \equiv \frac{1}{I_x}\frac{\partial L}{\partial v}, \quad L_p \equiv \frac{1}{I_x}\frac{\partial L}{\partial p}, \quad \dots \quad L_{\delta_a} \equiv \frac{1}{I_x}\frac{\partial L}{\partial \delta_a};$$
$$M_u \equiv \frac{1}{I_y}\frac{\partial M}{\partial u}, \quad M_w \equiv \frac{1}{I_y}\frac{\partial M}{\partial w}, \quad \dots \quad M_{\delta_T} \equiv \frac{1}{I_y}\frac{\partial M}{\partial \delta_T};$$
$$N_v \equiv \frac{1}{I_z}\frac{\partial N}{\partial v}, \quad N_p \equiv \frac{1}{I_z}\frac{\partial N}{\partial p}, \quad \dots \quad N_{\delta_a} \equiv \frac{1}{I_z}\frac{\partial N}{\partial \delta_a}.$$

- **Not all the stability derivatives are equally important in the linearized motion**

- Linearized (perturbed) equations of motion (not including range and altitude) become:

Longitudinal Dynamics

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} - X_u\right] u + g_0 \cos\Theta_0 \theta - X_w w = X_{\delta_e}\delta_e + X_{\delta_T}\delta_T$$

$$-Z_u u + \left[(1 - Z_{\dot{w}})\frac{\mathrm{d}}{\mathrm{d}t} - Z_w\right] w - [u_0 + Z_q] q + g_0 \sin\Theta_0 \theta = Z_{\delta_e}\delta_e + Z_{\delta_T}\delta_T$$

$$-M_u u - \left[M_{\dot{w}}\frac{\mathrm{d}}{\mathrm{d}t} + M_w\right] w + \left[\frac{\mathrm{d}}{\mathrm{d}t} - M_q\right] q = M_{\delta_e}\delta_e + M_{\delta_T}\delta_T$$

Longitudinal Rotational Kinematics

$$\boxed{\dot{\theta} = q}$$

$$\mathbf{A} = \begin{pmatrix} X_u & X_w & 0 & -g_0\cos\Theta_0 \\ \frac{Z_u}{1-Z_{\dot{w}}} & \frac{Z_w}{1-Z_{\dot{w}}} & \frac{u_0+Z_q}{1-Z_{\dot{w}}} & \frac{-g_0\sin\Theta_0}{1-Z_{\dot{w}}} \\ M_u + \frac{M_{\dot{w}}Z_u}{1-Z_{\dot{w}}} & M_w + \frac{M_{\dot{w}}Z_w}{1-Z_{\dot{w}}} & M_q + \frac{(u_0+Z_q)M_{\dot{w}}}{1-Z_{\dot{w}}} & \frac{-M_{\dot{w}}g_0\sin\Theta_0}{1-Z_{\dot{w}}} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} X_{\delta_e} & X_{\delta_T} \\ \frac{Z_{\delta_e}}{1-Z_{\dot{w}}} & \frac{Z_{\delta_T}}{1-Z_{\dot{w}}} \\ M_{\delta_e} + \frac{M_{\dot{w}}Z_{\delta_e}}{1-Z_{\dot{w}}} & M_{\delta_T} + \frac{M_{\dot{w}}Z_{\delta_T}}{1-Z_{\dot{w}}} \\ 0 & 0 \end{pmatrix}$$

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$

$$\boldsymbol{x}(t) = \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix}; \boldsymbol{u}(t) = \begin{bmatrix} \delta_E \\ \delta_T \end{bmatrix}$$

Note: The use w or $\alpha = w/U_0$ as state variable is equivalent

$$\boldsymbol{x}(t) = \begin{bmatrix} u \\ \alpha \\ q \\ \theta \end{bmatrix}$$

- Linearized (perturbed) equations of motion (not including range and altitude) become:

Lateral – Directional Dynamics

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} - Y_v\right] v - Y_p p + [u_0 - Y_r] r - g_0 \cos\Theta_0 \phi = Y_{\delta_r}\delta_r$$

$$-L_v v + \left[\frac{\mathrm{d}}{\mathrm{d}t} - L_p\right] p - \left[\frac{I_{xz}}{I_x}\frac{\mathrm{d}}{\mathrm{d}t} + L_r\right] r = L_{\delta_r}\delta_r + L_{\delta_a}\delta_a$$

$$-N_v v - \left[\frac{I_{xz}}{I_z}\frac{\mathrm{d}}{\mathrm{d}t} + N_p\right] p + \left[\frac{\mathrm{d}}{\mathrm{d}t} - N_r\right] r = N_{\delta_r}\delta_r + N_{\delta_a}\delta_a$$

$$\mathbf{A} = \begin{pmatrix} Y_v & Y_p & g_0\cos\Theta_0 & Y_r - u_0 \\ \frac{L_v + i_x N_v}{1 - i_x i_z} & \frac{L_p + i_x N_p}{1 - i_x i_z} & 0 & \frac{L_r + i_x N_r}{1 - i_x i_z} \\ 0 & 1 & 0 & 0 \\ \frac{N_v + i_z L_v}{1 - i_x i_z} & \frac{N_p + i_z L_p}{1 - i_x i_z} & 0 & \frac{N_r + i_z L_r}{1 - i_x i_z} \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} Y_{\delta_r} & 0 \\ \frac{L_{\delta_r} + i_x N_{\delta_r}}{1 - i_x i_z} & \frac{L_{\delta_a} + i_x N_{\delta_a}}{1 - i_x i_z} \\ 0 & 0 \\ \frac{N_{\delta_r} + i_z L_{\delta_r}}{1 - i_x i_z} & \frac{N_{\delta_a} + i_z L_{\delta_a}}{1 - i_x i_z} \end{pmatrix}$$

Lateral – Directional Rotational Kinematics

$$\begin{cases} \dot{\phi} = p + r\cos\theta_0 \\ \dot{\psi} = \dfrac{r}{\cos\theta_0} \end{cases}$$

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$

$$\boldsymbol{x}(t) = \begin{bmatrix} v \\ p \\ \phi \\ r \end{bmatrix}; \boldsymbol{u}(t) = \begin{bmatrix} \delta_A \\ \delta_R \end{bmatrix}$$

Note: The use v or $\beta = v/U_0$ as state variable is equivalent

$$\boldsymbol{x}(t) = \begin{bmatrix} \beta \\ p \\ \phi \\ r \end{bmatrix}$$

- **Relationship between stability derivatives and applied forces and moments**.
  1. Stability derivatives contain the contribution from all forces and moments due to perturbation from trim condition of all motion variables.
  2. They are evaluated at each selected equilibrium point (because Taylor series expansion)

X_u	$\frac{1}{m}\frac{\partial X}{\partial u}$	$\frac{1}{\text{sec}}$	$\frac{\rho SU}{m}(-C_D - C_{D_u})^a$
X_w	$\frac{1}{m}\frac{\partial X}{\partial w}$	$\frac{1}{\text{sec}}$	$\frac{\rho SU}{2m}(C_L - C_{D_\alpha})$
X_δ	$\frac{1}{m}\frac{\partial X}{\partial \delta}$	$\frac{\text{ft}}{\text{rad-sec}^2}$	$\frac{\rho SU^2}{2m}(-C_{D_\delta})$
Z_u	$\frac{1}{m}\frac{\partial Z}{\partial u}$	$\frac{1}{\text{sec}}$	$\frac{\rho SU}{m}(-C_L - C_{L_u})^c$
Z_w	$\frac{1}{m}\frac{\partial Z}{\partial w}$	$\frac{1}{\text{sec}}$	$\frac{\rho SU}{2m}(-C_{L_\alpha} - C_D)$
$Z_{\dot{w}}$	$\frac{1}{m}\frac{\partial Z}{\partial \dot{w}}$	$\frac{1}{1}$	$\frac{\rho Sc}{4m}(-C_{L_{\dot{\alpha}}})$
Z_q	$\frac{1}{m}\frac{\partial Z}{\partial q}$	$\frac{\text{ft}}{\text{rad-sec}}$	$\frac{\rho SUc}{4m}(-C_{L_q})$
Z_δ	$\frac{1}{m}\frac{\partial Z}{\partial \delta}$	$\frac{\text{ft}}{\text{rad-sec}^2}$	$\frac{\rho SU^2}{2m}(-C_{L_\delta})$
M_u	$\frac{1}{I_y}\frac{\partial M}{\partial u}$	$\frac{1}{\text{ft-sec}}$	$\frac{\rho SUc}{I_y}(C_M + C_{M_u})$
M_w	$\frac{1}{I_y}\frac{\partial M}{\partial w}$	$\frac{1}{\text{ft-sec}}$	$\frac{\rho SUc}{2I_y}C_{M_\alpha}$
$M_{\dot{w}}$	$\frac{1}{I_y}\frac{\partial M}{\partial \dot{w}}$	$\frac{1}{\text{ft}}$	$\frac{\rho Sc^2}{4I_y}C_{M_{\dot{\alpha}}}$
M_q	$\frac{1}{I_y}\frac{\partial M}{\partial q}$	$\frac{1}{\text{sec}}$	$\frac{\rho SUc^2}{4I_y}C_{M_q}$
M_δ	$\frac{1}{I_y}\frac{\partial M}{\partial \delta}$	$\frac{1}{\text{rad-sec}^2}$	$\frac{\rho SU^2c}{2I_y}C_{M_\delta}$

Y_v	$\frac{1}{mU}\frac{\partial Y}{\partial \beta}$	$\frac{1}{\text{sec}}$	$\frac{\rho SU}{2m}C_{Y_\beta}$
$Y_{\dot{v}}$	$\frac{1}{mU}\frac{\partial Y}{\partial \dot{\beta}}$	$\frac{1}{1}$	$\frac{\rho Sb}{4m}C_{Y_{\dot{\beta}}}$
Y_r^*	$\frac{1}{mU}\frac{\partial Y}{\partial r}$	$\frac{1}{\text{rad}}$	$\frac{\rho Sb}{4m}C_{Y_r}$
Y_p^*	$\frac{1}{mU}\frac{\partial Y}{\partial p}$	$\frac{1}{\text{rad}}$	$\frac{\rho Sb}{4m}C_{Y_p}$
Y_δ^*	$\frac{1}{mU}\frac{\partial Y}{\partial \delta}$	$\frac{1}{\text{rad-sec}}$	$\frac{\rho SU}{2m}C_{Y_\delta}$
N_β	$\frac{1}{I_z}\frac{\partial N}{\partial \beta}$	$\frac{1}{\text{sec}^2}$	$\frac{\rho SU^2b}{2I_z}C_{n_\beta}$
$N_{\dot{\beta}}$	$\frac{1}{I_z}\frac{\partial N}{\partial \dot{\beta}}$	$\frac{1}{\text{sec}}$	$\frac{\rho SUb^2}{4I_z}C_{n_{\dot{\beta}}}$
N_r	$\frac{1}{I_z}\frac{\partial N}{\partial r}$	$\frac{1}{\text{sec}}$	$\frac{\rho SUb^2}{4I_z}C_{n_r}$
N_p	$\frac{1}{I_z}\frac{\partial N}{\partial p}$	$\frac{1}{\text{sec}}$	$\frac{\rho SUb^2}{4I_z}C_{n_p}$
N_δ	$\frac{1}{I_z}\frac{\partial N}{\partial \delta}$	$\frac{1}{\text{rad-sec}^2}$	$\frac{\rho SU^2b}{2I_z}C_{n_\delta}$
L_β	$\frac{1}{I_x}\frac{\partial L}{\partial \beta}$	$\frac{1}{\text{sec}^2}$	$\frac{\rho SU^2b}{2I_x}C_{l_\beta}$
$L_{\dot{\beta}}$	$\frac{1}{I_x}\frac{\partial L}{\partial \dot{\beta}}$	$\frac{1}{\text{sec}}$	$\frac{\rho SUb^2}{4I_x}C_{l_{\dot{\beta}}}$
L_r	$\frac{1}{I_x}\frac{\partial L}{\partial r}$	$\frac{1}{\text{sec}}$	$\frac{\rho SUb^2}{4I_x}C_{l_r}$
L_p	$\frac{1}{I_x}\frac{\partial L}{\partial p}$	$\frac{1}{\text{sec}}$	$\frac{\rho SUb^2}{4I_x}C_{l_p}$
L_δ	$\frac{1}{I_x}\frac{\partial L}{\partial \delta}$	$\frac{1}{\text{rad-sec}^2}$	$\frac{\rho SU^2b}{2I_x}C_{l_\delta}$

- Example # 1: $$Z_w = -\frac{(C_{L\alpha} + C_{D0})QS}{mU_0}$$

- Example # 2:

$$\frac{\Delta M}{I_{yy}} = \frac{1}{2I_{yy}} C_m \rho S V^2 \overline{c} = M_u u + M_w w + M_{\dot{w}} \dot{w} + M_q q + M_{\delta_E} \delta_E$$

$$V \cong U_0 + u; w \simeq \frac{\alpha}{U_0}$$

$$M_w = \frac{1}{I_{yy}} \frac{\partial M}{\partial w}\bigg|_0 = \frac{1}{I_{yy} U_0} \frac{\partial M}{\partial \alpha}\bigg|_0 = \frac{1}{2I_{yy} U_0} \frac{\partial C_m \rho S U_0^2 \overline{c}}{\partial \alpha}\bigg|_0 =$$

$$= \frac{\rho S U_0 \overline{c}}{2I_{yy}} \frac{\partial C_m}{\partial \alpha}\bigg|_0 = \frac{\rho S U_0 \overline{c}}{2I_{yy}} C_{m\alpha}$$

$$M_{\delta_E} = \frac{\rho S U_0^2 \overline{c}}{2I_{yy}} C_{m\delta_E}$$

$$L_v = \frac{QSb}{I_x u_0} \mathbf{C}_{l\beta} \quad \Big| \quad N_v = \frac{QSb}{I_z u_0} \mathbf{C}_{n\beta}$$

- Example of linearized longitudinal equations using '**non dimensional**' aerodynamic coefficients derivatives

$$
\begin{aligned}
\dot{u} &= \left[C_{x_\alpha}\alpha + C_{x_{\delta_E}}\delta_E + (C_{x_u} + 2C_{x_1})\frac{u}{U_1}\right. \\
&+ \left. C_{x_q}\frac{\bar{c}q}{2U_1} + C_{x_{\dot{\alpha}}}\frac{\bar{c}\dot{\alpha}}{2U_1}\right]\frac{\bar{q}_1 S}{m} - g\theta \\
U_1\dot{\alpha} &= U_1 q + \left[C_{z_\alpha}\alpha + C_{z_{\delta_E}}\delta_E + (C_{z_u} + 2C_{z_1})\frac{u}{U_1}\right. \\
&+ \left. C_{z_q}\frac{\bar{c}q}{2U_1} + C_{z_{\dot{\alpha}}}\frac{\bar{c}\dot{\alpha}}{2U_1}\right]\frac{\bar{q}_1 S}{m} \\
\dot{\theta} &= q \\
\dot{q} &= \left[C_{m_\alpha}\alpha + C_{m_{\delta_E}}\delta_E + (C_{m_u} + 2C_{m_1})\frac{u}{U_1}\right. \\
&+ \left. C_{m_q}\frac{\bar{c}q}{2U_1} + C_{m_{\dot{\alpha}}}\frac{\bar{c}\dot{\alpha}}{2U_1}\right]\frac{\bar{q}_1 S\bar{c}}{I_{yy}}
\end{aligned}
$$

$$
\begin{aligned}
&\dot{u} = X_u u + X_w w + X_{\delta_E}\delta_E + X_{\delta_T}\delta_T - g_0\cos\Theta\theta \\
&(1 - Z_{\dot{w}})\dot{w} = Z_u u + Z_w w + (U_0 + Z_q)q + Z_{\delta_E}\delta_E + Z_{\delta_T}\delta_T \\
&\dot{\theta} = q \\
&\dot{q} = M_u u + M_w w + M_{\dot{w}}\dot{w} + M_{\delta_E}\delta_E + M_{\delta_T}\delta_T
\end{aligned}
$$

- Large Commercial Aircraft data

Table B7.1 Geometric Data for the Boeing B747-200 Aircraft

Wing Surface (ft^2)	S	5,500
Mean Aerodynamic Chord (MAC) (ft)	c	27.3
Wing Span (ft)	b	196

Table B7.2 Flight Conditions Data for the Boeing B747−200 Aircraft

		Approach	Cruise (low)	Cruise (high)
Altitude (ft)	h	0	20,000	40,000
Mach Number	M	0.198	0.65	0.90
True Airspeed (ft/sec)	V_{P_1}	221	673	871
Dynamic Pressure (lbs/ft^2)	$\bar{q}$	58	287.2	222.8
Location of CG - % MAC	x_{CG}	0.25	0.25	0.25
Steady state angle of attack (deg)	α_1	8.5	2.5	2.4

Table B7.3 Mass and Inertial Data for the Boeing B747-200 Aircraft

		Approach	Cruise (low)	Cruise (high)
Mass (lbs)	m	564,000	636,636	636,636
Moment of Inertia x-axis (slug ft^2)	I_{XX_B}	13,700,000	18,200,000	18,200,000
Moment of Inertia y-axis (slug ft^2)	I_{YY_B}	30,500,000	33,100,000	33,100,000
Moment of Inertia z-axis (slug ft^2)	I_{ZZ_B}	43,100,000	49,700,000	49,700,000
Product of inertia xz-plane (slug ft^2)	I_{XZ_B}	830,000	970,000	970,000

33100000 slug foot² = 44877574.53865 kilogram meter²

Table B7.5 Longitudinal Dimensional Stability Derivatives for the Boeing B747-200 Aircraft

Cruise (high)
$X_u = -0.0218,\ X_{T_u} = -0.0604,\ X_\alpha = 1.2227,\ X_{\delta_E} = 0$
$Z_u = -0.0569,\ Z_\alpha = -339.0,\ Z_{\dot{\alpha}} = -7.666,\ Z_q = -7.474,\ Z_{\delta_E} = -18.341$
$M_u = -0.0001,\ M_{T_u} = 0.0,\ M_\alpha = -1.616,\ M_{T_\alpha} = 0,$ $M_{\dot{\alpha}} = -0.1425,\ M_q = -0.4038,\ M_{\delta_E} = -1.2124$

Table B7.8 Lateral Directional Dimensional Stability Derivatives for the Boeing B747-200 Aircraft

Cruise (high)
$Y_\beta = -55.023,\ Y_P = 0.0,\ Y_r = 0.0,\ Y_{\delta_A} = 0,\ Y_{\delta_R} = 7.336$
$L_\beta = -2.114,\ L_P = -0.5054,\ L_r = 0.1932,\ L_{\delta_A} = 0.1717,\ L_{\delta_R} = 0.1057$
$N_\beta = 0.7725,\ N_{T_\beta} = 0,\ N_P = -0.0141,\ N_r = -0.1521,\ N_{\delta_A} = 0.0087,\ N_{\delta_R} = -0.4828$

- Military Basic Trainer

Table B8.1 Geometric Data for the SIAI Marchetti S-211 Aircraft

Wing Surface (ft^2)	S	136
Mean Aerodynamic Chord (MAC) (ft)	c	5.4
Wing Span (ft)	b	26.3

Table B8.2 Flight Conditions Data for the SIAI Marchetti S-211 Aircraft

		Approach	Cruise (low)	Cruise (high)
Altitude (ft)	h	0	25,000	35,000
Mach Number	M	0.111	0.60	0.60
True Airspeed (ft/sec)	V_{P_1}	124	610	584
Dynamic Pressure (lbs/ft^2)	$\bar{q}$	18.2	198.0	125.7
Location of CG - % MAC	x_{CG}	0.25	0.25	0.25
Steady-state angle of attack (deg)	α_1	8	0	0.9

Table B8.3 Mass and Inertial Data for the SIAI Marchetti S-211 Aircraft

		Approach	Cruise (low)	Cruise (high)
Mass (lbs)	m	3,500	4,000	4,000
Moment of Inertia x-axis ($slug\ ft^2$)	I_{XX_B}	750	800	800
Moment of Inertia y-axis ($slug\ ft^2$)	I_{YY_B}	4,600	4,800	4,800
Moment of Inertia z-axis ($slug\ ft^2$)	I_{ZZ_B}	5,000	5,200	5,200
Product of inertia xz-plane ($slug\ ft^2$)	I_{XZ_B}	200	200	200

Table B8.5 Longitudinal Dimensional Stability Derivatives for the SIAI Marchetti S-211 Aircraft

Approach
$X_u = -0.078,\ X_{T_u} = -0.0055,\ X_\alpha = 6.26,\ X_{\delta_E} = 0$
$Z_u = -0.534,\ Z_\alpha = -119,\ Z_{\dot{\alpha}} = -1.492,\ Z_q = -4.48,\ Z_{\delta_E} = -8.91$
$M_u = 0.0,\ M_{T_u} = 0.0,\ M_\alpha = -1.75,\ M_{T_\alpha} = 0.0,$ $M_{\dot{\alpha}} = -0.445,\ M_q = -1.0,\ M_{\delta_E} = -2.625$

Table B8.8 Lateral Directional Dimensional Stability Derivatives for the Marchetti S-211 Aircraft

Approach
$Y_\beta = -21.47,\ Y_P = -0.0242,\ Y_r = 1.43,\ Y_{\delta_A} = 0,\ Y_{\delta_R} = 5.94$
$L_\beta = -11.77,\ L_P = -3.12,\ L_r = 5,\ L_{\delta_A} = 9.25,\ L_{\delta_R} = 2.52$
$N_\beta = 2.1,\ N_{T_\beta} = 0,\ N_P = -0.042,\ N_r = -0.432,\ N_{\delta_A} = -0.394,\ N_{\delta_R} = -1.445$

Mach Loop Typhoon Mach Loop F 15 Death Valley FLY BY Fly Over

❑ Longitudinal Dynamics Response

$$\dot{\boldsymbol{x}}_{\boldsymbol{LO}}(t) = \begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = A_{LO}\boldsymbol{x}(t) + B_{LO}\,\boldsymbol{u}_{\boldsymbol{LO}}(t); \boldsymbol{u}_{\boldsymbol{LO}}(t) = \begin{bmatrix} \delta_E \\ \delta_T \end{bmatrix}$$

- Longitudinal linearized motion is characterized, traditionally, by a 4° order system, with the system matrix $\boldsymbol{A}$ providing the 4 eigenvalues.
- Numerical values of stability derivatives depend on the particular equilibrium condition (flight condition) selected for linearization.
- Propulsive and Aerodynamic control are usually decoupled, in that propulsion acts primarily on the longitudinal translation axis alone, and a set point input, while aerodynamic control affect all three longitudinal axes, and it is used for transient response and trim .
- **Although the time response can have different dynamic characteristics, depending on the four eigenvalues, traditional stable dynamic behavior shows the presence of 2 damped oscillations, whose desirable properties are generally specified by certification for a given aircraft, given flight condition.**
- **Note**: Two additional eigenvalues are present if the linearized range and altitude kinematic equations are included

$$\begin{cases} \dot{x}_{DR} = u + \Theta_0(v + w) \approx u \\ \dot{h} = U_0\theta - w \end{cases}$$

- Alternate state vectors for longitudinal motion dynamics

  - Standard state vector in Body Axes

$$\begin{bmatrix} \dot{u} = \dfrac{X}{m} - g\sin\theta = f_1 \\ \dot{w} = \dfrac{Z}{m} + g\cos\phi\cos\theta = f_2 \\ \dot{q} = \dfrac{M}{I_{yy}} = f_3 \\ \dot{\theta} = q = f_4 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix}$$

  - Replace Pitch Angle by Angle of Attack, and express translation in Velocity Axes

$$\gamma = \theta - \alpha$$

$$\begin{bmatrix} \dot{V} = \dfrac{1}{m}\left[T\cos\alpha - D - mg\sin\gamma\right] = f_1 \\ \dot{\gamma} = \dfrac{1}{mV}\left[T\sin\alpha + L - mg\cos\gamma\right] = f_2 \\ \dot{q} = \dfrac{M}{I_{yy}} = f_3 \\ \dot{\alpha} = \dot{\theta} - \dot{\gamma} = q - \dfrac{1}{mV}\left[T\sin\alpha + L - mg\cos\gamma\right] = f_4 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} V \\ \gamma \\ q \\ \alpha \end{bmatrix}$$

- Slow Dynamics
- Fast Dynamics

  - Replace Pitch Angle by Angle of Attack, and express translation in Velocity Axes

$$\gamma = \theta - \alpha$$

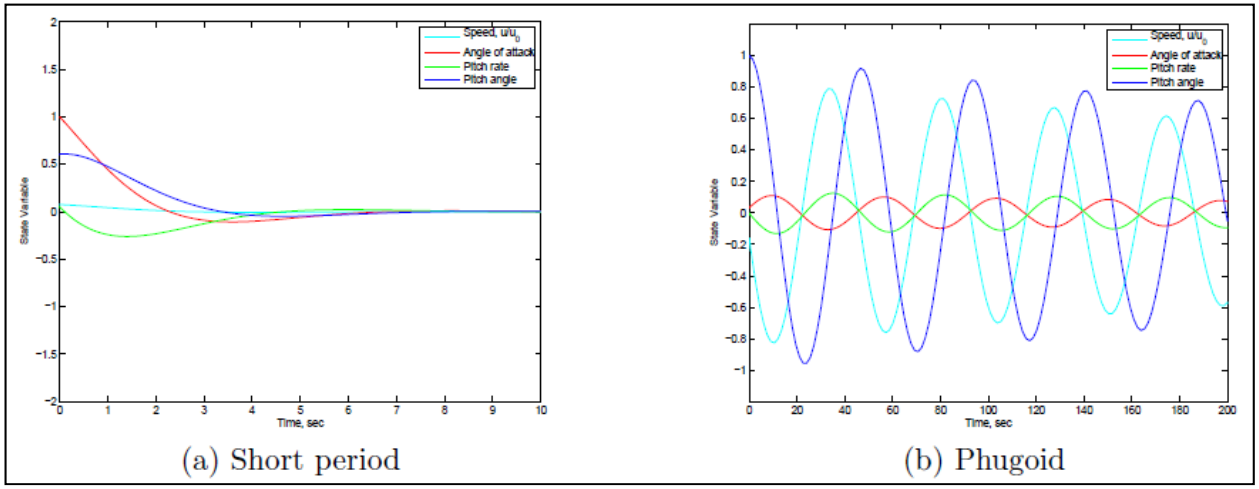
$$\begin{bmatrix} \dot{V} = \dfrac{1}{m}\left[T\cos\alpha - D - mg\sin\gamma\right] = f_1 \\ \dot{\gamma} = \dfrac{1}{mV}\left[T\sin\alpha + L - mg\cos\gamma\right] = f_2 \\ \dot{q} = \dfrac{M}{I_{yy}} = f_3 \\ \dot{\theta} = q = f_4 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} V \\ \dot{\gamma} \\ \dot{q} \\ \dot{\theta} \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix}$$

- From linear systems theory, there are 2 natural modes, whose frequency properties are defined by the eigenvalues, and whose amplitude properties are defined by the associate eigenvectors (typical parameters include time to half or time to double the amplitude):

$$\boldsymbol{x}(t) = \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} = \sum_{i=1}^{2} \boldsymbol{A_i} e^{\mathrm{Re}[\lambda_i]t} \sin\left(\mathrm{Im}[\lambda_i]t + \boldsymbol{B_i}\right) = \sum_{i=1}^{4} \boldsymbol{v}_i e^{\lambda_i t} \boldsymbol{\mu}_i^T \boldsymbol{x}_0$$

- The first mode represents a high-frequency, highly-damped oscillation called the **short-period mode**, and the second mode represents a low-frequency, lightly-damped oscillation called the **phugoid mode**.



(a) Short period (b) Phugoid

- **Example:** Commercial airliner flying at low cruising altitude (23,000 ft), and Mach = 0.86

```
XU   = -0.00795   (1/sec)
XW   = 0.02195   (1/sec)
XQ   = -0.61551   (ft/rad*sec)
XWP  = -0.00027   (1/1)
ZU   = -0.07325   (1/sec)
ZW   = -0.92669   (1/sec)
ZQ   = -12.58496   (ft/rad*sec)
ZWP  = -0.00560   (1/1)
MU   = -0.00003   (1/ft*sec)
MW   = -0.00434   (1/ft*sec)
MQ   = -0.85000   (1/sec)
MWP  = -0.00038   (1/ft)
Malfa   = -3.82681   (1/sec^2)
MalfaP = -0.33438   (1/sec)

X:    0.00249    0.00021
Z:    -20.62474    -0.00002
M:    -2.34000    0.00000
```

```
a =
                u           w           q        teta
   u      -0.007931      0.0222     -0.8519     -32.15
   w       -0.07284     -0.9215       863.6          0
   q     -2.579e-006    -0.003994     -1.178          0
   teta          0           0           1          0

b =
             delta_e     delta_t
   u        0.008103   0.0002062
   w          -20.51  -2.088e-005
   q          -2.332   8.049e-007
   teta            0           0
```

- Consider the free unforced response

$$\begin{cases} \lambda_{SP} = -1.0499 \pm 1.8528j \\ \xi_{SP} = 0.4930, \omega_{SP} = 2.1296 \\ \mathrm{P}_{SP} = \dfrac{2\pi}{\omega_{SP}} = 2.95\,\mathrm{sec} \\ T_{SP}^{1/2} = \dfrac{0.693}{1.0499} \approx 0.66\,\mathrm{sec} \end{cases}$$

$$\begin{cases} \lambda_{PH} = -0.0037 \pm 0.0451j \\ \xi_{PH} = 0.0824; \omega_{PH} = 0.0452 \\ \mathrm{P}_{PH} = \dfrac{2\pi}{\omega_{PH}} = 139\,\mathrm{sec}; \\ T_{PH}^{1/2} = \dfrac{0.693}{0.0037} \approx 187.3\,\mathrm{sec} \end{cases}$$

```
-1.0500 + 1.8528i   0.0000 + 0.0000i
 0.0000 + 0.0000i  -1.0500 - 1.8528i
 0.0000 + 0.0000i   0.0000 + 0.0000i
 0.0000 + 0.0000i   0.0000 + 0.0000i

 0.0066 + 0.0000i   0.0066 - 0.0000i
 1.0000 + 0.0000i   1.0000 + 0.0000i
-0.0001 + 0.0021i  -0.0001 - 0.0021i
 0.0009 - 0.0004i   0.0009 + 0.0004i
```

```
 0.0000 + 0.0000i   0.0000 + 0.0000i
 0.0000 + 0.0000i   0.0000 + 0.0000i
-0.0037 + 0.0453i   0.0000 + 0.0000i
 0.0000 + 0.0000i  -0.0037 - 0.0453i

-0.9998 + 0.0000i  -0.9998 + 0.0000i
 0.0189 + 0.0003i   0.0189 - 0.0003i
-0.0001 + 0.0000i  -0.0001 - 0.0000i
 0.0001 + 0.0014i   0.0001 - 0.0014i
```

- State vector evolution is function of eigenvectors (amplitude), and eigenvalues (time)

$$\begin{bmatrix} u(t) \\ w(t) \\ q(t) \\ \theta(t) \end{bmatrix} \propto \begin{bmatrix} 0.0066 + 0.0000j \\ 1.0000 + 0.0000j \\ -0.0001 + 0.0021j \\ 0.0009 - 0.0004j \end{bmatrix} e^{(-1.0500 \; + \; 1.8528j)t} + \begin{bmatrix} 0.0066 - 0.0000j \\ 1.0000 - 0.0000j \\ -0.0001 - 0.0021j \\ 0.0009 + 0.0004j \end{bmatrix} e^{(-1.0500 \; - \; 1.8528j)t} +$$
$$+ \begin{bmatrix} -0.9998 + 0.0000j \\ 0.0189 + 0.0003j \\ -0.0001 + 0.0000j \\ 0.0001 + 0.0014j \end{bmatrix} e^{(-0.0037 \; + \; 0.0453j)t} + \begin{bmatrix} -0.9998 - 0.0000j \\ 0.0189 - 0.0003j \\ -0.0001 - 0.0000j \\ 0.0001 - 0.0014j \end{bmatrix} e^{(-0.0037 \; - \; 0.0453j)t}$$

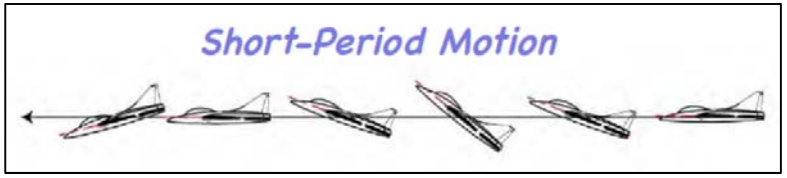
$$\begin{cases} \xi_{SP} = 0.4930, \omega_{SP} = 2.1296 & \omega_{SP}^{d} = 1.853, T_{SP}^{1/2} \approx 0.66 \sec \\ \xi_{PH} = 0.0824; \omega_{PH} = 0.0452 & \omega_{PH}^{d} = 0.045, T_{PH}^{1/2} \approx 187.3 \sec \end{cases}$$

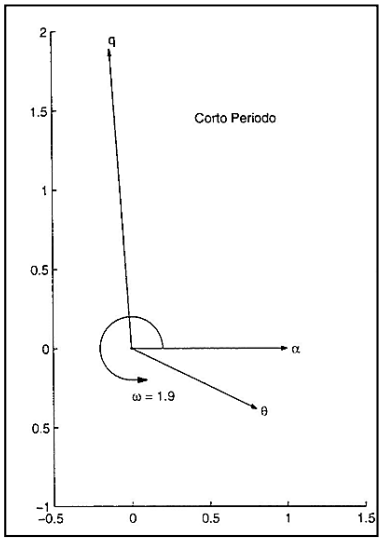
$$\begin{bmatrix} 0.0036 = u \,/\, U_0 \\ 0.57 = \alpha \\ 1.0 = q \\ 0.47 = \theta \end{bmatrix}$$

- Normalized Short Period Eigenvector

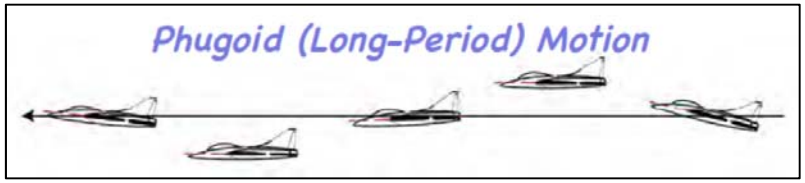
$$\begin{bmatrix} 0.8571 = u \,/\, U0 \\ 0.0183 = \alpha \\ 0.071 = q \\ 1.0 = \theta \end{bmatrix}$$

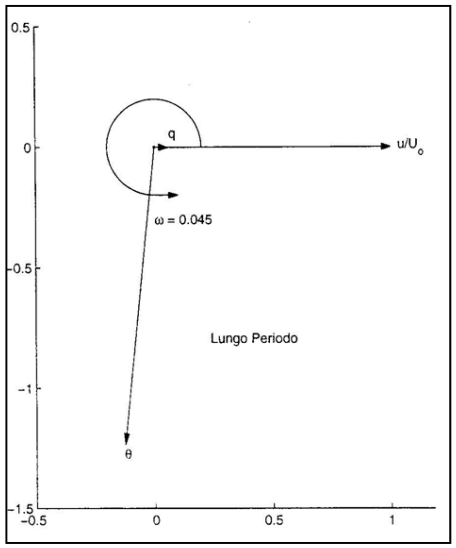
- Normalized Phugoid Eigenvector





- **The short period, motion involves rapid changes to the angle of attack and pitch attitude at approximately constant airspeed.**





- **Phugoid mode involves a trade between kinetic and potential energy. In this mode, the aircraft, at nearly constant angle of attack, climbs and slows down, then dives, losing altitude while picking up speed.**

- Common practice in the study of aircraft forced response is the use of frequency response between inputs (elevator, throttle in this case), and state variables

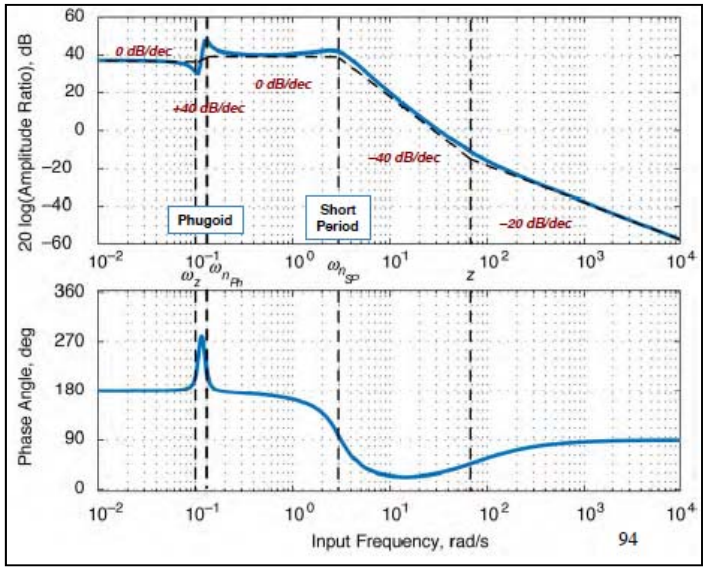
- **Frequency Response Analysis:** Transfer functions can be extracted from state space representation

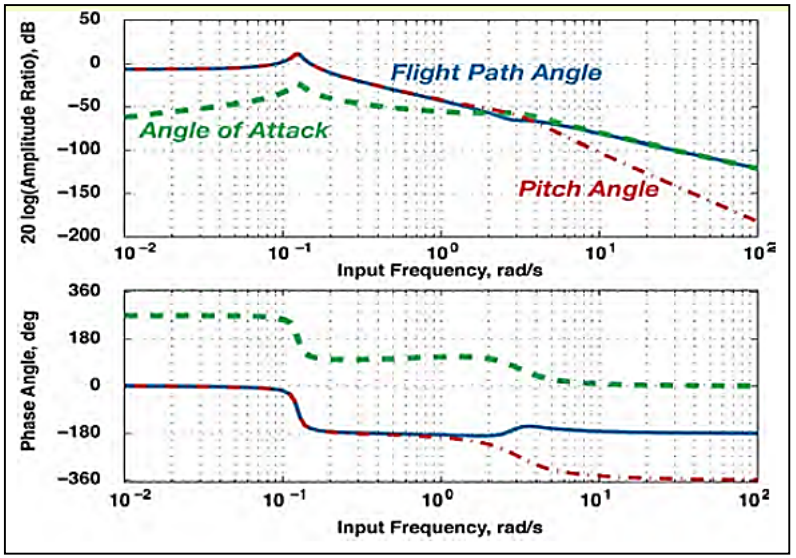
$$G_{LON}(s) = C(sI - A)^{-1}B = \frac{Cadj(sI - A)B}{\det(sI - A)} \qquad \det(sI - A) = as^4 + bs^3 + cs^2 + ds + e$$

$$\begin{aligned}
a &= U_1 - Z_{\dot{\alpha}} \\
b &= -[(U_1 - Z_{\dot{\alpha}})M_q + Z_\alpha + M_{\dot{\alpha}}(U_1 + Z_q) - X_u(U_1 - Z_{\dot{\alpha}}) \\
c &= Z_\alpha M_q - M_\alpha(U_1 + Z_q) + X_u[(U_1 - Z_{\dot{\alpha}})M_q + Z_\alpha \\
&+ M_{\dot{\alpha}}(U_1 + Z_q)] - X_\alpha Z_u \\
d &= -X_u[Z_\alpha M_q - M_\alpha(U_1 + Z_q)] + X_\alpha[Z_u M_q \\
&- M_u(U_1 + Z_q)] + g[Z_u M_{\dot{\alpha}} + M_u(U_1 - Z_{\dot{\alpha}})] \\
e &= g[Z_u M_\alpha - M_u Z_\alpha]
\end{aligned}$$

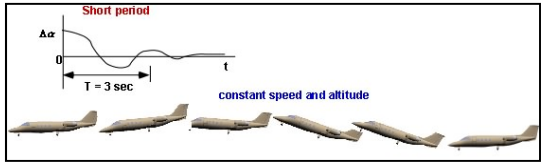
$$N_\delta^\theta(s) = A_\theta s^2 + B_\theta s + C_\theta = A_\theta(s + \frac{1}{T_{\theta 1}})(s + \frac{1}{T_{\theta 2}})$$

$$N_\delta^{w=\frac{\alpha}{U_0}}(s) = A_w s^3 + B_w s^2 + C_w s + D_w = \begin{cases} A_w(s + \frac{1}{T_{w1}})(s + \frac{1}{T_{w2}})(s + \frac{1}{T_{w3}}) \\ A_w(s + \frac{1}{T_{w1}})(s^2 + 2\xi_w \omega_w s + \omega_w^2) \end{cases}$$





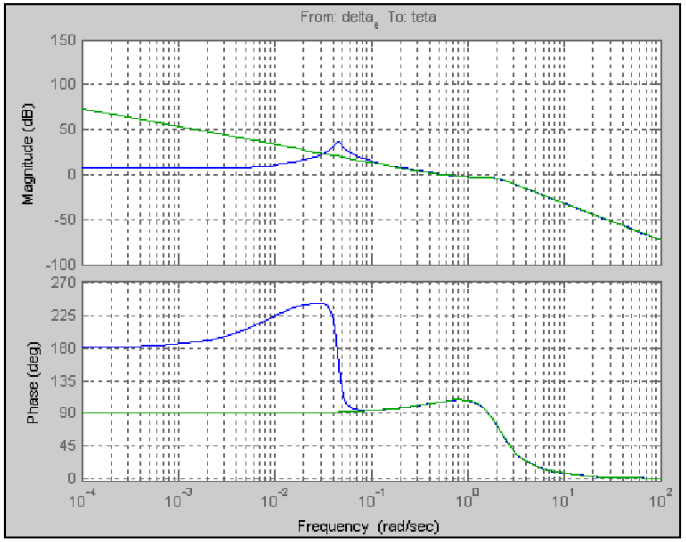
- Due to the frequency separation, in many cases we can use second order approximate dynamics.



❑ **Short Period Approximation:** The short period mode typically occurs so quickly that it proceeds at essentially constant vehicle speed. A useful approximation for the mode can thus be developed by setting $u = 0$

$$\begin{cases} (1 - Z_{\dot{w}})\dot{w} = Z_w w + (U_0 + Z_q)q + Z_\delta \delta \\ -M_{\dot{w}}\dot{w} + \dot{q} = M_w w + M_q q + M_\delta \delta \\ w = U_0 \alpha \end{cases}$$

$$G(s) = \frac{\theta(s)}{\delta(s)}$$



- The transfer functions denominator is

$$\lambda^2 - (Z_w + M_q + u_0 M_{\dot{w}})\lambda + Z_w M_q - u_0 M_w$$

$$\lambda^2 - \left(M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0}\right)\lambda - M_\alpha + \frac{Z_\alpha M_q}{u_0}$$

$$\omega_n = \sqrt{-M_\alpha + \frac{Z_\alpha M_q}{u_0}}$$

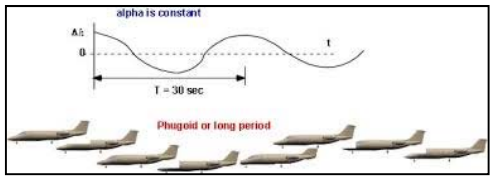
$$\zeta = -\frac{M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0}}{2\omega_n}$$

$$\begin{cases} N_\delta^w(s) = Z_\delta s + (U_0 M_\delta - Z_\delta M_q) \\ N_\delta^q(s) = (M_\delta + Z_\delta M_{\dot{w}})s + (Z_\delta M_w - M_\delta Z_w) \end{cases}$$

SHORT PERIOD TEST

- **Phugoid Approximation:** During the Phugoid motion, the angle of attack is approximately constant. Hence, an approximate result can be obtained by setting $\alpha = 0$ and ignoring the moment equation. Phugoid motion involves fairly large oscillatory changes in forward speed, pitch attitude, and altitude, with approximately constant angle of attack;

$$\begin{cases} \dot{u} = X_u u - g\theta \\ -U_0 q = Z_u u + Z_q q + Z_\delta \delta \\ q = \dot{\theta} \end{cases}$$



$$G(s) = \frac{\theta(s)}{\delta(s)}$$

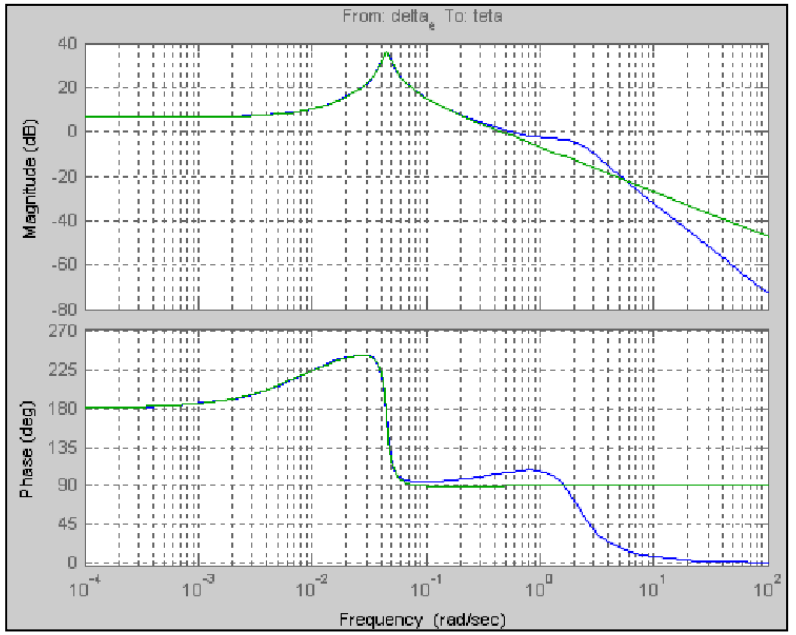
- The transfer functions denominator is

$$\lambda^2 - X_u\lambda - \frac{g_0}{u_0}Z_u = 0$$

$$\omega_n = \sqrt{-\frac{g_0}{u_0}Z_u} \qquad \omega_n = \sqrt{2}\frac{g_0}{u_0}$$

$$\zeta = \frac{-X_u}{2\omega_n} \qquad \zeta = \frac{1}{\sqrt{2}}\frac{C_{D0}}{C_{L0}}$$

$$\begin{cases} N_\delta^u(s) = gZ_\delta \\ N_\delta^\theta(s) = Z_\delta(s - X_u) \end{cases}$$



Phugoid

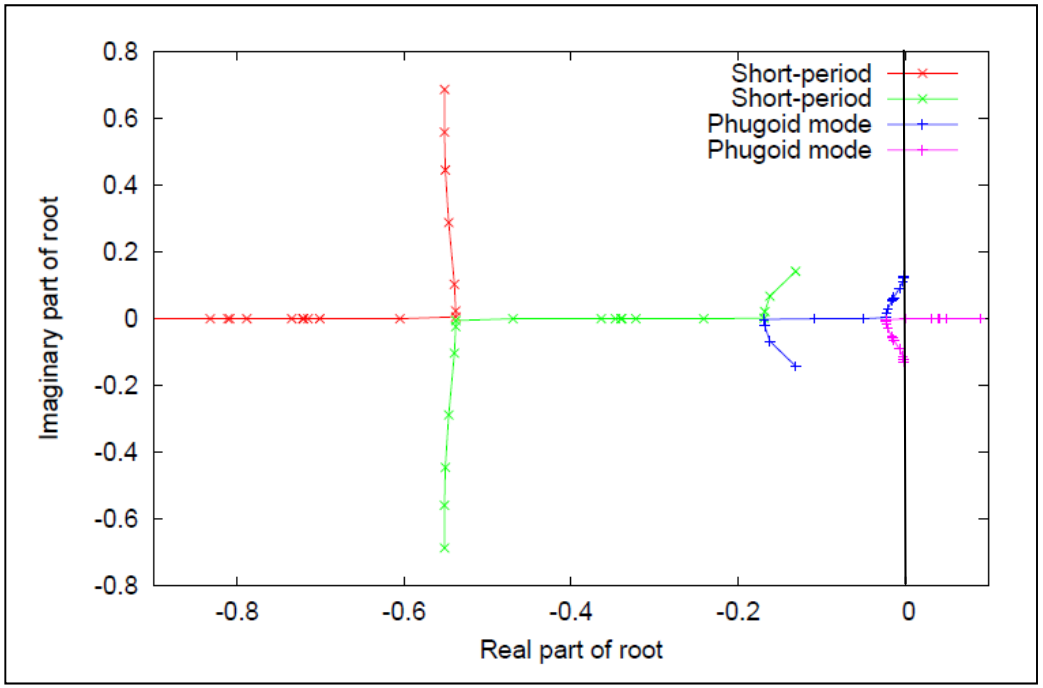
- **Root Locus Analysis:** Recall that longitudinal static stability is affected by the static margin (> 0 for stability). We can draw a variation of poles location as static margin changes (location of center of mass closer to NP).

$$C_{m\alpha} = C_{L\alpha}\left(\frac{X_{CG}}{\overline{c}} - \frac{X_{AC}}{\overline{c}}\right)$$

$$M_w = \frac{\rho S U_0 \overline{c}}{2I_{yy}} C_{m\alpha}$$

$$\det(sI - A) = as^4 + bs^3 + cs^2 + ds + e$$

$$\begin{aligned}
a &= U_1 - Z_{\dot{\alpha}} \\
b &= -[(U_1 - Z_{\dot{\alpha}})M_q + Z_\alpha + M_{\dot{\alpha}}(U_1 + Z_q) - X_u(U_1 - Z_{\dot{\alpha}}) \\
c &= Z_\alpha M_q - M_\alpha(U_1 + Z_q) + X_u[(U_1 - Z_{\dot{\alpha}})M_q + Z_\alpha \\
&+ M_{\dot{\alpha}}(U_1 + Z_q)] - X_\alpha Z_u \\
d &= -X_u[Z_\alpha M_q - M_\alpha(U_1 + Z_q)] + X_\alpha[Z_u M_q \\
&- M_u(U_1 + Z_q)] + g[Z_u M_{\dot{\alpha}} + M_u(U_1 - Z_{\dot{\alpha}})] \\
e &= g[Z_u M_\alpha - M_u Z_\alpha]
\end{aligned}$$



Locus of roots of longitudinal plant matrix for Boeing 747 in level flight at M = 0.25 as standard seal level conditions as functions of c.g. location for values of static margin ranging from 0.22 to -.05. As the static margin is reduced, the roots of both oscillatory modes coalesce on the real axis; one of the phugoid roots moves to the right and becomes unstable, while the other moves to the left and joins with one of the short period roots to form a third oscillatory mode.

Luogo delle Radici non - Standard

- Cosa succede nel caso in cui si vuole tracciare l'andamento dei poli in anello chiuso al variare di un parametro che non è il guadagno di anello?
  - **Esempio:**

$$G(s) = \frac{s^2 + s + 1}{s^3 + 4s^2 + \boldsymbol{k}s + 1}$$

- Come variano i poli del sistema $G(s)$ al variare di $\boldsymbol{k}$?

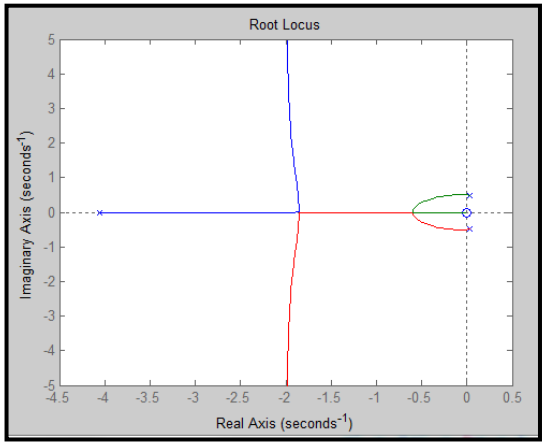
$$s^3 + 4s^2 + \boldsymbol{k}s + 1 = 0 \qquad \left(s^3 + 4s^2 + 1\right) + \boldsymbol{k}s = 0 \qquad \frac{\left(s^3 + 4s^2 + 1\right)}{\left(s^3 + 4s^2 + 1\right)} + \frac{\boldsymbol{k}s}{\left(s^3 + 4s^2 + 1\right)} = 0$$

$$1 + \frac{\boldsymbol{k}s}{\left(s^3 + 4s^2 + 1\right)} = 0$$

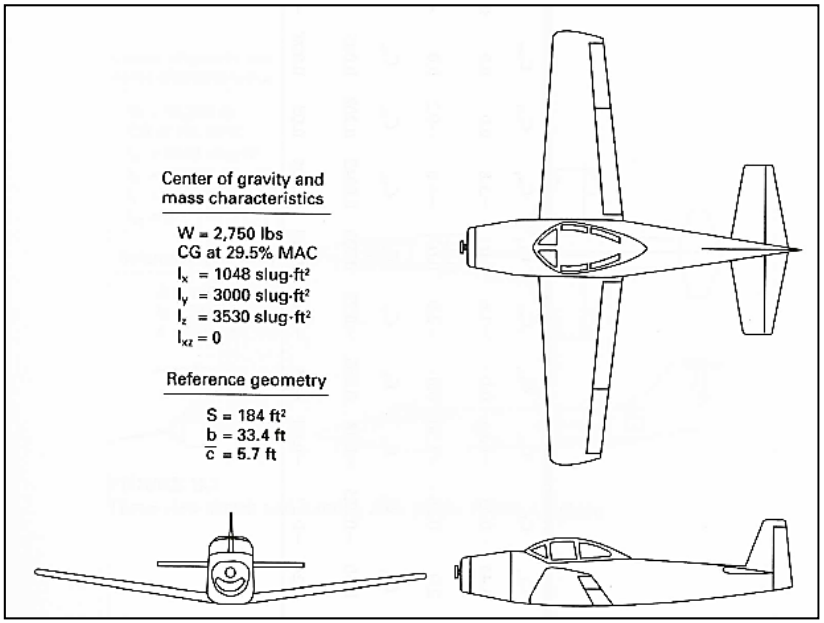
$$1 + \boldsymbol{k}G_{ol}^{Eq}(s) = 0$$

$$G_{ol}^{Eq}(s) = \frac{s}{\left(s^3 + 4s^2 + 1\right)}$$

- Tracciare il luogo delle radici per il nuovo sistema



- **Problem:** Study the Longitudinal dynamic properties of Navion general aviation aircraft at sea level (h = 0 ft, M = 0.158, V_0 = 176 ft/sec).



General aviation airplane: NAVION

Longitudinal	C_L	C_D	C_{L_α}	C_{D_α}	C_{m_α}	$C_{L_{\dot\alpha}}$	$C_{m_{\dot\alpha}}$	C_{L_q}	C_{m_q}	C_{L_M}	C_{D_M}	C_{m_M}	$C_{L_{\delta_e}}$	$C_{m_{\delta_e}}$
M = 0.158														
Sea level	0.41	0.05	4.44	0.33	−0.683	0.0	−4.36	3.8	−9.96	0.0	0.0	0.0	0.355	−0.923

Atmospheric Flight Dynamics

$$X_u = \frac{-(C_{D_u} + 2C_{D_0})QS}{mu_0} \qquad X_w = \frac{-(C_{D_\alpha} - C_{L_0})QS}{mu_0}$$

$$Z_u = \frac{-(C_{L_u} + 2C_{L_0})QS}{mu_0}$$

$$Z_w = \frac{-(C_{L_\alpha} + C_{D_0})QS}{mu_0} \qquad Z_{\dot{w}} = C_{z\dot{\alpha}} \frac{\bar{c}}{2u_0} QS/(u_0 m)$$

$$Z_\alpha = u_0 Z_w \qquad Z_{\dot{\alpha}} = u_0 Z_{\dot{w}}$$

$$Z_q = C_{Z_q} \frac{\bar{c}}{2u_0} QS/m \qquad Z_{\delta e} = C_{Z_{\delta e}} QS/m$$

$$M_u = C_{m_u} \frac{(QS\bar{c})}{u_0 I_y}$$

$$M_w = C_{m_\alpha} \frac{(QS\bar{c})}{u_0 I_y} \qquad M_{\dot{w}} = C_{m\dot{\alpha}} \frac{\bar{c}}{2u_0} \frac{QS\bar{c}}{u_0 I_y}$$

$$M_\alpha = u_0 M_w \qquad M_{\dot{\alpha}} = u_0 M_{\dot{w}}$$

$$M_q = C_{m_q} \frac{\bar{c}}{2u_0} (QS\bar{c})/I_y \qquad M_{\delta e} = C_{m\delta e}(QS\bar{c})/I_y$$

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{w} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & u_0 & 0 \\ M_u + M_{\dot{w}}Z_u & M_w + M_{\dot{w}}Z_w & M_q + M_{\dot{w}}u_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix} + \begin{bmatrix} X_\delta & X_{\delta_T} \\ Z_\delta & Z_{\delta_T} \\ M_\delta + M_{\dot{w}}Z_\delta & M_{\delta_T} + M_{\dot{w}}Z_{\delta_T} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta\delta_T \end{bmatrix}$$

$$\Delta\alpha = \frac{\Delta w}{V_0}$$

- Units are: ft, sec, rad, lbs

$$\begin{bmatrix} X_u = -0.446 & Z_u = -0.361 & M_u = 0 \\ X_w = 0.0357 & Z_w = -1.978 & M_w = -0.0488 \\ X_{\dot{w}} = 0 & Z_{\dot{w}} = 0 & M_{\dot{w}} = -0.0049 \\ X_q = 0 & Z_q = 0 & M_q = -2.006 \end{bmatrix} \qquad \begin{bmatrix} X_{\delta E} = 0 \\ Z_{\delta E} = -0.855 \\ M_{\delta E} = -0.066 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.0446 & 0.0357 & 0 & -32.2 \\ -0.361 & -1.978 & 168.8 & 0 \\ 0.0018 & -0.0389 & -2.885 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ -0.855 \\ -0.066 \\ 0 \end{bmatrix} \delta_E$$

- Consider the input - free response

```
>> eig(A)

ans =

  -2.4373 + 2.5209i
  -2.4373 - 2.5209i
  -0.0165 + 0.2141i
  -0.0165 - 0.2141i
```

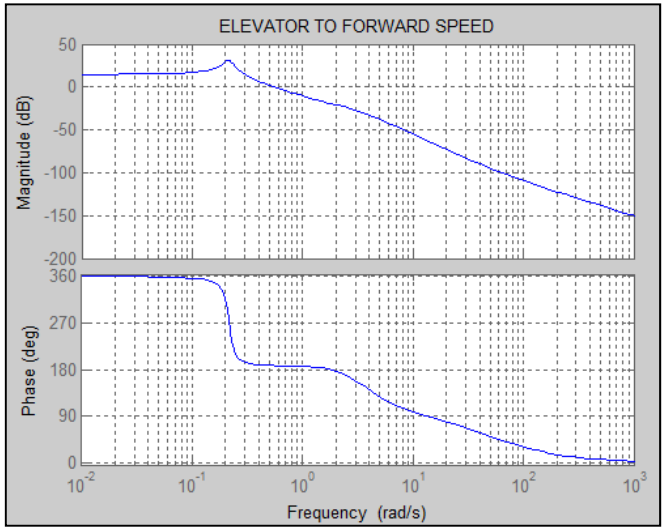
Phugoid (long period)	Short period
$t_{1/2} = 0.69/\lvert\eta\rvert = \frac{0.69}{-0.0171}$	$t_{1/2} = 0.69/\lvert\eta\rvert = \frac{0.69}{-2.5}$
$t_{1/2} = 40.3$ s	$t_{1/2} = 0.28$ s
Period $= 2\pi/\omega = 2\pi/0.213$	Period $= 2\pi/\omega = 2\pi/2.59$
Period $= 29.5$ s	Period $= 2.42$ s
Number of cycles to half amplitude	**Number of cycles to half amplitude**
$N_{1/2} = \frac{t_{1/2}}{P} = 0.110\frac{\omega}{\lvert\eta\rvert}$	$N_{1/2} = 0.110\frac{\omega}{\lvert\eta\rvert}$
$= \frac{(0.110)(0.213)}{\lvert-0.0171\rvert}$	$= \frac{(0.110)(2.59)}{\lvert-2.5\rvert}$
$N_{1/2} = 1.37$ cycles	$N_{1/2} = 0.11$ cycles

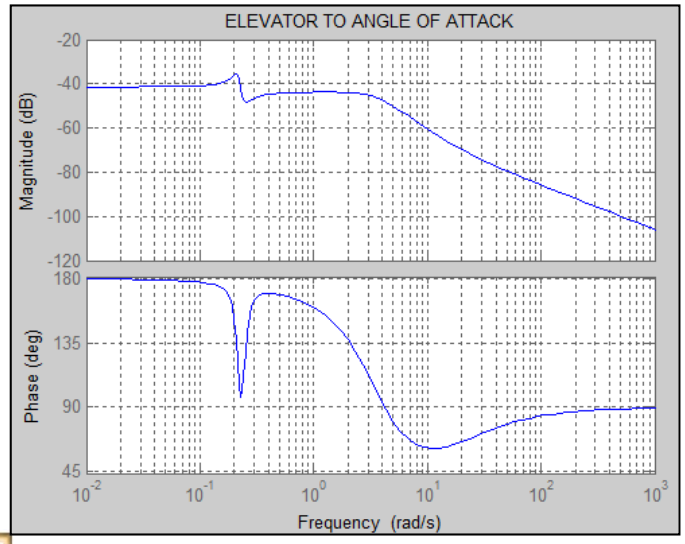
$$\xi_{SP} = 0.6951 \quad \xi_{PH} = 0.0807$$
$$\omega_{SP} = 3.5965 \quad \omega_{PH} = 0.2045$$

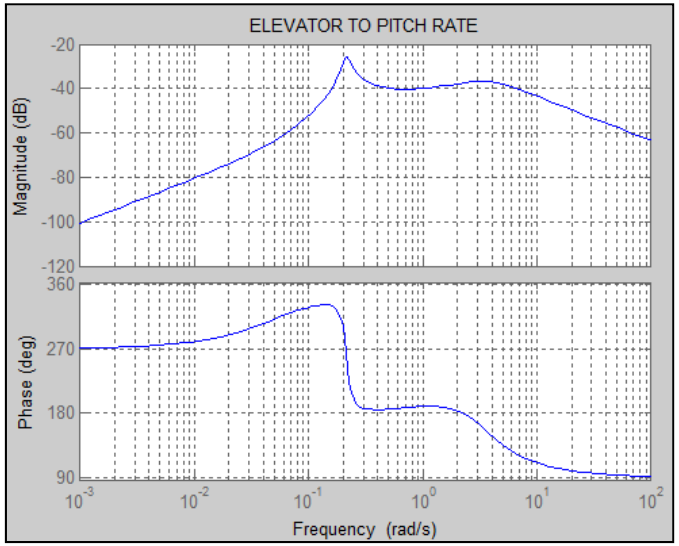
```
v =

  -0.0320 - 0.0012i   -0.0320 + 0.0012i   -0.9982 + 0.0000i   -0.9982 + 0.0000i
  -0.9994 + 0.0000i   -0.9994 + 0.0000i    0.0602 + 0.0012i    0.0602 - 0.0012i
   0.0027 - 0.0149i    0.0027 + 0.0149i   -0.0014 + 0.0001i   -0.0014 - 0.0001i
  -0.0036 + 0.0024i   -0.0036 - 0.0024i    0.0009 + 0.0066i    0.0009 - 0.0066i


d =

  -2.4373 + 2.5209i    0.0000 + 0.0000i    0.0000 + 0.0000i    0.0000 + 0.0000i
   0.0000 + 0.0000i   -2.4373 - 2.5209i    0.0000 + 0.0000i    0.0000 + 0.0000i
   0.0000 + 0.0000i    0.0000 + 0.0000i   -0.0165 + 0.2141i    0.0000 + 0.0000i
   0.0000 + 0.0000i    0.0000 + 0.0000i    0.0000 + 0.0000i   -0.0165 - 0.2141i
```

- Frequency Response Analysis

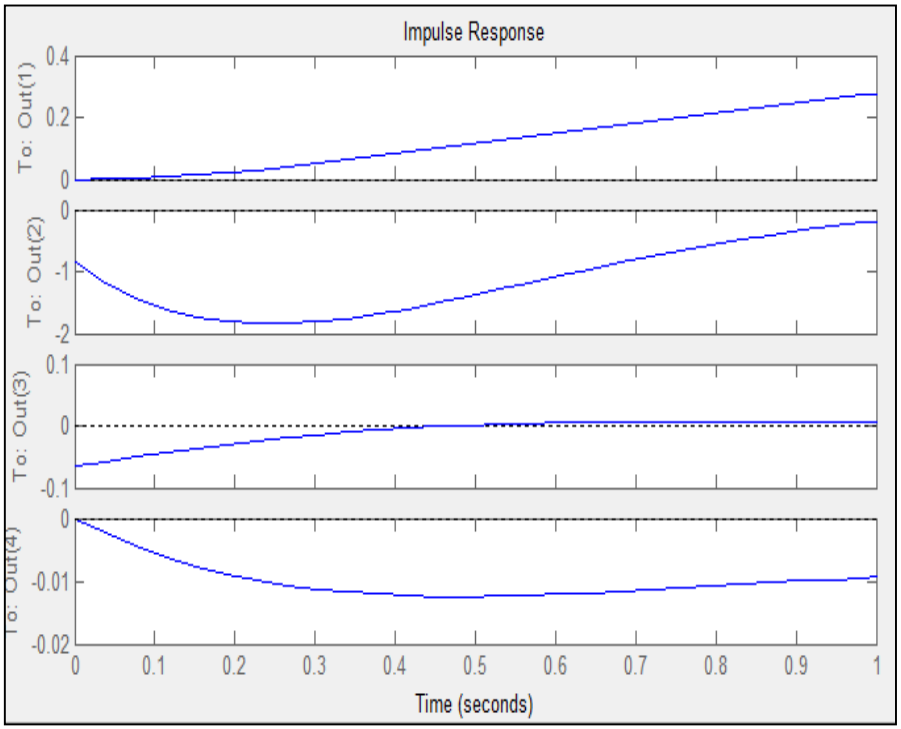
```
btot =

         0         0   -0.0305    1.6394    3.1327
         0   -0.8550  -13.6456   -0.6069   -0.8168
         0   -0.0660   -0.1002   -0.0052         0
         0         0   -0.0660   -0.1002   -0.0052


atot =

    1.0000    4.9076   12.5026    0.6317    0.5668
```

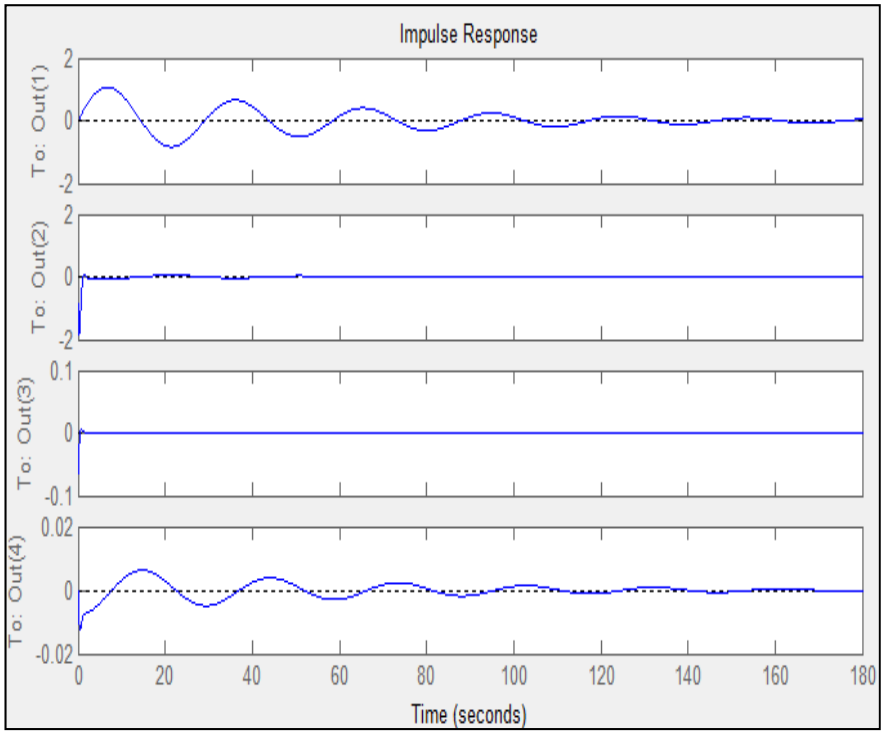






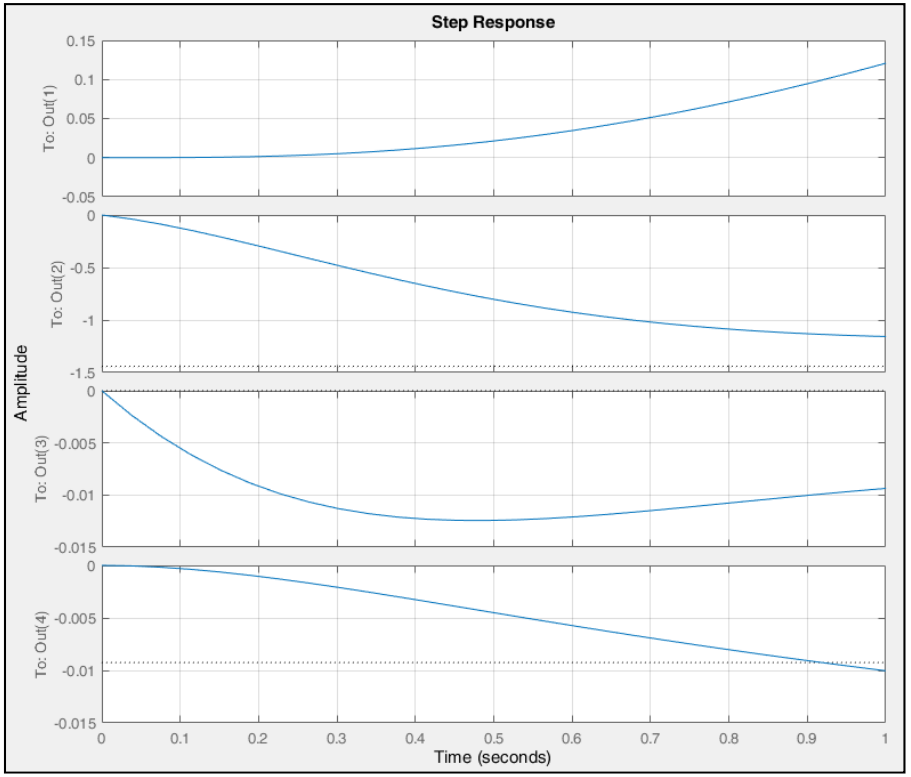
- Time Response (Impulse)

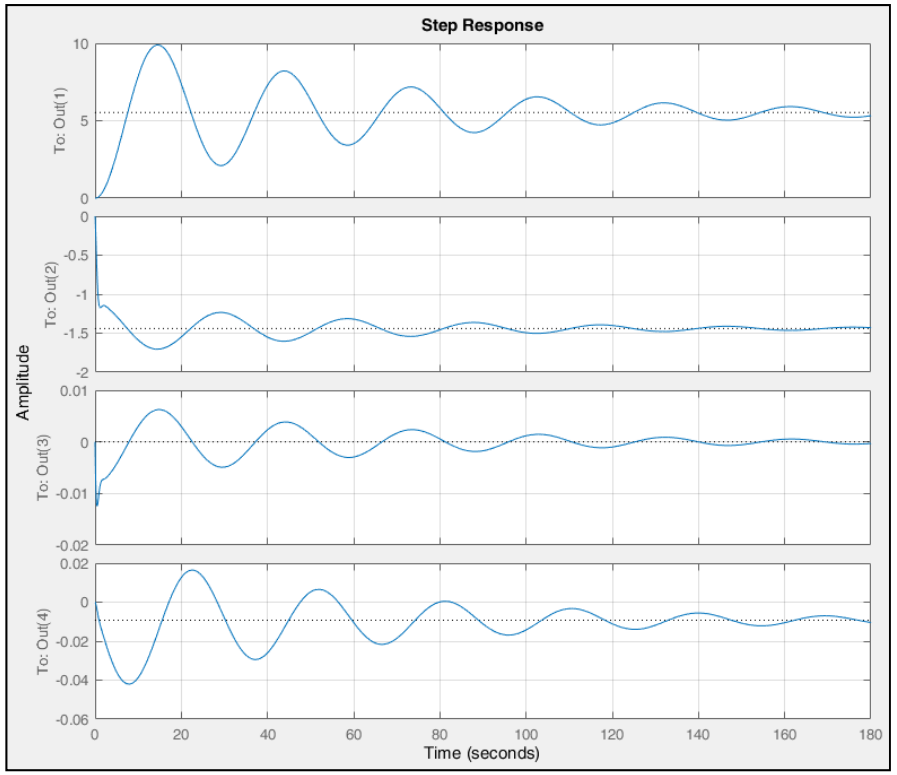




$$\begin{bmatrix} u \\ w = 176 \cdot \alpha \\ q \\ \theta \end{bmatrix}$$

- Time Response (Unit Step)





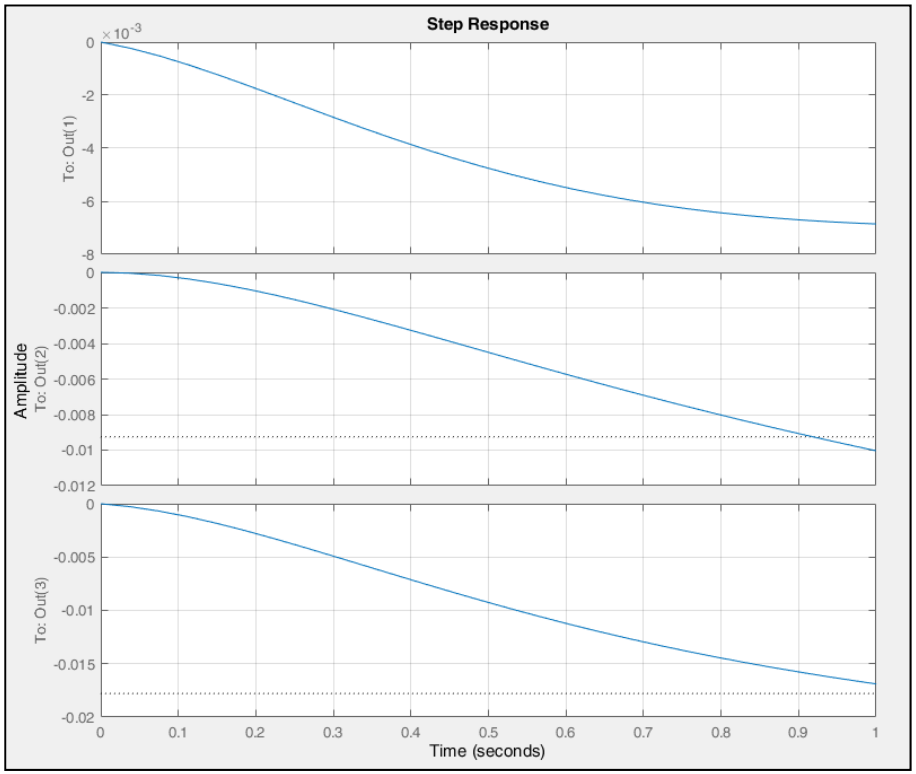
$$\begin{bmatrix} u \\ w = 176 \cdot \alpha \\ q \\ \theta \end{bmatrix}$$

Atmospheric Flight Dynamics

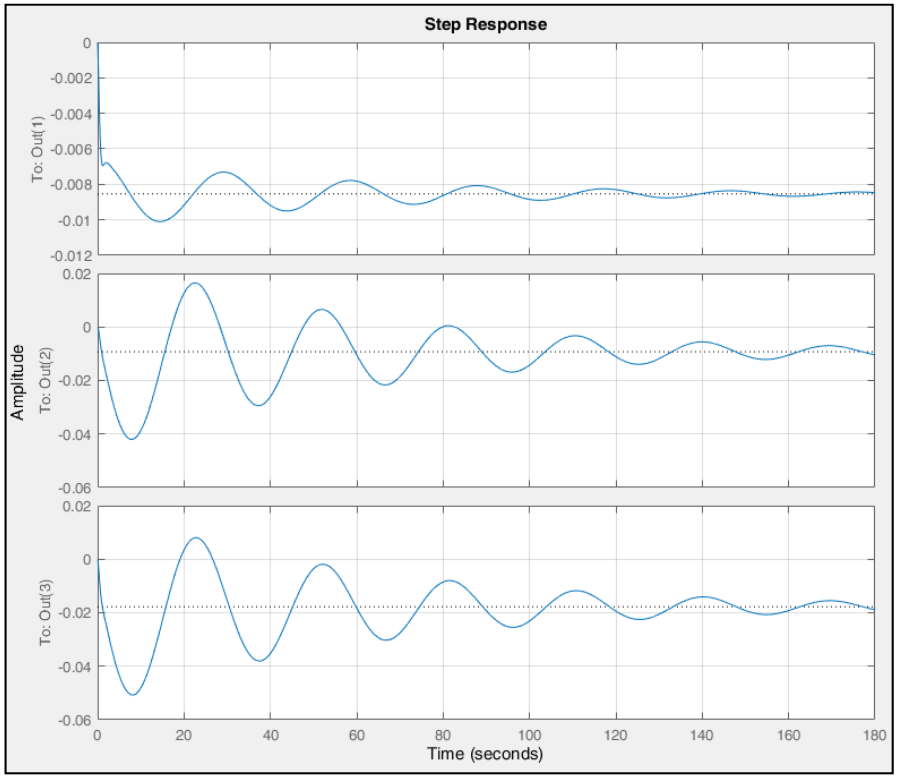
- Initially, pitch angle and angle of attack are the same in magnitude, thus flight path is zero

$$\gamma = \theta - \alpha$$

- As time increases, the angle of attack goes to zero, and the flight path angle rotates with pitch angle



$$\begin{bmatrix} \alpha \\ \theta \\ \gamma \\ 0 \end{bmatrix}$$



$$\begin{cases} \theta \approx \alpha \\ \gamma \approx 0 \end{cases}$$

$$\begin{cases} \alpha \to 0 \\ \gamma \to \theta \end{cases}$$

- Short Period Approximation

$$\begin{cases}(1-Z_{\dot w})\dot w = Z_w w + (U_0 + Z_q)q + Z_\delta \delta \\ -M_{\dot w}\dot w + \dot q = M_w w + M_q q + M_\delta \delta \\ w = U_0 \alpha \end{cases}$$

$$\begin{bmatrix} X_u = -0.446 & Z_u = -0.361 & M_u = 0 \\ X_w = 0.0357 & Z_w = -1.978 & M_w = -0.0488 \\ X_{\dot w} = 0 & Z_{\dot w} = 0 & M_{\dot w} = -0.0049 \\ X_q = 0 & Z_q = 0 & M_q = -2.006 \end{bmatrix} \qquad \begin{bmatrix} X_{\delta E} = 0 \\ Z_{\delta E} = -0.855 \\ M_{\delta E} = -0.066 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix}\dot\alpha \\ \dot q\end{bmatrix} = \begin{bmatrix}-1.978 & 1 \\ -6.60 & -2.8331\end{bmatrix}\begin{bmatrix}\alpha \\ q\end{bmatrix} + \begin{bmatrix}-0.0046 \\ -0.0618\end{bmatrix}\delta_E$$

$$\omega_n = \sqrt{-M_\alpha + \frac{Z_\alpha M_q}{u_0}}$$

$$\zeta = -\frac{M_q + M_{\dot\alpha} + \frac{Z_\alpha}{u_0}}{2\omega_n}$$

```
>> eig(ASP)

ans =

  -2.4055 + 2.5332i
  -2.4055 - 2.5332i
```

Short period

$t_{1/2} = 0.69/|\eta| = \dfrac{0.69}{-2.5}$

$t_{1/2} = 0.28$ s

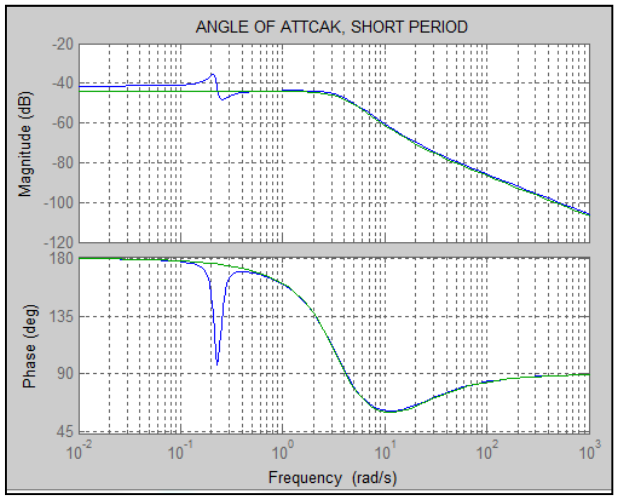
Period $= 2\pi/\omega = 2\pi/2.59$

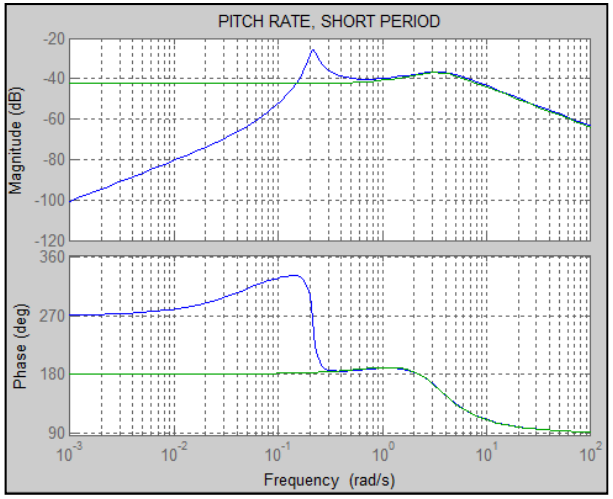
Period $= 2.42$ s

Number of cycles to half amplitude

$$N_{1/2} = 0.110\frac{\omega}{|\eta|} = \frac{(0.110)(2.59)}{|-2.5|}$$

$N_{1/2} = 0.11$ cycles





- Phugoid Approximation

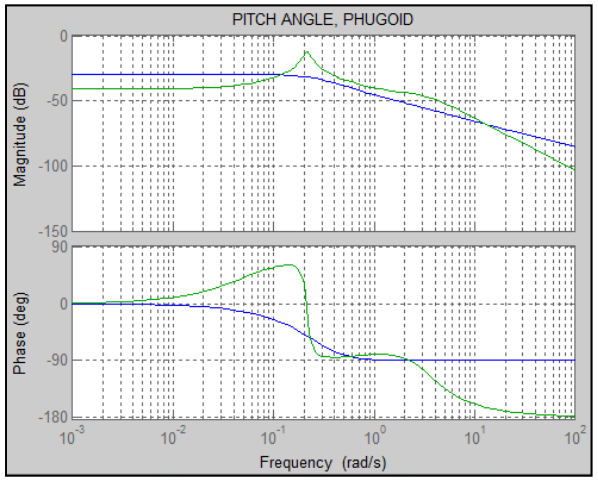
$$\begin{cases} \dot{u} = X_u u - g\theta \\ -U_0 q = Z_u u + Z_q q + Z_\delta \delta \\ q = \dot{\theta} \end{cases}$$

$$\begin{bmatrix} \dot{u} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.446 & -32.2 \\ 0.0021 & 0 \end{bmatrix} \begin{bmatrix} u \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0.0051 \end{bmatrix} \delta_E$$

```
>> eig(APH)

ans =

  -0.2230 + 0.1338i
  -0.2230 - 0.1338i
```

$$\omega_n = \sqrt{-\frac{g_0}{u_0} Z_u} \qquad \omega_n = \sqrt{2}\frac{g_0}{u_0}$$

$$\zeta = \frac{-X_u}{2\omega_n} \qquad \zeta = \frac{1}{\sqrt{2}} \frac{C_{D0}}{C_{L0}}$$



Phugoid
(long period)

$$t_{1/2} = 0.69/|\eta| = \frac{0.69}{-0.0171}$$

$t_{1/2} = 40.3$ s

Period $= 2\pi/\omega = 2\pi/0.213$

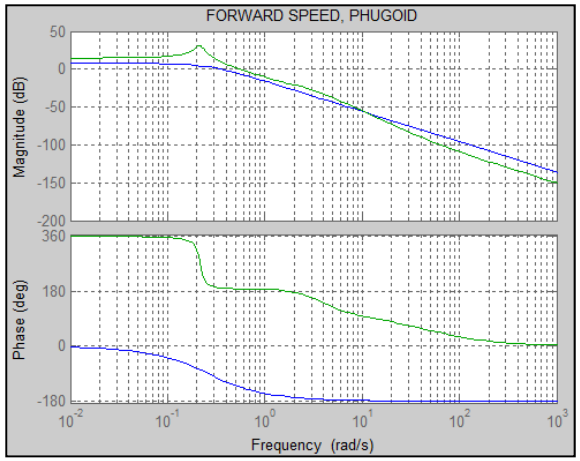
Period $= 29.5$ s

Number of cycles to half amplitude

$$N_{1/2} = \frac{t_{1/2}}{P} = 0.110\frac{\omega}{|\eta|}$$

$$= \frac{(0.110)(0.213)}{|-0.0171|}$$

$N_{1/2} = 1.37$ cycles



Influence of stability derivatives on the long- and short-period motions

Stability derivative	Mode affected	How affected
$M_q + M_{\dot{\alpha}}$	Damping of short-period mode of motion	Increasing $M_q + M_{\dot{\alpha}}$ increases damping
M_α	Frequency of short-period mode of motion	Increasing M_α or static stability increases the frequency
X_u	Damping of the phugoid or long-period mode of motion	Increasing X_u increases damping
Z_u	Frequency of phugoid mode of motion	Increasing Z_u increases the frequency

- **Linearized Lateral – Directional Equations of motion**: Coupling between inputs and state variables is more significant compared to longitudinal motion

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} - Y_v\right] v - Y_p p + [u_0 - Y_r] r - g_0 \cos\Theta_0 \phi = Y_{\delta_r}\delta_r$$

$$-L_v v + \left[\frac{\mathrm{d}}{\mathrm{d}t} - L_p\right] p - \left[\frac{I_{xz}}{I_x}\frac{\mathrm{d}}{\mathrm{d}t} + L_r\right] r = L_{\delta_r}\delta_r + L_{\delta_a}\delta_a$$

$$-N_v v - \left[\frac{I_{xz}}{I_z}\frac{\mathrm{d}}{\mathrm{d}t} + N_p\right] p + \left[\frac{\mathrm{d}}{\mathrm{d}t} - N_r\right] r = N_{\delta_r}\delta_r + N_{\delta_a}\delta_a$$

Variable	Y	L	N
v	$Y_v = \frac{QS}{mu_0}\mathbf{C}_{y_\beta}$	$L_v = \frac{QSb}{I_x u_0}\mathbf{C}_{l_\beta}$	$N_v = \frac{QSb}{I_z u_0}\mathbf{C}_{n_\beta}$
p	$Y_p = \frac{QSb}{2mu_0}\mathbf{C}_{y_p}$	$L_p = \frac{QSb^2}{2I_x u_0}\mathbf{C}_{l_p}$	$N_p = \frac{QSb^2}{2I_z u_0}\mathbf{C}_{n_p}$
r	$Y_r = \frac{QSb}{2mu_0}\mathbf{C}_{y_r}$	$L_r = \frac{QSb^2}{2I_x u_0}\mathbf{C}_{l_r}$	$N_r = \frac{QSb^2}{2I_z u_0}\mathbf{C}_{n_r}$

- Equations of motion can be written in terms of non – dimensional stability derivatives as well

$L_{\delta_a} \equiv \frac{1}{I_x}\frac{\partial L}{\partial \delta_a} = \frac{QSb}{I_x}\mathbf{C}_{l_{\delta_a}}$
$L_{\delta_r} \equiv \frac{1}{I_x}\frac{\partial L}{\partial \delta_r} = \frac{QSb}{I_x}\mathbf{C}_{l_{\delta_r}}$
$N_{\delta_a} \equiv \frac{1}{I_z}\frac{\partial N}{\partial \delta_a} = \frac{QSb}{I_z}\mathbf{C}_{n_{\delta_a}}$
$N_{\delta_r} \equiv \frac{1}{I_z}\frac{\partial N}{\partial \delta_r} = \frac{QSb}{I_z}\mathbf{C}_{n_{\delta_r}}$

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \qquad \boldsymbol{x}(t) = \begin{bmatrix} v \\ p \\ \phi \\ r \end{bmatrix}; \boldsymbol{u}(t) = \begin{bmatrix} \delta_A \\ \delta_R \end{bmatrix}$$

Note: It is equivalent to use v or $U_0\beta$ as state variable

- Expanding the dimensional system matrices yields:

$$\mathbf{A} = \begin{pmatrix} Y_v & Y_p & g_0 \cos\Theta_0 & Y_r - u_0 \\ \frac{L_v + i_x N_v}{1 - i_x i_z} & \frac{L_p + i_x N_p}{1 - i_x i_z} & 0 & \frac{L_r + i_x N_r}{1 - i_x i_z} \\ 0 & 1 & 0 & 0 \\ \frac{N_v + i_z L_v}{1 - i_x i_z} & \frac{N_p + i_z L_p}{1 - i_x i_z} & 0 & \frac{N_r + i_z L_r}{1 - i_x i_z} \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} Y_{\delta_r} & 0 \\ \frac{L_{\delta_r} + i_x N_{\delta_r}}{1 - i_x i_z} & \frac{L_{\delta_a} + i_x N_{\delta_a}}{1 - i_x i_z} \\ 0 & 0 \\ \frac{N_{\delta_r} + i_z L_{\delta_r}}{1 - i_x i_z} & \frac{N_{\delta_a} + i_z L_{\delta_a}}{1 - i_x i_z} \end{pmatrix}$$

- For many vehicles, the product of inertias $I_{xx} I_{zz}$ is negligible compared to unity, thus an approximated form becomes (**see primed derivatives next**):

$$A = \begin{pmatrix} Y_v & Y_p & g_0 \cos\theta_0 & Y_r - U_0 \\ L_v + I_{xx} N_v & L_p + I_{xx} N_p & 0 & L_r + I_{xx} N_r \\ 0 & 0 & 1 & 0 \\ N_v + I_{zz} L_v & N_p + I_{zz} L_p & 0 & N_p + I_{zz} L_p \end{pmatrix} \qquad B = \begin{pmatrix} Y_{\delta r} & 0 \\ L_{\delta r} + I_{xx} N_{\delta r} & L_{\delta a} + I_{xx} N_{\delta a} \\ 0 & 0 \\ N_{\delta r} + I_{zz} L_{\delta r} & N_{\delta a} + I_{zz} L_{\delta a} \end{pmatrix}$$

- **Primed Derivatives (A'_i):** The cross – product of inertia terms in the lateral – directional equations can be eliminated by a simple mathematical procedure:

$$\begin{aligned} I_A &\triangleq I_{xz} / I_{xx} \\ I_B &\triangleq I_{xz} / I_{zz} \end{aligned}$$

$$\begin{cases} L'_\beta = L_\beta + I_B N_\beta & N'_\beta = N_\beta + I_A L_\beta \\ L'_p = L_p + I_B N_p & N'_p = N_p + I_A L_p \\ L'_r = L_r + I_B N_r & N'_r = N_r + I_A L_r \\ L'_{\delta_A} = L_{\delta_A} + I_B N'_{\delta_A} & N'_{\delta_A} = N_{\delta_A} + I_A L_{\delta_A} \\ L'_{\delta_R} = L_{\delta_R} + I_B N_{\delta_R} & N'_{\delta_R} = N_{\delta_R} + I_A L_{\delta_R} \end{cases}$$

$$\begin{cases} \dot{v} = Y_v v + U_0 r - \left(g \cos\gamma_0\right)\phi + Y_{\delta_R}\delta_R \\ \dot{p} = L'_v v + L'_p p + L'_r r + L'_{\delta_A}\delta_A + L'_{\delta_R}\delta_R \\ \dot{r} = N'_v v + N'_p p + N'_r r + N'_{\delta_A}\delta_A + N'_{\delta_R}\delta_R \end{cases}$$

$$\begin{cases} \dot{\phi} = p + r\tan\theta_0 \\ \dot{\psi} = r\sec\theta_0 \end{cases}$$

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$

$$\boldsymbol{x}(t) = \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix}; \boldsymbol{u}(t) = \begin{bmatrix} \delta_A \\ \delta_R \end{bmatrix}$$

$$A = \begin{bmatrix} Y_v & 0 & U_0 & -g\cos\theta_0 \\ L'_v & L'_p & L'_r & 0 \\ N'_v & N'_p & N'_r & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & Y_{\delta_R} \\ L'_{\delta_A} & L'_{\delta_R} \\ N'_{\delta_A} & N'_{\delta_R} \\ 0 & 0 \end{bmatrix}$$

- Augmented state vector including kinematic equations

$$\boldsymbol{x}_{LA}(t) = \begin{bmatrix} v = U_0\beta \\ y_{CR} \\ p \\ r \\ \phi \\ \psi = \chi - \beta \end{bmatrix} \qquad \begin{cases} \dot{v} = Y_v v + U_0 r - \left(g\cos\theta_0\right)\phi + Y_{\delta_R}\delta_R \\ \dot{y}_{CR} \approx v \\ \dot{p} = L'_v v + L'_p p + L'_r r + L'_{\delta_A}\delta_A + L'_{\delta_R}\delta_R \\ \dot{r} = N'_v v + N'_p p + N'_r r + N'_{\delta_A}\delta_A + N'_{\delta_R}\delta_R \\ \dot{\phi} = p + r\tan\theta_0 \\ \dot{\psi} = r\sec\theta_0 \end{cases}$$

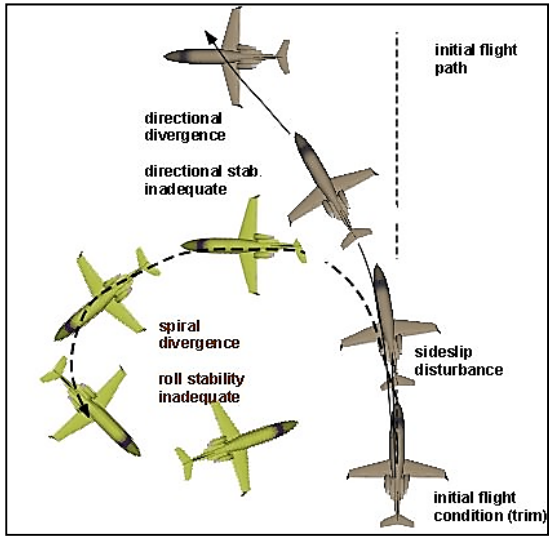
- Side velocity
- Crossrange
- Body axis roll rate
- Body axis yaw rate
- Body axis roll angle
- Inertial axis yaw angle

- Traditional lateral – directional dynamics are characterized by **3 natural modes** associated to the 4 eigenvalues and relative eigenvectors. Consider the free response:

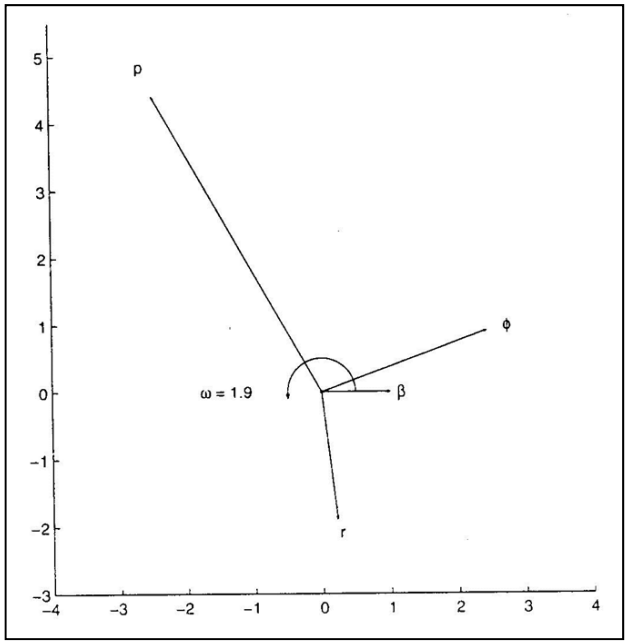
$$\boldsymbol{x}(t) = \begin{bmatrix} v \\ p \\ \phi \\ r \end{bmatrix} = \boldsymbol{A} e^{\mathrm{Re}[\lambda_{1,2}]t} \sin\left(\mathrm{Im}[\lambda_{1,2}]t + \boldsymbol{B}\right) + \boldsymbol{C} e^{\lambda_3 t} + \boldsymbol{D} e^{\lambda_4 t} = \sum_{i=1}^{4} \boldsymbol{v}_i \boldsymbol{\mu}_i^T \boldsymbol{x}_0 e^{\lambda_i t}$$

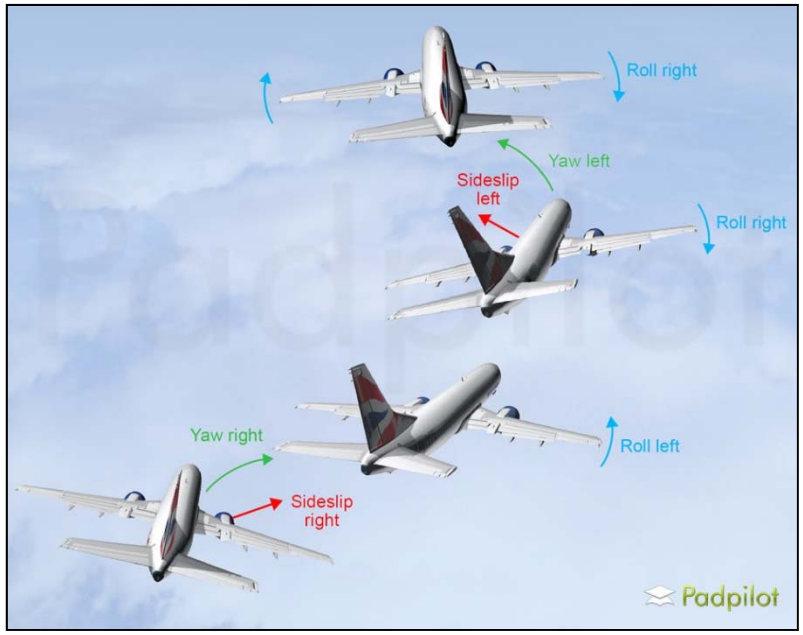
- **Roll Mode:** A first-order convergent mode of relatively short time constant. The roll mode consists of almost pure rolling motion and is generally a non-oscillatory motion showing how rolling motion is damped. The primary stability derivative involved in the roll mode is $\boldsymbol{L_p}$, and the primary control power is provided by the ailerons with $\boldsymbol{L_{\delta A}}$

- **Spiral Mode:** the spiral mode is usually very slow and often of not critical importance for piloted aircraft. It is almost a pure yaw motion combined with induced roll ($\boldsymbol{N_p}$). A 747 has a nonoscillatory spiral mode that damps to half amplitude in 95 seconds under typical conditions, while many airplanes have unstable spiral modes that require pilot input from time to time to maintain heading.



Dutch Roll Mode: is a coupled roll and yaw motion that is often not sufficiently damped for good handling. The response consists of a combined rolling, sideslipping, yawing motion reminiscent of a (Dutch) speed-skater. Transport aircraft often require active yaw dampers to suppress this motion. High directional stability (C_{n_β}) tends to stabilize the Dutch-Roll mode while reducing the stability of the spiral mode. Conversely large effective dihedral (rolling moment due to sideslip, C_{l_β}) stabilizes the spiral mode while destabilizing the Dutch-Roll motion. Because sweep produces effective dihedral and because low wing airplanes often have excessive dihedral to improve ground clearance, Dutch-Roll motions are often poorly damped on swept-wing aircraft.





DUTCH ROLL

- **Example:** Boeing 747 aircraft in a powered approach at Mach 0.25 and at standard sea-level conditions.

$$V = 279.1 \text{ ft/sec}, \quad \rho = 0.002377 \text{ slug/ft}^3 \quad 1\,\text{slug} = 1\,\frac{\text{lb}_F \cdot \text{s}^2}{\text{ft}}$$
$$S = 5,500. \text{ ft}^2, \quad \bar{c} = 27.3 \text{ ft}$$
$$W = 564,032. \text{ lb}, \quad I_y = 32.3 \times 10^6 \text{ slug-ft}^2$$
$$W = 564,032. \text{ lbf}, \quad b = 195.7 \text{ ft}$$
$$I_x = 14.3 \times 10^6 \text{ slug ft}^2, \quad I_z = 45.3 \times 10^6 \text{ slug ft}^2, \quad I_{xz} = -2.23 \times 10^6 \text{ slug ft}^2$$

$$\begin{array}{lll} C_{y_\beta} = -.96 & C_{y_p} = 0.0 & C_{y_r} = 0.0 \\ C_{l_\beta} = -.221 & C_{l_p} = -.45 & C_{l_r} = 0.101 \\ C_{n_\beta} = 0.15 & C_{n_p} = -.121 & C_{n_r} = -.30 \end{array} \qquad \begin{array}{lll} Y_v = -0.0999, & Y_p = 0.0, & Y_r = 0.0 \\ L_v = -0.0055, & L_p = -1.0994, & L_r = 0.2468 \\ N_v = 0.0012, & N_p = -.0933, & N_r = -.2314 \end{array}$$

$$\mathbf{A} = \begin{pmatrix} -0.0999 & 0.0000 & 32.174 & -279.10 \\ -0.0057 & -1.0932 & 0.0 & 0.2850 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0015 & -.0395 & 0.0 & -.2454 \end{pmatrix} \qquad \boldsymbol{x}(t) = \begin{bmatrix} v\left[\frac{ft}{\text{sec}}\right] \\ p\left[\frac{rad}{\text{sec}}\right] \\ \phi[rad] \\ r\left[\frac{rad}{\text{sec}}\right] \end{bmatrix}$$

$$\lambda_{\text{DR}} = -.08066 \pm \imath\ 0.7433$$
$$\lambda_{\text{roll}} = -1.2308$$
$$\lambda_{\text{spiral}} = -.04641$$

$$\mathbf{A} = \begin{pmatrix} -0.0999 & 0.0000 & 0.1153 & -1.0000 \\ -1.6038 & -1.0932 & 0.0 & 0.2850 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.4089 & -.0395 & 0.0 & -.2454 \end{pmatrix} \qquad \boldsymbol{x}(t) = \begin{bmatrix} \beta = \frac{v}{U_0}[rad] \\ p\left[\frac{rad}{\text{sec}}\right] \\ \phi[rad] \\ r\left[\frac{rad}{\text{sec}}\right] \end{bmatrix}$$

The damping ratio of the Dutch Roll mode is thus given by

$$\zeta_{\text{DR}} = \sqrt{\frac{1}{1+\left(\frac{\eta}{\xi}\right)^2_{\text{DR}}}} = \sqrt{\frac{1}{1+\left(\frac{0.7433}{0.08066}\right)^2}} = 0.1079$$

and the undamped natural frequency of the mode is

$$\omega_{n_{\text{DR}}} = \frac{-\xi_{\text{DR}}}{\zeta_{\text{DR}}} = \frac{0.08066}{0.1079} = 0.7477 \text{ sec}^{-1}$$

The period of the Dutch Roll mode is then given by

$$T_{\text{DR}} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\pi}{0.7477\sqrt{1-0.1079^2}} = 8.45 \text{ sec}$$

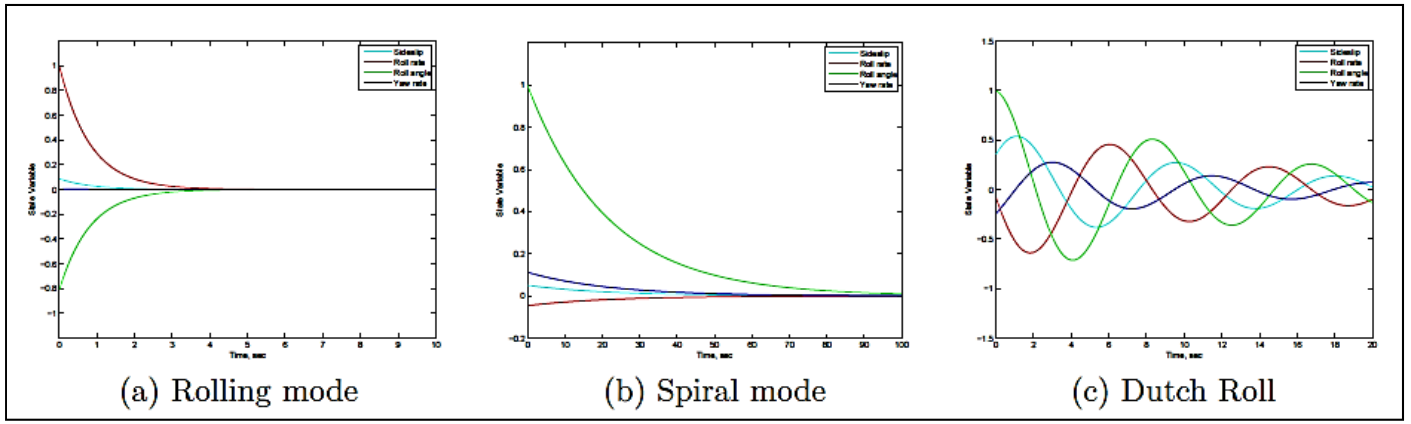
and the number of cycles to damp to half amplitude is

$$N_{1/2_{\text{DR}}} = \frac{\ln 2}{2\pi}\frac{\sqrt{1-\zeta^2}}{\zeta} = \frac{\ln 2}{2\pi}\frac{\sqrt{1-0.1079^2}}{0.1079} = 1.016$$

Thus, the period of the Dutch Roll mode is seen to be on the same order as that of the longitudinal short period mode, but is much more lightly damped.

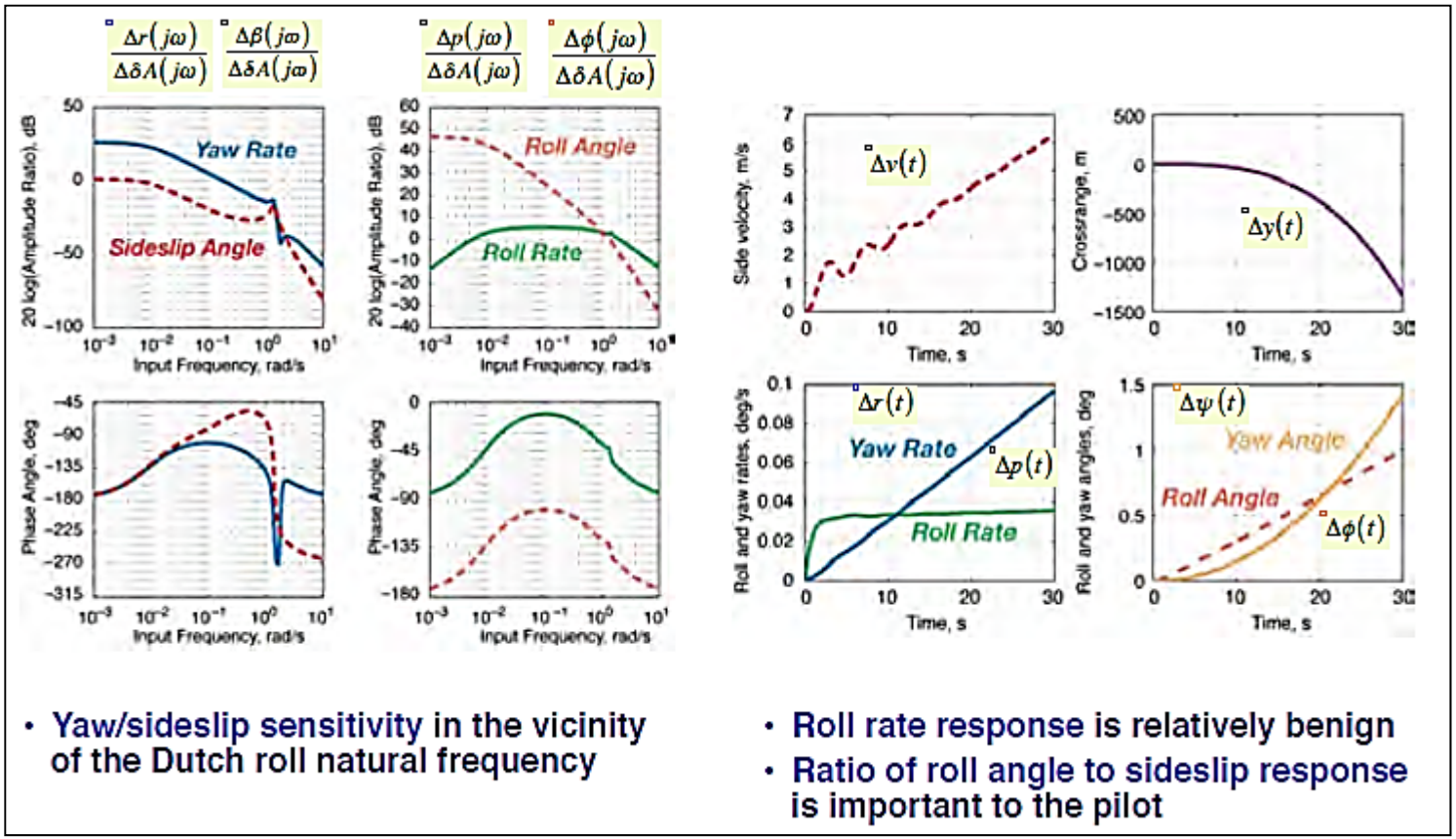
$$t_{1/2_{\text{roll}}} = \frac{\ln 2}{-\xi_{\text{roll}}} = \frac{\ln 2}{1.2308} = 0.563 \text{ sec}$$

$$t_{1/2_{\text{spiral}}} = \frac{\ln 2}{-\xi_{\text{spiral}}} = \frac{\ln 2}{0.04641} = 14.93 \text{ sec}$$



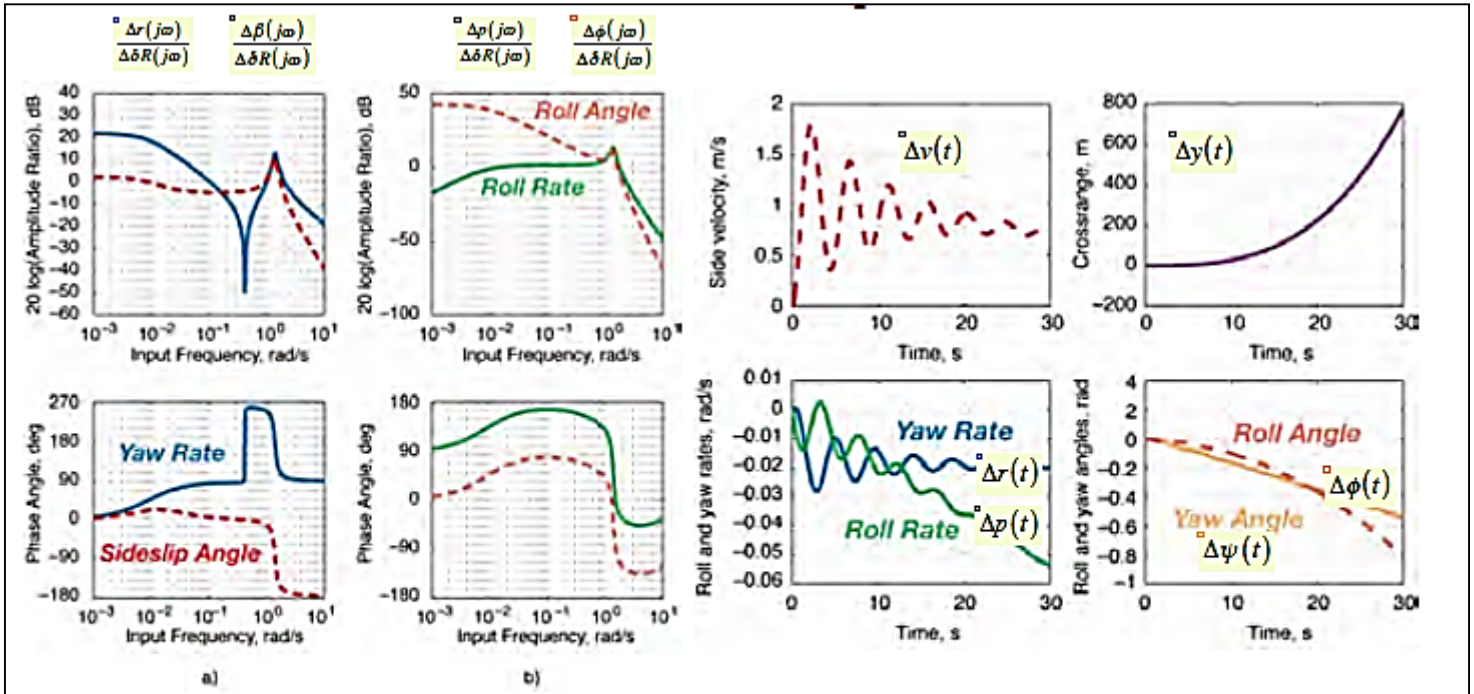
(a) Rolling mode (b) Spiral mode (c) Dutch Roll

- **Frequency Response Analysis:** if we assume availability of all state variables from measurements, the transfer function matrix has dimensions 4x2. Here are typical responses to ailerons input.



- Yaw/sideslip sensitivity in the vicinity of the Dutch roll natural frequency
- Roll rate response is relatively benign
- Ratio of roll angle to sideslip response is important to the pilot

- **Frequency Response Analysis:** if we assume availability of all state variables from measurements, the transfer function matrix has dimensions 4x2. Here are typical responses to rudder input



- Yaw response variability near and below the Dutch roll natural frequency
- Significant roll rate response near the Dutch roll natural frequency
- Lightly damped yaw/sideslip response would be hard to control precisely

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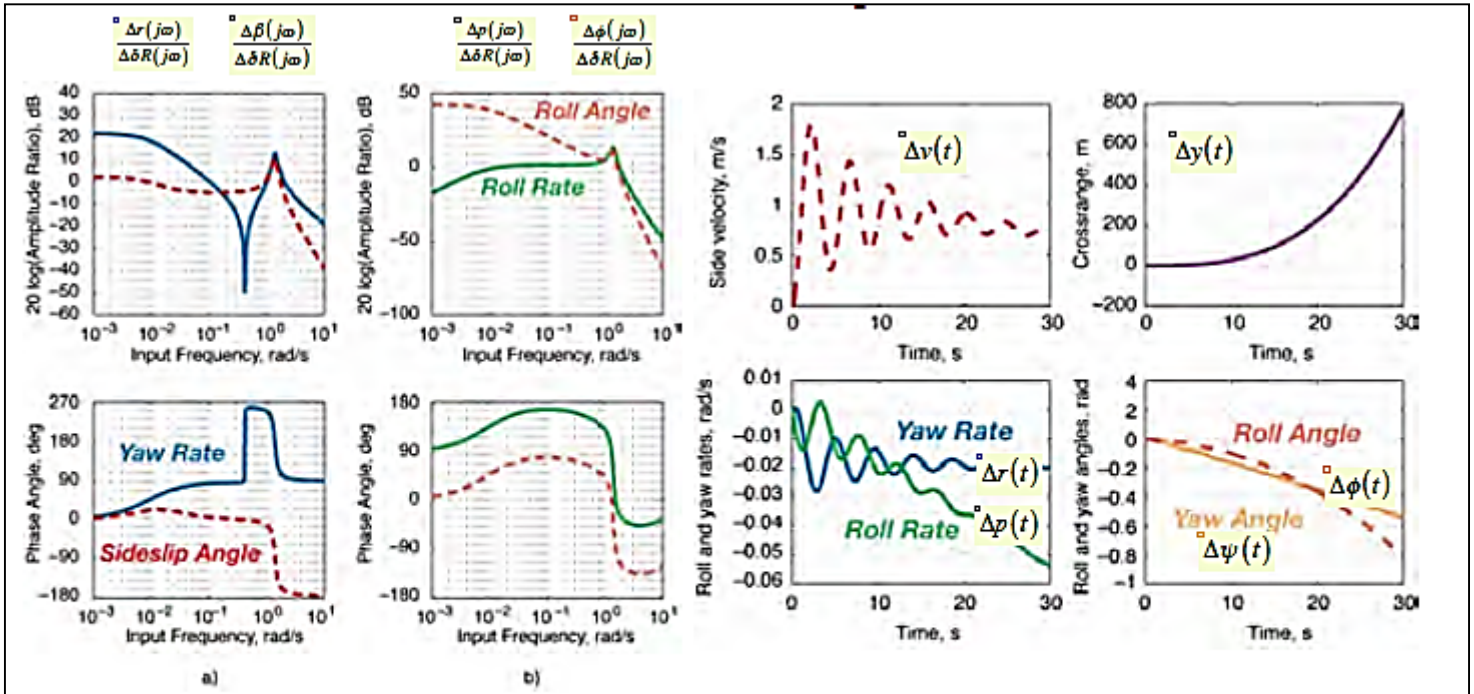


- Yaw response variability near and below the Dutch roll natural frequency
- Significant roll rate response near the Dutch roll natural frequency
- Lightly damped yaw/sideslip response would be hard to control precisely

❑ Lateral – Directional Modes Approximation:

❑ **Roll Mode Approximation:** at low angles of attack (no coning motion), we can reduce the dynamics to pure roll rate variation:

$$\begin{cases} (1 - I_{xx} I_{zz})\dot{p} = L_p p + I_{xx} N_p p + L_{\delta_A} \delta_A \\ \lambda_R = \dfrac{L_p + I_{xx} N_p}{(1 - I_{xx} I_{zz})} \approx L_p^{'} \end{cases}$$

- Example Boeing 747 power approach, M = 0.25

$$\lambda = \frac{-1.0994 + (-.1559)(-.0933)}{1 - (-.1559)(-.0492)} \sec^{-1} = -1.093 \sec^{-1}$$

$$\lambda_{full} = -1.238$$

❑ **Spiral Mode Approximation:** The spiral mode consists of a slow rolling/yawing motion for which the sideslip is relatively small. The roll rate is quite small compared to the yaw rate, so a reasonable approximation is to set roll rate equal to zero:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 0 = \frac{L_v + i_x N_v}{1 - i_x i_z} v + \frac{L_r + i_x N_r}{1 - i_x i_z} r \qquad v \approx -\frac{L_r + i_x N_r}{L_v + i_x N_v} r \qquad v \approx -\frac{L_r}{L_v} r$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{N_v + i_z L_v}{1 - i_x i_z} v + \frac{N_r + i_z L_r}{1 - i_x i_z} r$$

$$\dot{r} = \left(N_r - \frac{L_r N_v}{L_v} \right) r + N_{\delta_A} \delta_A + N_{\delta_R} \delta_R$$

- Spiral Mode eigenvalue approximation

$$\lambda = N_r - \frac{L_r N_v}{L_v}$$

- According to this approximation, the spiral mode is stabilized by yaw damping N_r. Also, since stable dihedral effect corresponds to negative L_v and weathercock stability N_v and roll due to yaw rate L_r are always positive, the second term is destabilizing; thus increasing weathercock destabilizes the spiral mode while increasing dihedral effect stabilizes it.

  - Example Boeing 747 power approach, M = 0.25 $\lambda = -.2314 - \frac{0.2468}{-.0055}(0.0012) = -.178$

$$\lambda_{full} = -04641$$

- **Dutch Roll Mode Approximation:** The Dutch Roll mode is particularly difficult to approximate because it usually involves significant perturbations in all four state variables. The most useful approximations require neglecting either the roll component or simplifying the sideslip component by assuming the vehicle c. g. travels in a straight line

.

$$\psi \approx -\beta \text{ or } r = -\frac{\dot{v}}{U_0}$$

- With the approximation: $1 - I_{xx}I_{zz} \sim 1$

$$\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -U_0 \\ L_v & L_p & L_r \\ N_v & N_p & N_r \end{bmatrix} \begin{bmatrix} v \\ p \\ r \end{bmatrix} + \begin{bmatrix} Y_{\delta_R} & 0 \\ L_{\delta_R} & L_{\delta_A} \\ N_{\delta_R} & N_{\delta_A} \end{bmatrix} \begin{bmatrix} \delta_R \\ \delta_A \end{bmatrix}$$

$$\lambda^3 - (L_p + N_r)\lambda^2 + (L_p N_r + u_0 N_v - L_r N_p)\lambda + u_0 (L_v N_p - L_p N_v) = 0$$

- This is still a cubic equation, for which there is no general closed-form solution. A useful approach to cubic equations that have a lightly damped oscillatory mode is Bairstow's approximation, which proceeds as follows. If a general cubic equation:

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

has a lightly damped oscillatory mode, its undamped natural frequency can be approximated as

$$a_2\lambda^2 + a_0 \approx 0 \quad \text{or} \quad \lambda^2 \approx -\frac{a_0}{a_2}$$

$$a_2\lambda^2 + \left(a_1 - \frac{a_0}{a_2}\right)\lambda + a_0 = 0$$

- In our case, we obtain:

$$\lambda^2 - \left[\frac{L_p N_r + u_0 N_v - L_r N_p}{L_p + N_r} + \frac{u_0 \left(L_v N_p - L_p N_v\right)}{\left(L_p + N_r\right)^2}\right]\lambda + \frac{u_0 \left(L_p N_v - L_v N_p\right)}{L_p + N_r} = 0$$

$$\boxed{\omega_n^2 = \frac{u_0 \left(L_p N_v - L_v N_p\right)}{L_p + N_r}}$$

Since N_p is usually negative, both terms in the numerator have the same sign for stable dihedral. Thus, increasing either weathercock stability N_v or dihedral effect L_v increases the natural frequency of the motion.

$$2\zeta\omega_n = \frac{-L_p N_r - u_0 N_v + L_r N_p}{L_p + N_r} + \frac{u_0 \left(-L_v N_p + L_p N_v\right)}{\left(L_p + N_r\right)^2}$$

For most aircraft, the ratio N_r/L_p is small,

$$\boxed{2\zeta\omega_n \approx -N_r\left(1 + \frac{u_0}{L_p^2} N_v\right) + \frac{N_p}{L_p}\left(L_r - \frac{u_0}{L_p} L_v\right)}$$

Yaw damping is thus seen to contribute to positive ζ and be stabilizing, and weathercock stability N_v augments this effect. Since both N_p and L_p usually are negative, however, the dihedral effect L_v is seen to destabilize the Dutch Roll mode.[5]

[5] Recall that there are two contributions from the wing to the yaw-due-to-roll derivative N_p; profile drag contributes to positive N_p, while induced drag contributes to negative N_p. At low values of lift coefficient (i.e., high speeds) the profile drag contribution can dominate, in which case N_p becomes positive. In this case, increased dihedral effect can improve damping of the Dutch Roll mode. Consistent with this observation is the fact that Dutch Roll tends to be a more serious problem at low speeds.

- For our example of the Boeing 747 in powered approach at M = 0.25,

$$\omega_n = \left[\frac{(279.1)\left[(-1.0994)(0.0012) - (-.0055)(-.0933)\right]}{-1.0994 - .2314}\ \text{sec}^{-2}\right]^{1/2} = 0.620\ \text{sec}^{-1}$$

and

$$\zeta = -\frac{\frac{(-1.0994)(-.2314)-(.2468)(-.0933)+(279.1)(0.0012)}{-1.0994-.2314} + (279.1)\frac{(-.0055)(-.0933)-(-1.0994)(0.0012)}{(-1.0994-.2314)^2}}{2(0.620)}$$

$$= 0.138$$

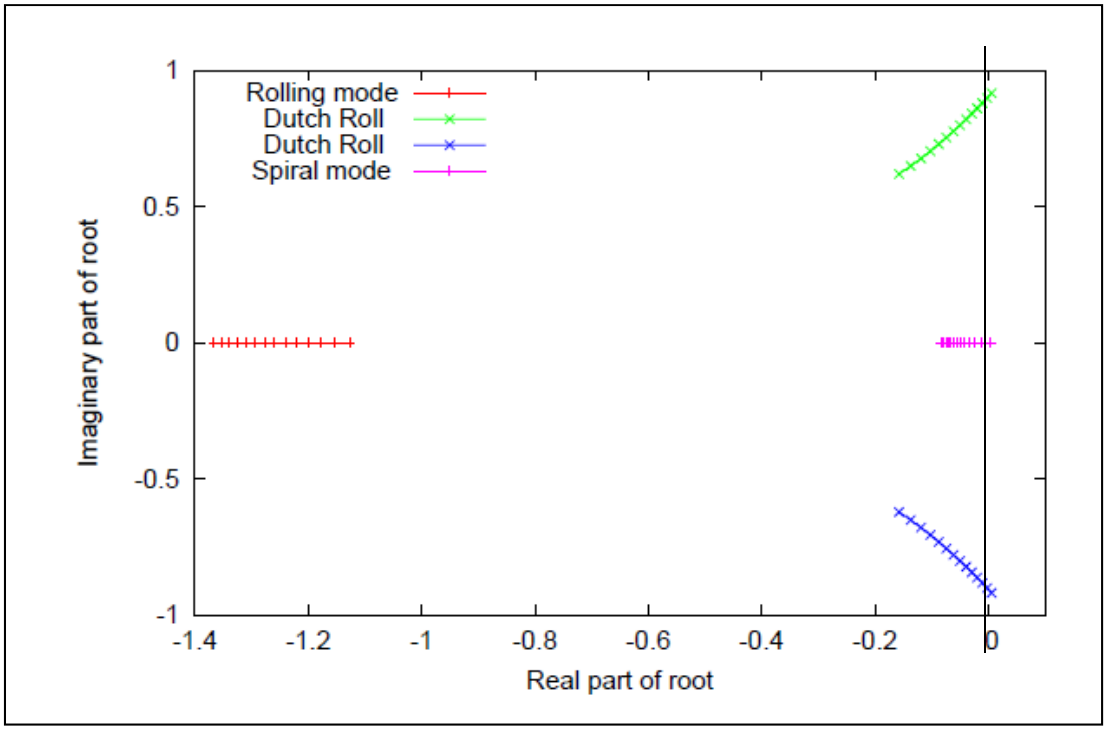
$$\lambda_{DR} = -0.0856 \pm j0.6141$$

$$\lambda_{\text{DR}} = -.08066 \pm \imath\ 0.7433$$

- Influence of stability derivatives on lateral – directional eigenvalues

$$L_v = \frac{QSb}{I_x u_0} C_{l\beta}$$

- The dihedral effect $C_{l\beta}$ is increased from -.041 to -.561.

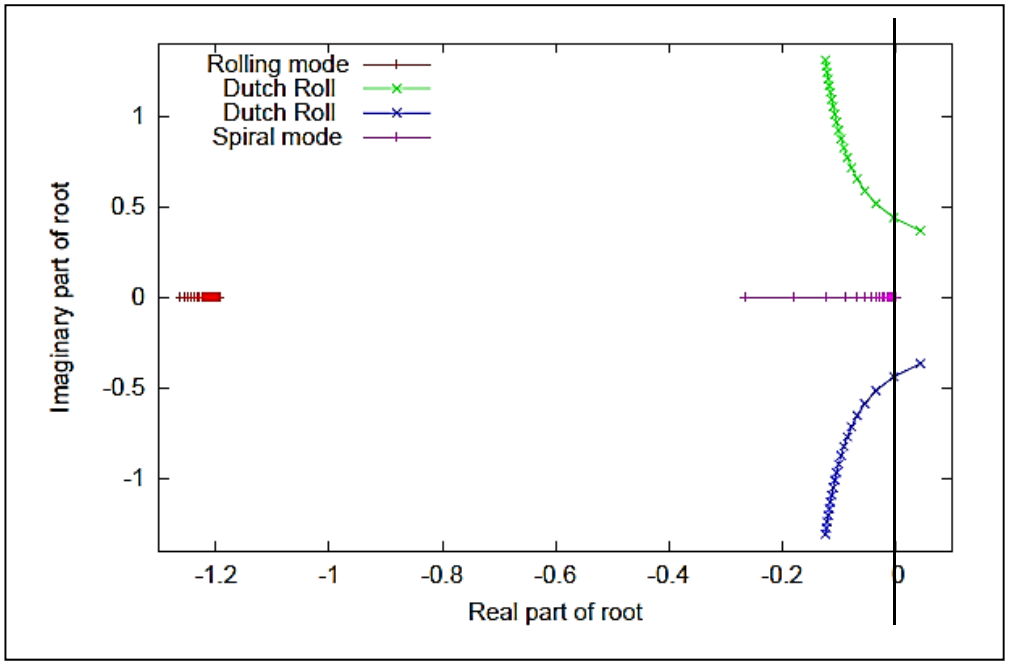


all other stability derivatives are held fixed at their nominal values. Rolling and spiral modes become increasingly stable as dihedral effect is increased; spiral mode becomes stable at approximately $C_{l\beta} = -.051$. Dutch Roll mode becomes less stable as dihedral effect is increased and becomes unstable at approximately $C_{l\beta} = -.532$.

- Influence of stability derivatives on lateral – directional eigenvalues

$$N_v = \frac{QSb}{I_z u_0} \mathbf{C}_{n\beta}$$

- Weathercock stability coefficient $C_{n\beta}$ is increased from -.07 to 0.69.



all other stability derivatives are held fixed at their nominal values. Rolling and spiral modes become less stable as weathercock stability is increased; spiral mode becomes unstable at approximately $\mathbf{C}_{n\beta} = 0.6567$. Dutch roll mode becomes increasingly stable as weathercock stability is increased, but is unstable for less than about $\mathbf{C}_{n\beta} = -.032$.

❑ Summary of Natural Modes

Simulation of Natural Modes

- **External Disturbances:** primary influence due to gusts and turbulence. Using linearized models, the influence of disturbances is added to the equations of motion.

  - Example of introduction of vertical gust w_g in the longitudinal equations of motion

$$w_g(t) = \frac{k}{T}(1 - \cos 2\pi \frac{t}{T})$$

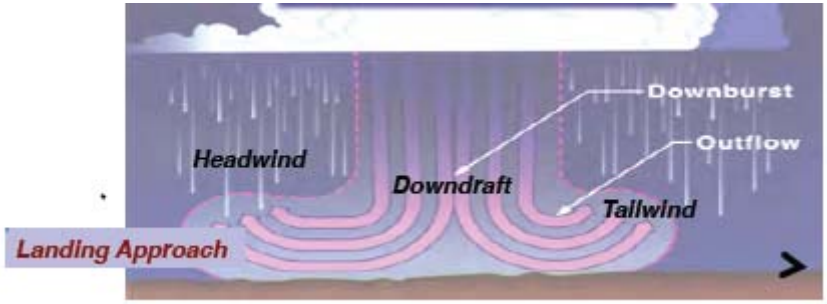
$$\dot{u} + W_0 q + g\cos\Theta_0\theta = X_u(u-u_g) + X_w(w-w_g) + X_{\dot{w}}(\dot{w}-\dot{w}_g) + X_q(q-q_g) + \sum_i X_{\delta_i}\delta_i \quad (3.1)$$

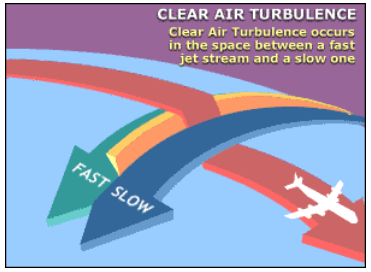
$$\dot{w} - U_0 q + g\sin\Theta_0\theta = Z_u(u-u_g) + Z_w(w-w_g) + Z_{\dot{w}}(\dot{w}-\dot{w}_g) + Z_q(q-q_g) + \sum_i Z_{\delta_i}\delta_i \quad (3.2)$$

$$\dot{q} = M_u(u-u_g) + M_w(w-w_g) + M_{\dot{w}}(\dot{w}-\dot{w}_g) + M_q(q-q_g) + \sum_i M_{\delta_i}\delta_i \quad (3.3)$$

  - Turbulence models use velocity characteristics represented by white noise processes with specified power spectra (i.e. Dryden Spectra), according to turbulence severity, aircraft type, flight condition, etc.

Atmospheric Flight Dynamics

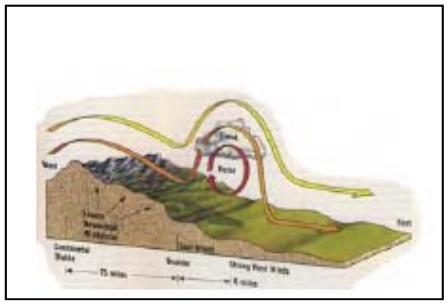


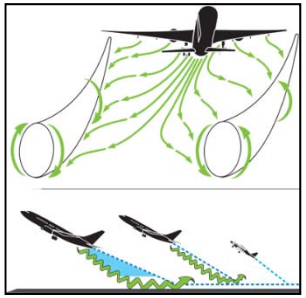


IIT MADRAS

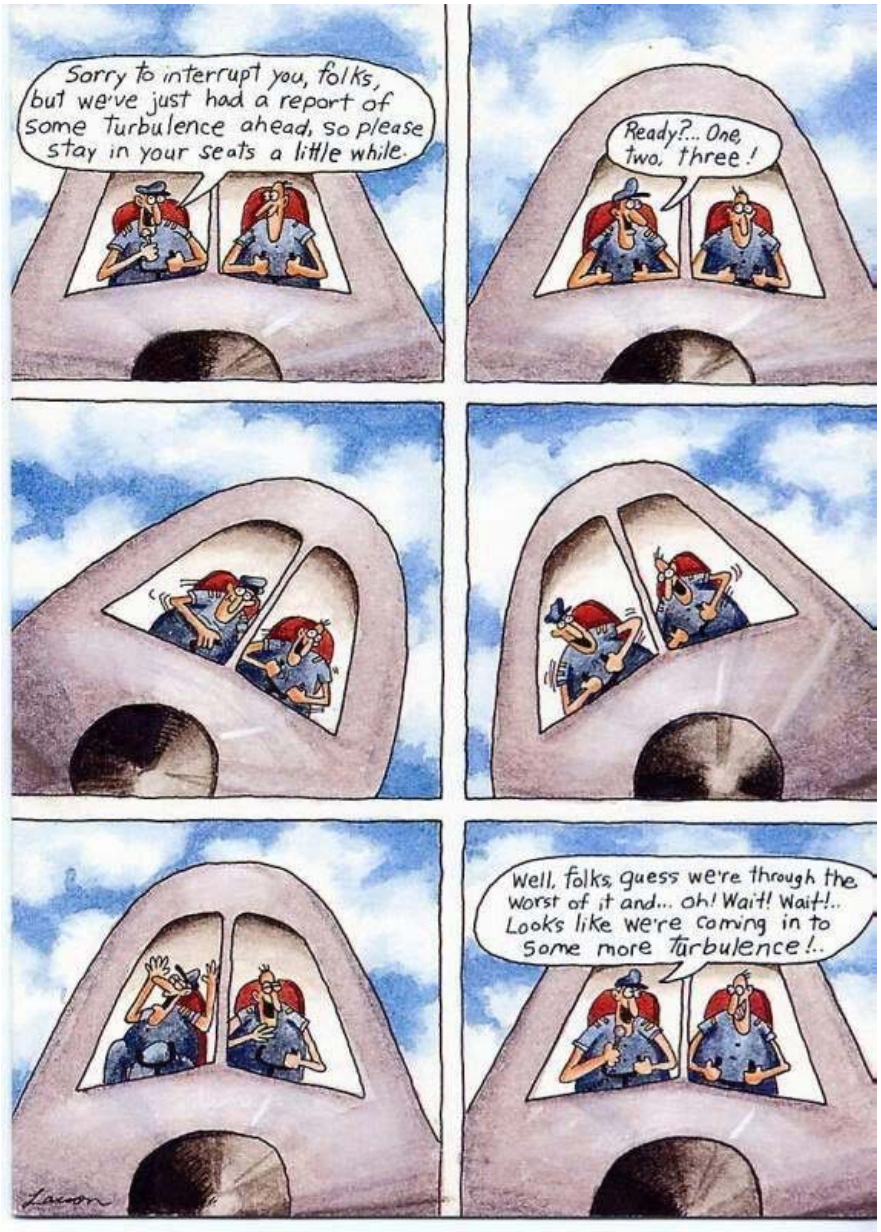
Denver to Baltimore

SCHIPHOL ATC TOWER





Atmospheric Flight Dynamics



Glossary

Altitude	Quota
Flight Envelope	Inviluppo di volo
Lift, Drag	Portanza, Resistenza
Wind, Velocity, Stability Axes	Assi vento, velocità, stabilità
Steady Pull up	Cabrata
Steay condition	Condizione non accelerata
Maneuver	Manovra
Still Atmosphere	Atmosfera a riposo
Dynamic Pressure	Pressione dinamica
Camber	Curvatura
Airfoil	Profilo (alare)
Straight level flight	Volo livellato
Climd, Descent	Salita, discesa
Boundary Layer	Strato limite
Flight Path	Traiettoria di volo

Glossary

Inviscid, Irrotational Flow	Fluido non viscoso, irrotazionale
Steady Coordinated Turn	Virata coordinata
Leading, Trailing Edges	Bordo di attacco, di uscita
Angle of attack, Sideslip	Angoli di attacco e di derapata
Downwash	Downwash
True, Ground, Indicated, Calibrated Airspeed	Velocità vera, al suolo, indicata, calibrata
Heading	Direzione
Bank	Inclinazione
Aspect Ratio	Allungamento alare
Longitudinal, Lateral-Directional Stability	Stabilità longitudinale, latero-direzionale
Wing Sweep, Dihedral Angle	Freccia alare, Angolo diedro
Trim, Static Margin	Equilibrio, Margine statico
Stability Derivative	Derivate di stabilità (derivate aerodinamiche)
Elevator, Horizontal Stabilizer	Equilibratore, piano di coda orizzontale

Glossary

Ailerons, Rudder, Fin, Tail	Alettoni, timone, deriva, coda
Lift Curve Slope, Aerodynamic Center	Pendenza curva di portanza, Centro aerodinamico
Weathercock Effect	Effetto banderuola
Short Period, Phugoid	Corto Periodo, Fugoide
Roll, Spiral, Dutch-Roll	Rollio, spirale, Dutch-Roll
Gust, Turbulence	Raffica, turbolenza
Wake	Scia
Neutral Point	Punto neutro
Vorticity	Vorticità
Horseshoe Vortices	Vortici a ferro di cavallo
Slats, Flaps	Slats, flaps
Winglet, Spoliler	Aletta, Spoiler
Load Factor, Turn Rate	Fattore di carico, Velocità di curvatura (long.)
Handling Qualities	Qualità di volo
Pitch, Roll, Yaw	Beccheggio, Rollio, Imbardata

Glossary

The brief history of 737 max problems:
Viewer advisory

https://www.youtube.com/watch?v=H2tuKiiznsY