**Task 6 - Safety Review and Licensing**
**On the Job Training on Stress Analysis**

# Fracture Mechanics: Linear Elastic Fracture Mechanics 1/2

Davide Mazzini – **Ciro Santus**

BEYOND THE RESEARCH

UNIVERSITÀ DI PISA

**Pisa (Italy)**
**June 15 – July 14, 2015**

## Content

- <u>Stress singularity</u>
  - Notch degenerating into a crack
  - Multi-axial stress at notch root/ crack tip
  - The Williams problem
- Linear Elastic Fracture Mechanics (LEFM)
  - The Westergaard stress function
  - Definition and calculation of the Stress Intensity Factors (SIFs)
  - LEFM Validity limitations

**Books on Fracture Mechanics**

T.L. Anderson, Fracture Mechanics: Fundamentals and Applications, third edition. CRC Press 2005.

D. Broek. The Practical Use of Fracture Mechanics. Kluwer 1989.

*… and many many others*

- Experities and similitude (up to 1700)
- Elastic evaluations (nominal solutions) (Eulero, Cauchy, De SaintVenant, 1800)
- Stress concentrations (Kirsch, Inglis, 1900)
- Theory of plasticity (Prandtl, 1920)
- Sharp tip defects (Griffith, 1922)

- Griffith’s energy approach for brittle materials (1930)
- Practical relevance (1940-1950)
- Definition of K, extension to metallic materials, complete develpment of the Linear Elastic Fracture Mechanics (LEFM) (Williams, Irwin, 1950)
- Application of the LEFM to Fatigue (Paris, 1960)
- Extension to ductile materials (Elatic Plastic Fracture Mechanics EPFM) (Irwin, Dugdale, Barenblatt, Wells, Landes, Rice, 1960)
- Dynamics and crack arrest (DFM), viscous and (NLFM) (AA.VV. 1980)
- Engineering applications, standards for design and testing, NDT, corrosion, anisotropic materials, Damage Tolerant approaches, .. (ASTM, ASME, ESIS, BS)

PERGAMON

Engineering Fracture Mechanics 69 (2002) 533–553

Engineering Fracture Mechanics

www.elsevier.com/locate/engfracmech

# The past, present, and future of fracture mechanics

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**Abstract**

The science of fracture mechanics was born and came to maturity in the 20th century. Its literature is now vast. Perhaps the most successful application of fracture mechanics is to fatigue. However, this short paper is limited to a core topic of fracture, the initiation and propagation of fracture under monotonic loading at low strain rates.

As the author was invited to present this paper on the occasion of his 65th birthday, it is a somewhat personal view of the development and future of fracture mechanics, but it is hoped that it will interest many, especially the young researchers at the beginning of their careers. 

# ***Liberty ships – World War II***

2500 liberty ships, hull assembled by the innovative process of welding

700 experienced heavy structural damage, 145 completely destroyed, many lost (complete breakage of the hull)

**Post-failure analysis**

- Failure at low stress (sometimes with the ship in the arbor)
- Quite "brittle" fractures
- Failure more frequent in winter time (ductile to brittle transition temperature)
- Effect of the technological process (metallurgical, geometrical: weld crack-like defects)

*Fracture mechanics was born to understand these failure!*

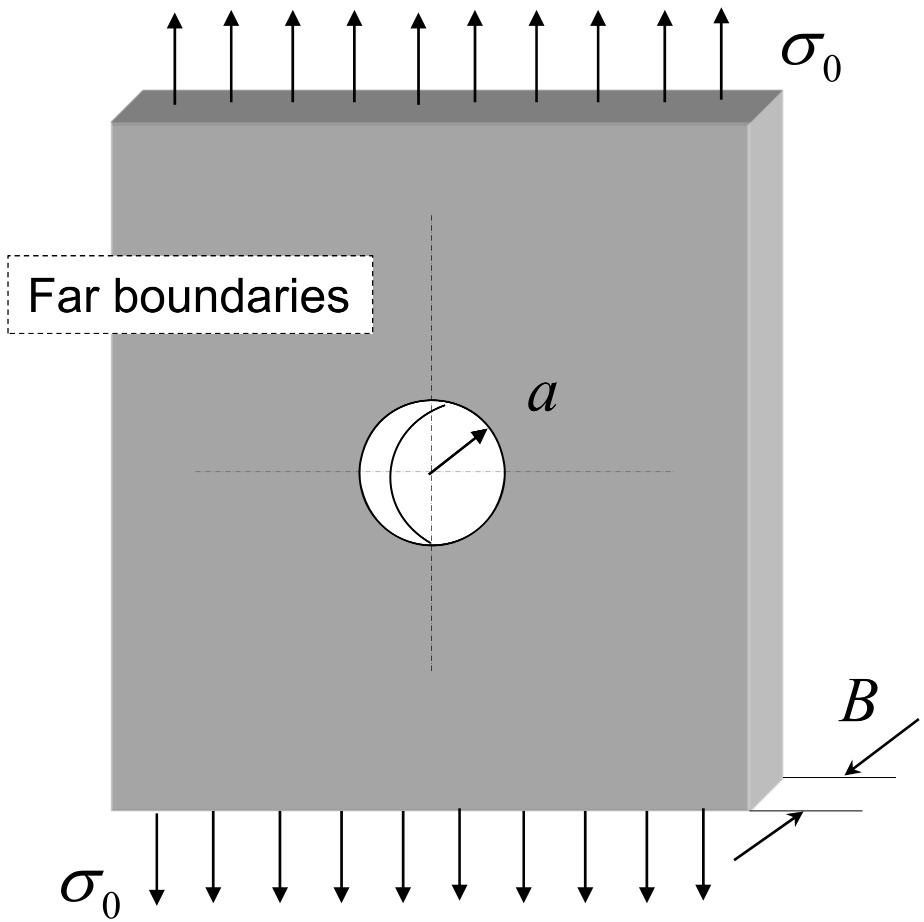
**Circular hole in a flat plate**

Complete analytical solution

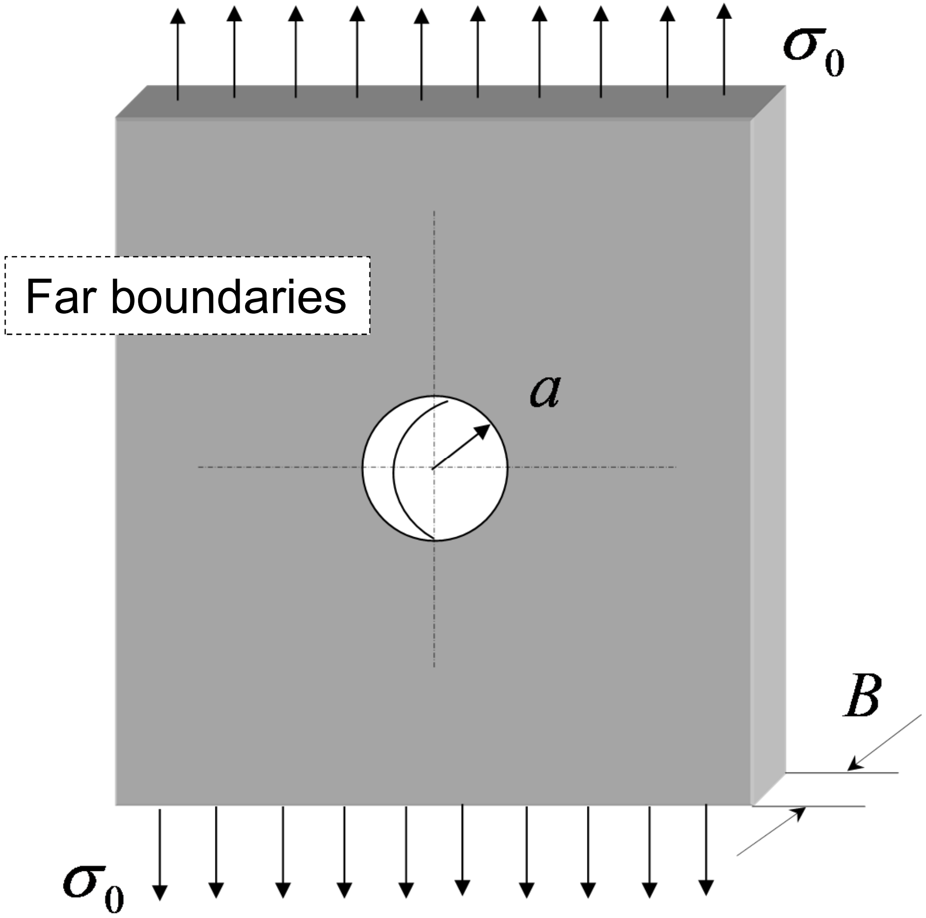
Plane stress solution if *a*>>*B*

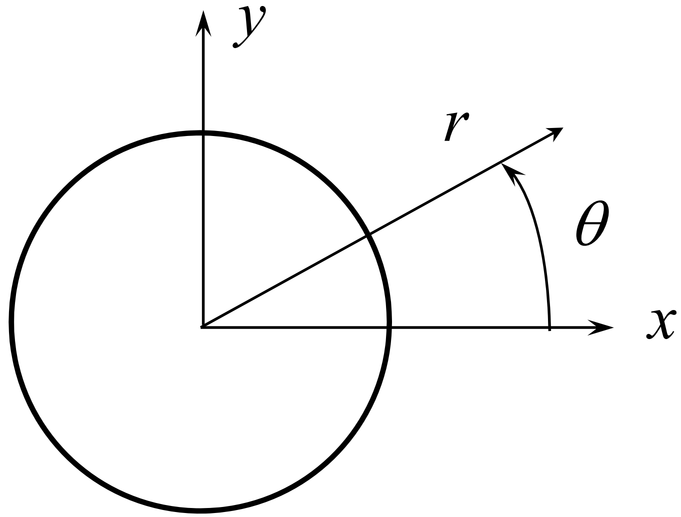
Plane strain if *a*<<*B*

Extension to other problems



## Circular hole in a flat plate



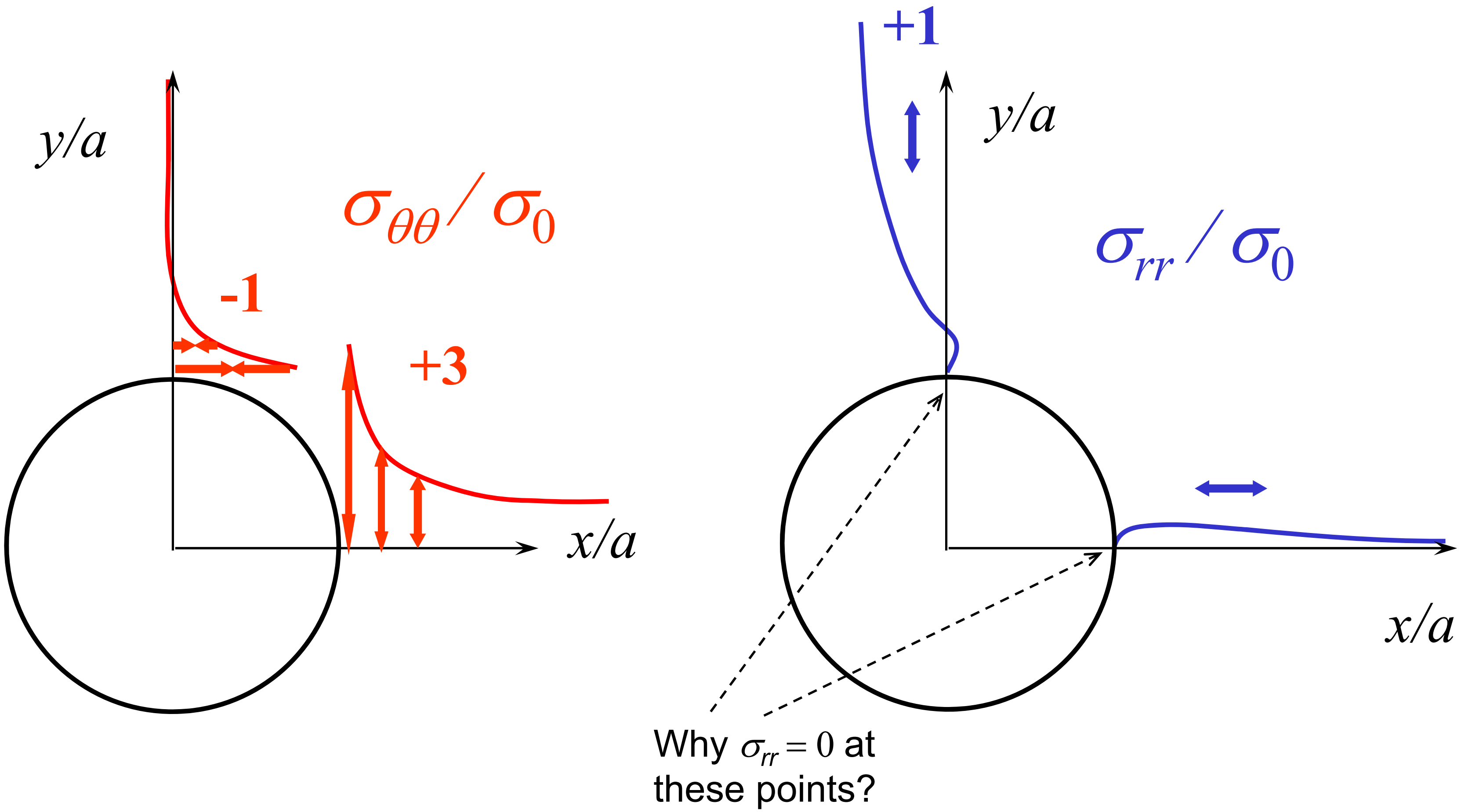




$$\sigma_{rr} = \frac{\sigma_0}{2}\left\{\left(1-\frac{a^2}{r^2}\right)\left[1-\left(1-3\frac{a^2}{r^2}\right)\cos 2\theta\right]\right\}$$

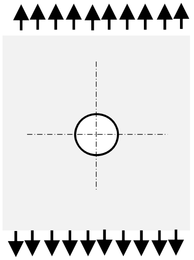
$$\sigma_{\theta\theta} = \frac{\sigma_0}{2}\left[\left(1+\frac{a^2}{r^2}\right)+\left(1+3\frac{a^4}{r^4}\right)\cos 2\theta\right]$$

$$\sigma_{r\theta} = \frac{\sigma_0}{2}\left(1-\frac{a^2}{r^2}\right)\left(1+3\frac{a^2}{r^2}\right)\sin 2\theta$$

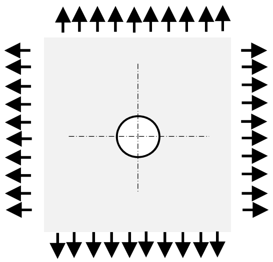
## Circular hole in a flat plate



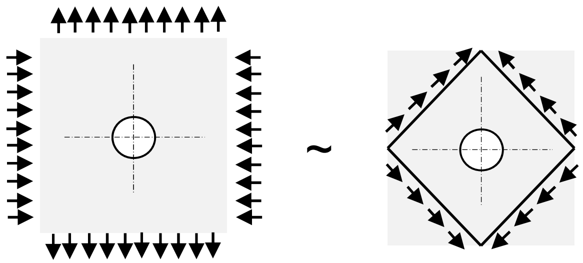
## Circular hole in a flat plate, bi-axial loading



Uniaxial $K_t = 3$

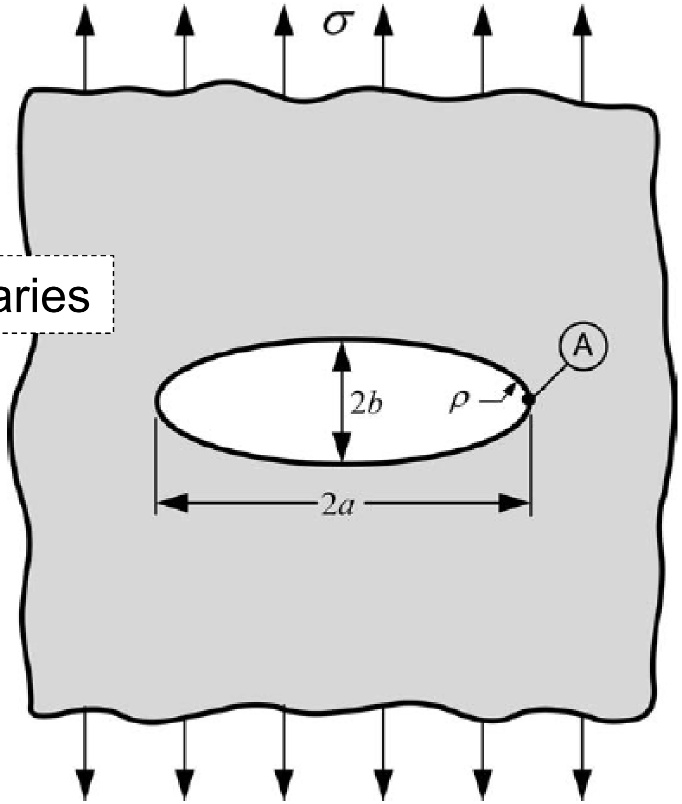


Equibiaxial $K_t = 2$



Pure shear $K_t = 4$

## Elliptical hole in a flat plate




Far boundaries

Problem definition:

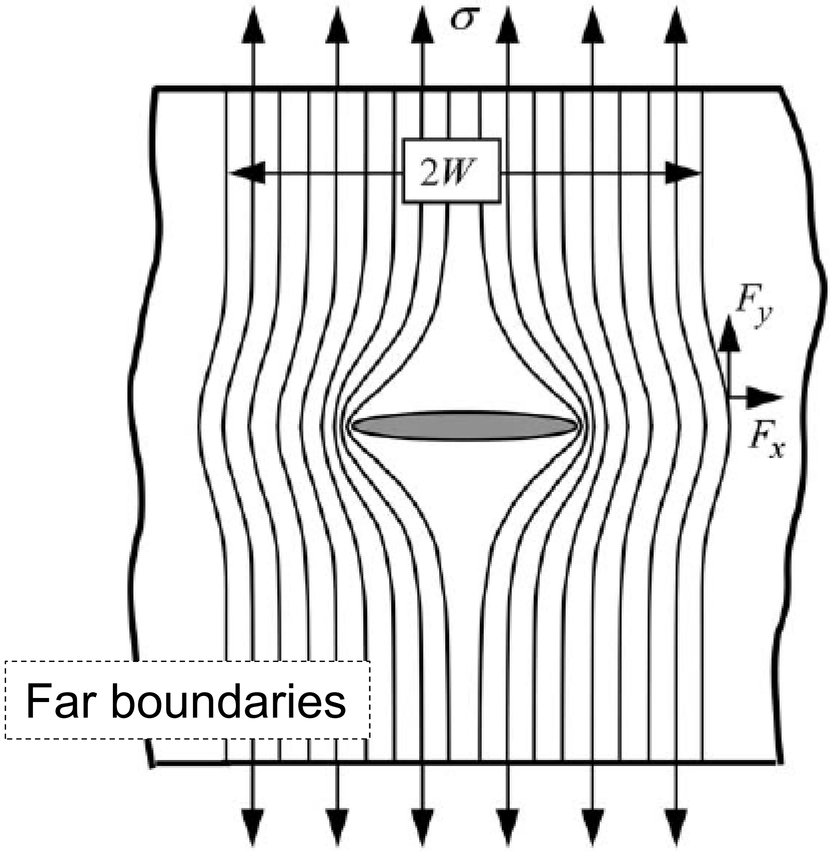
Geometry

$a, b$

Load, nominal stress

$\sigma$ (far field stress)

## Elliptical hole in a flat plate




Stress concentration:

$$\sigma_{\mathrm{A}} = \sigma\left(1 + \frac{2a}{b}\right)$$

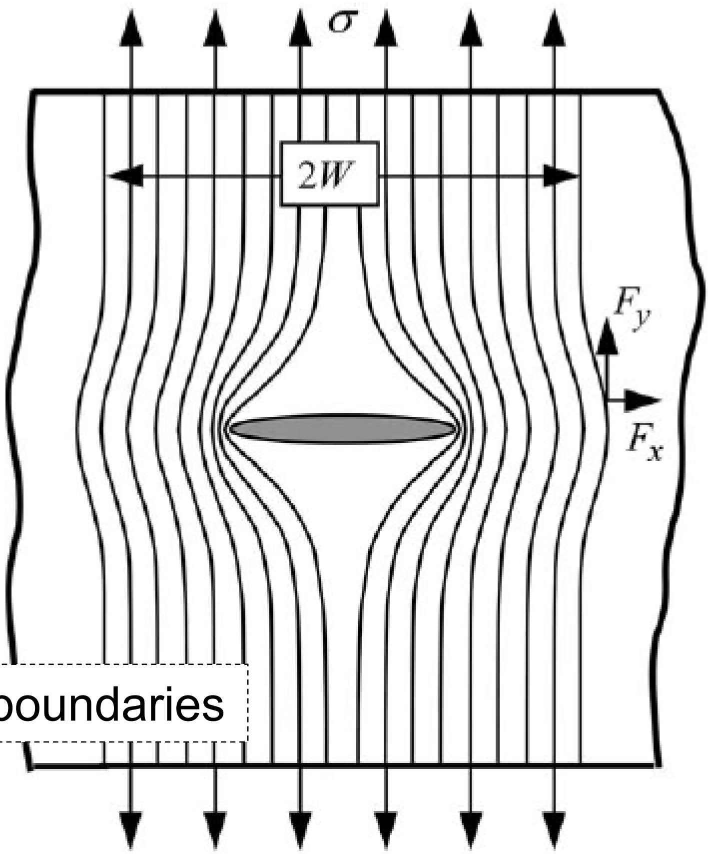
$$K_{\mathrm{t}} = \frac{\sigma_{\mathrm{A}}}{\sigma} = 1 + \frac{2a}{b}$$

Kirsch solution for central hole

$$b = a$$

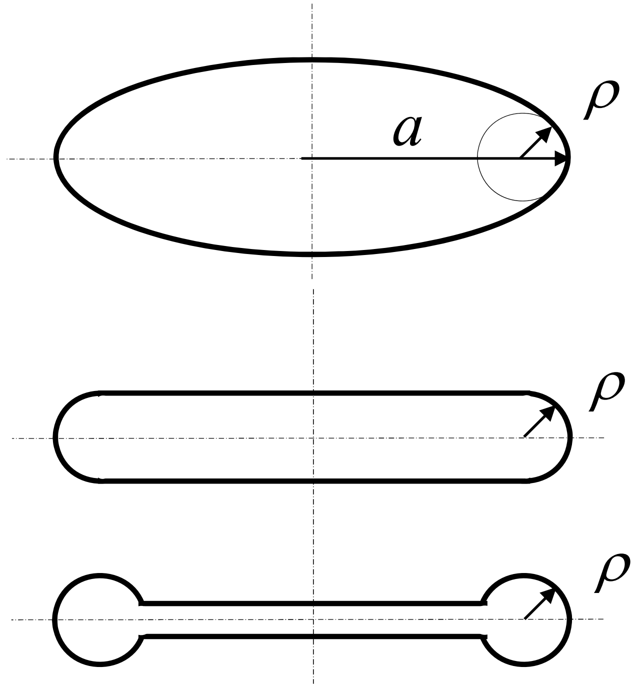
$$K_{\mathrm{t}} = 3$$

## Elliptical hole in a flat plate

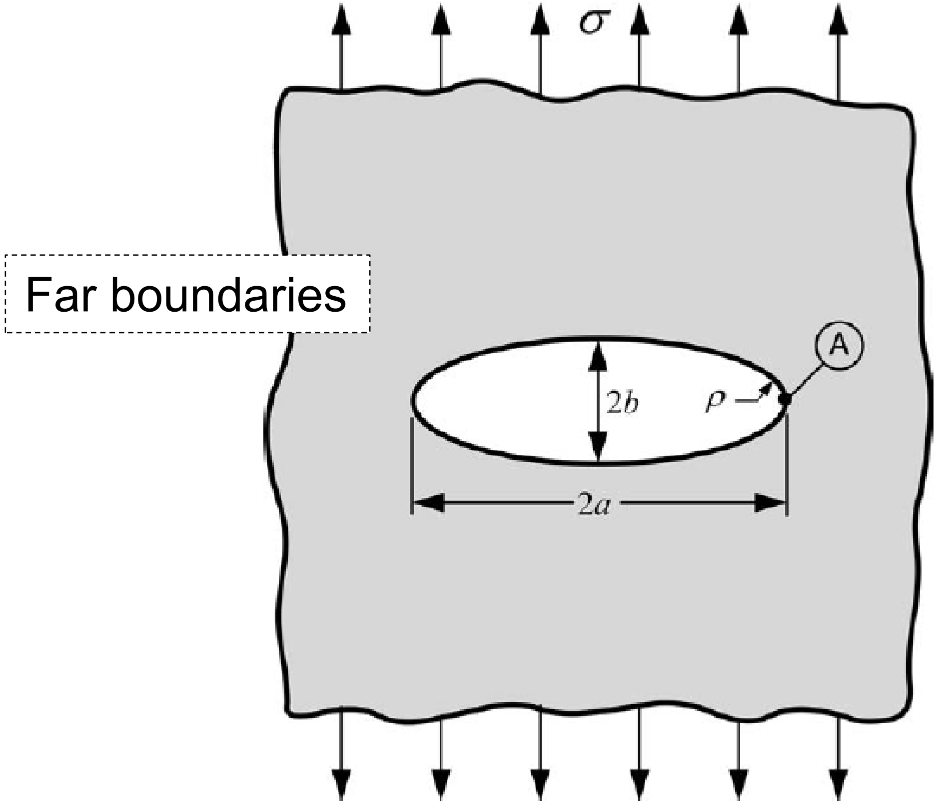



More significant, local radius:

$$\rho = \frac{b^2}{a}$$



## Elliptical hole in a flat plate




$$K_t = 1 + \frac{2a}{b}$$

being: $\rho = \frac{b^2}{a}$

then:

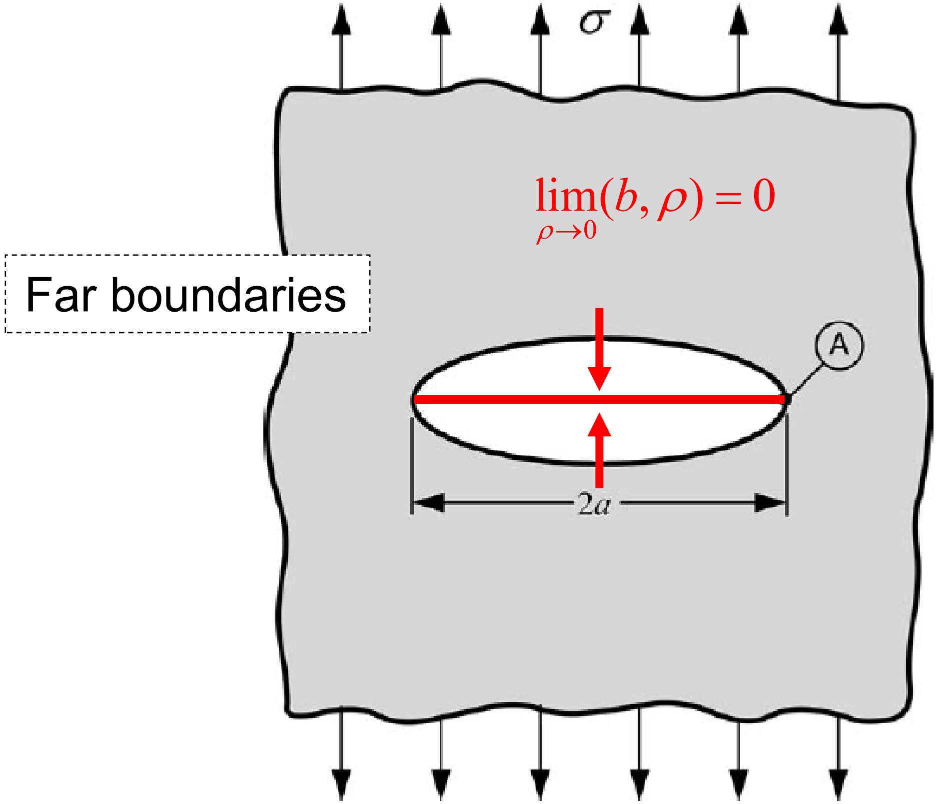
$$K_t = 1 + 2\sqrt{\frac{a}{\rho}}$$

$\rho, a$ are more properly defining the local geometry

when: $\rho << a$

$$K_t \approx 2\sqrt{\frac{a}{\rho}}$$

## Elliptical hole in a flat plate



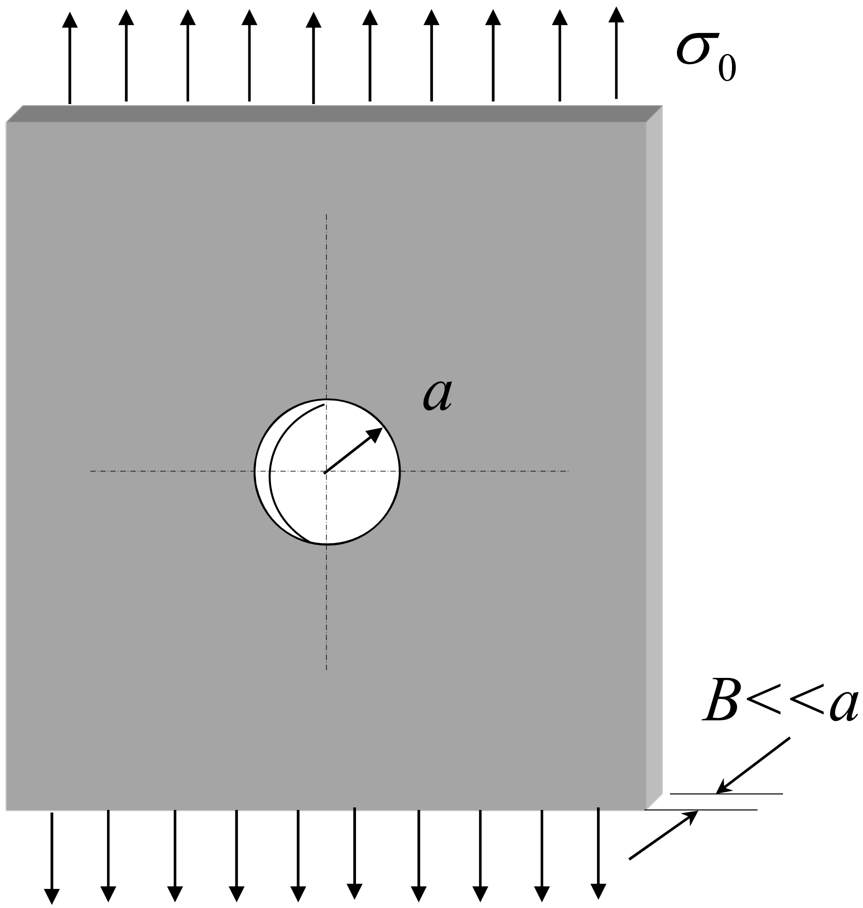

Limit:

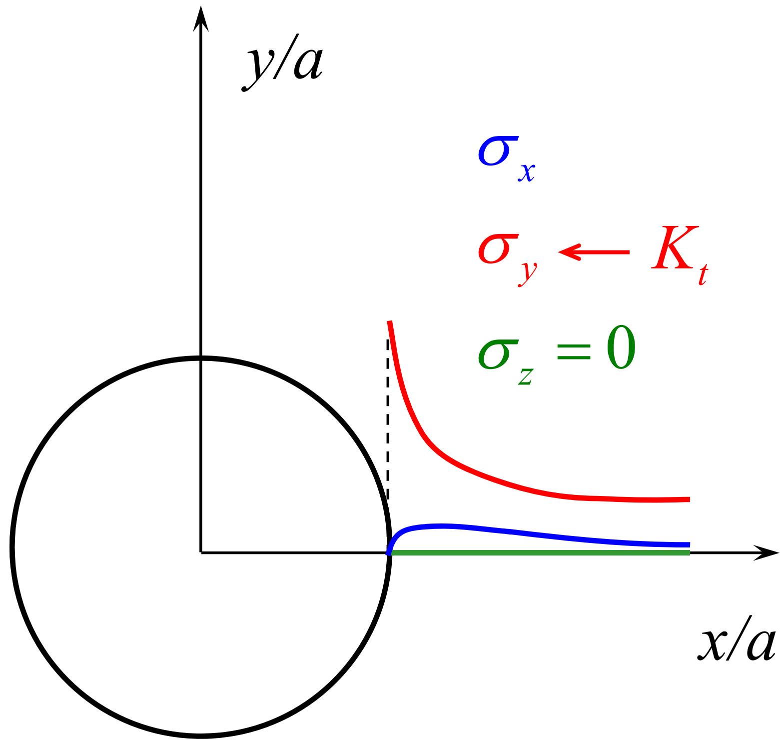
$$\lim_{\rho \to 0} K_t = \lim_{\rho \to 0} \left( 2\sqrt{\frac{a}{\rho}} \right) = \infty$$

and the power of singularity is

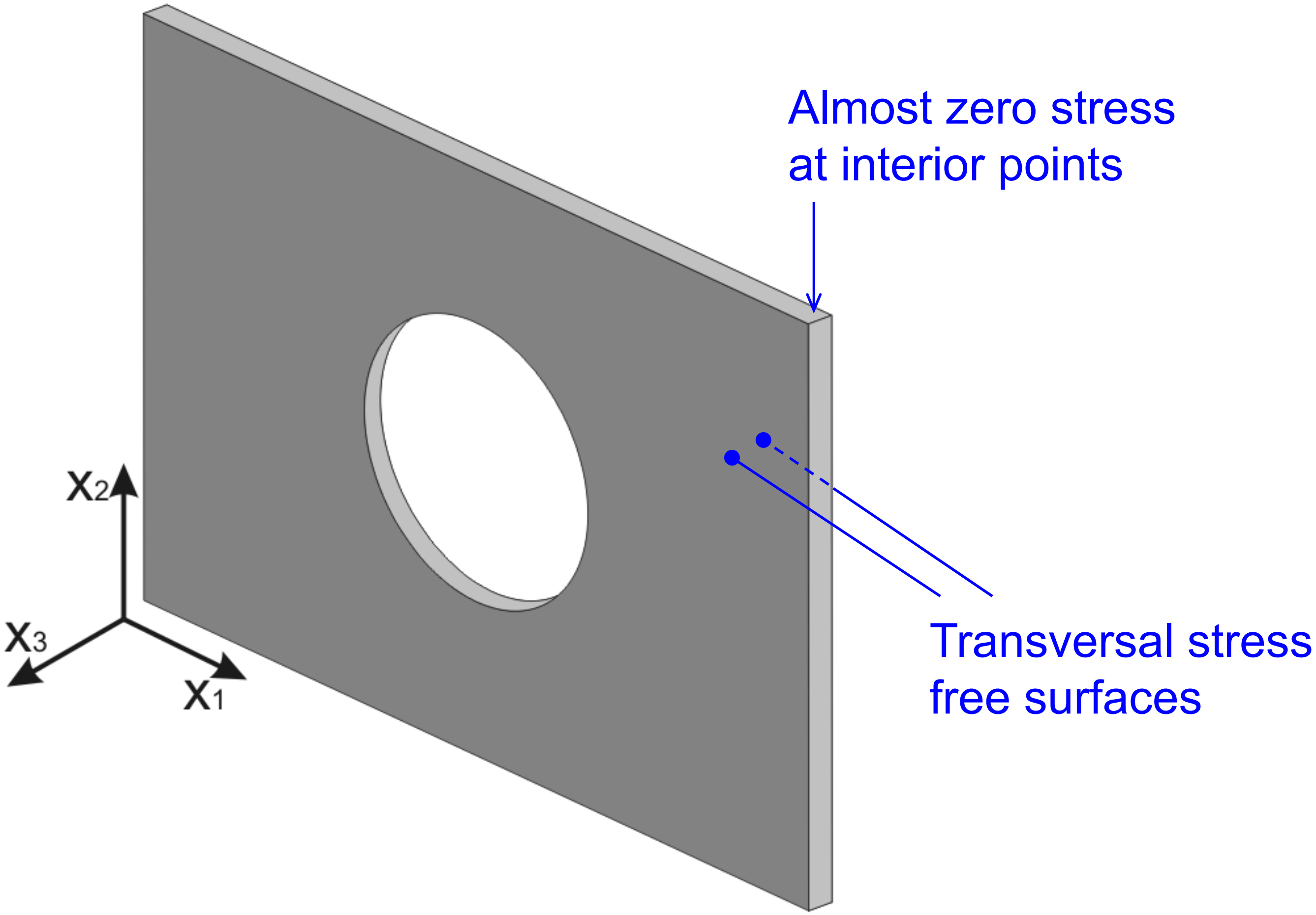
the square root of the local radius

## Stress components in-plane, *plane* **stress**

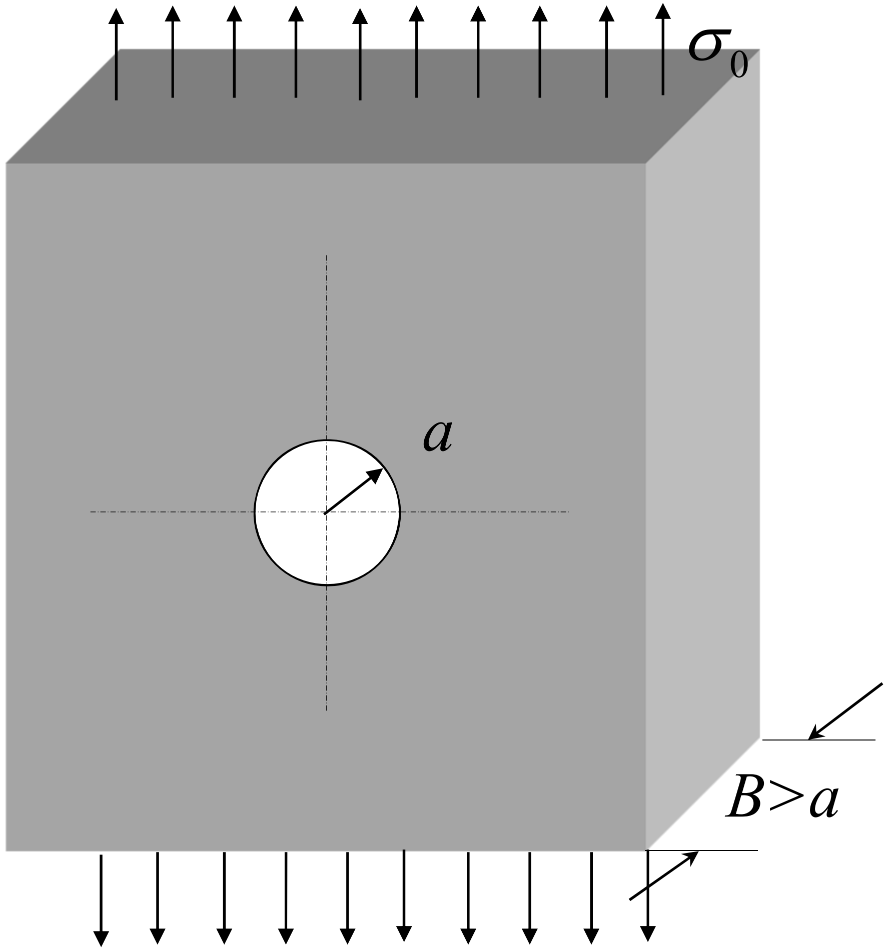


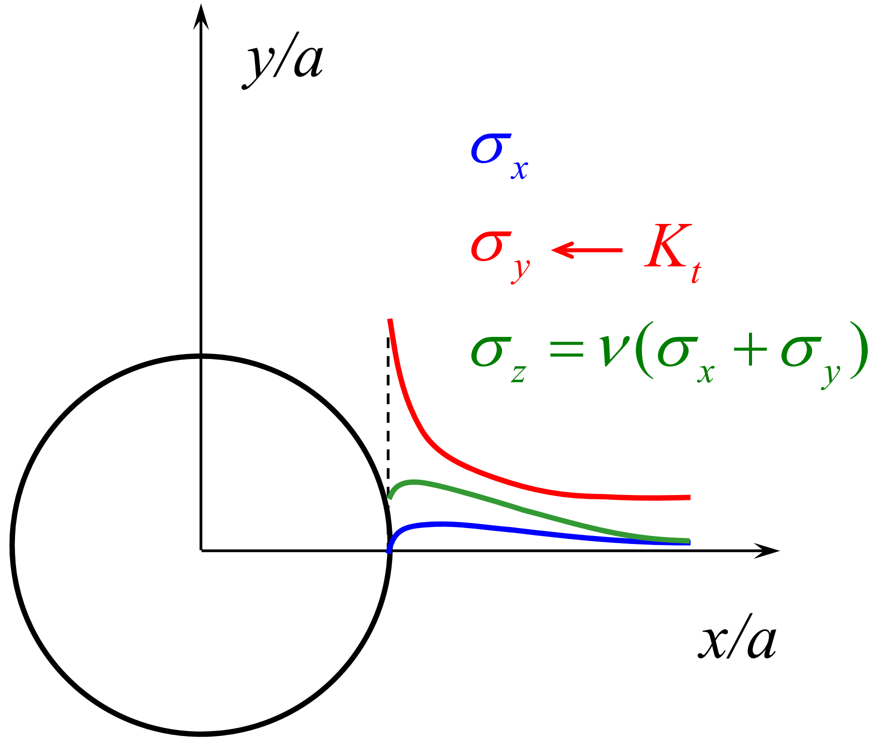



***Plane stress***

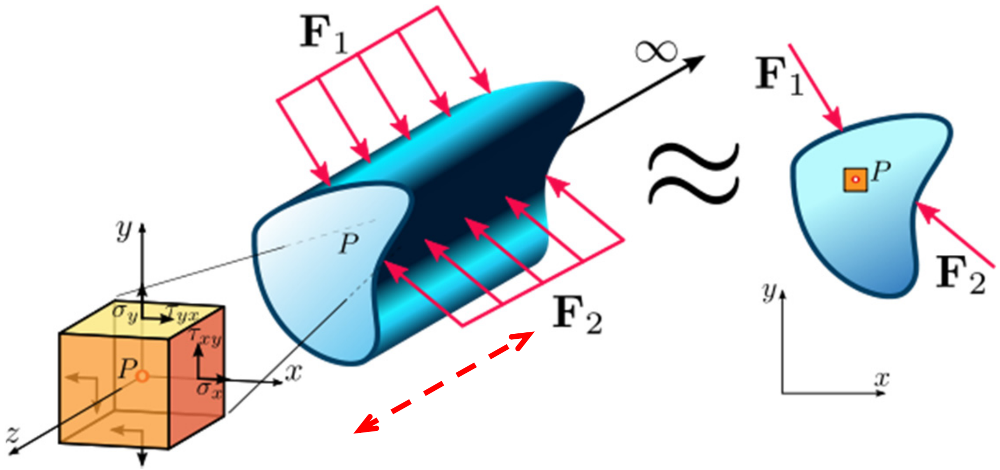


**Stress components in-plane, *plane strain (approx.)***



## *Plane strain*




Zero transversal displacement:

$\varepsilon_z = 0$

$$\frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix}$$

After imposing $\varepsilon_z = 0$

$$\frac{1}{E}(-\nu\sigma_x - \nu\sigma_y + \sigma_z) = 0$$

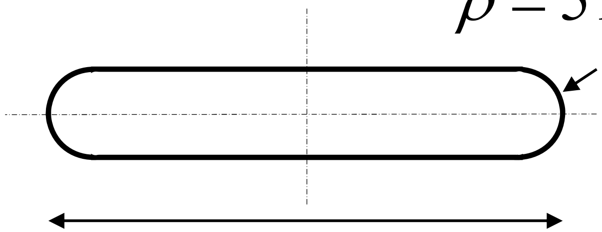
$\rightarrow$

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

**Inglis notch-like, *plane stress***

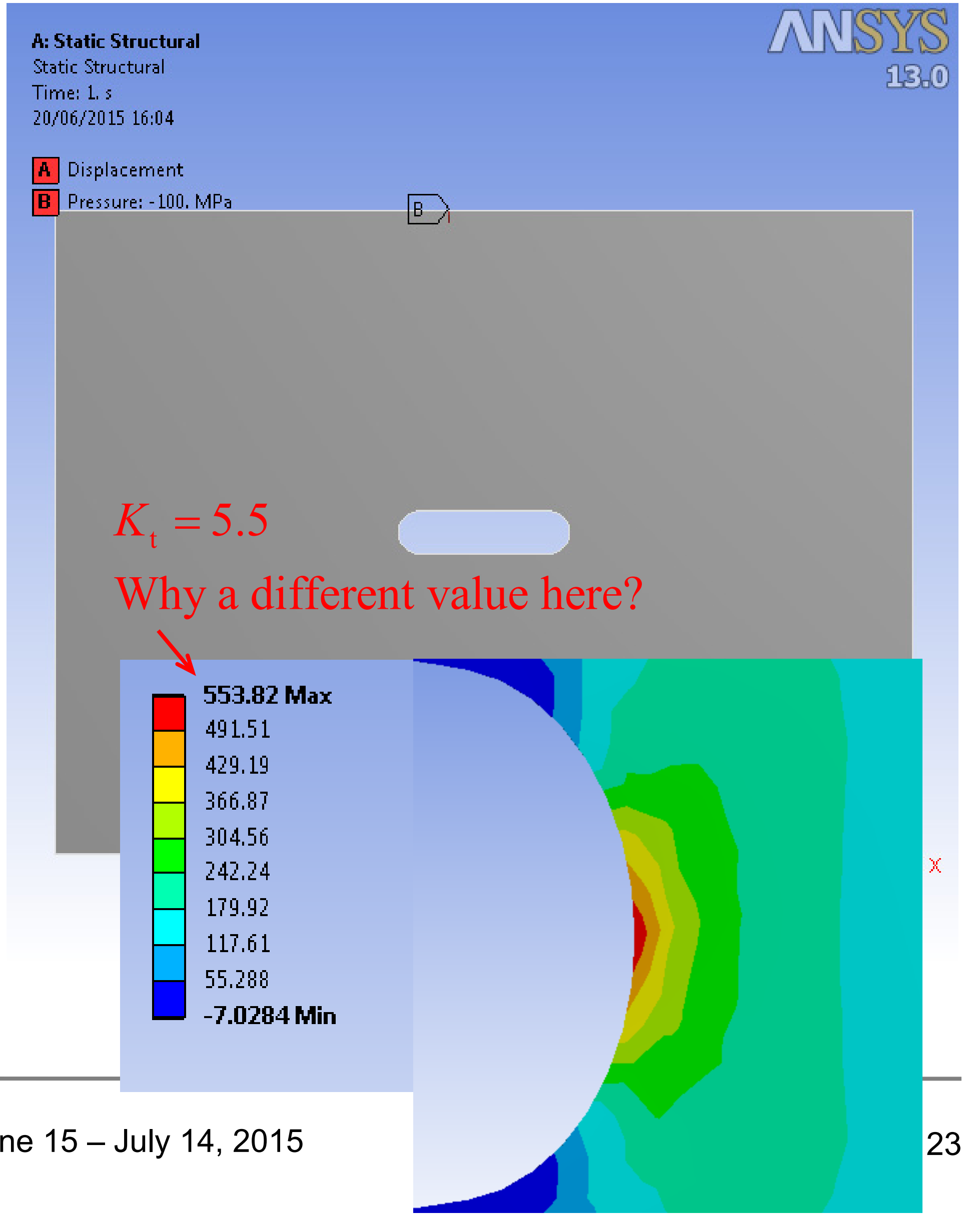
***ANSYS Wb***

$\rho = 5\,\mathrm{mm}$



$2a = 40\,\mathrm{mm}$

$$K_t = 1 + 2\sqrt{\frac{a}{\rho}} = 1 + 2\sqrt{\frac{20}{5}} = 5$$



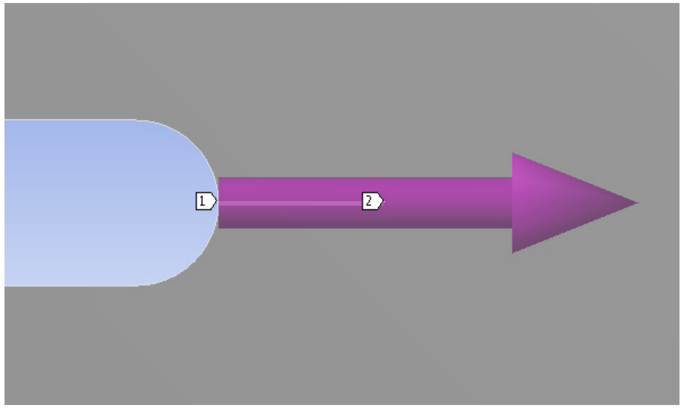

$K_t = 5.5$

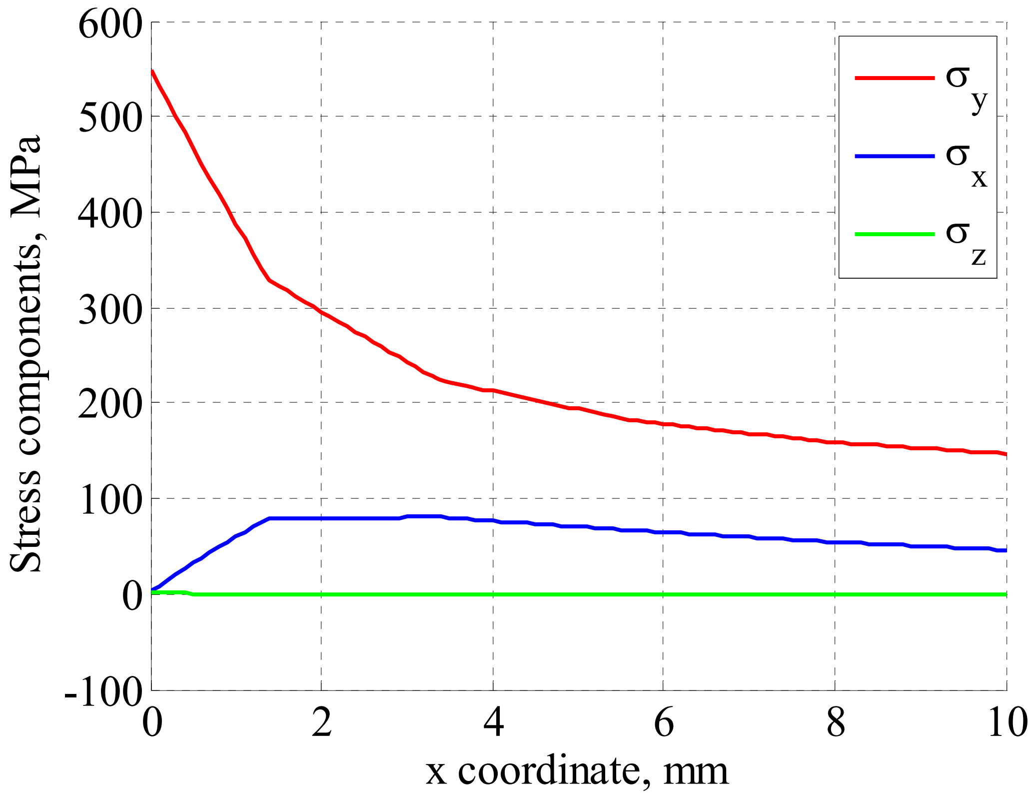
Why a different value here?

## Inglis notch-like, *plane stress*

***ANSYS Wb***

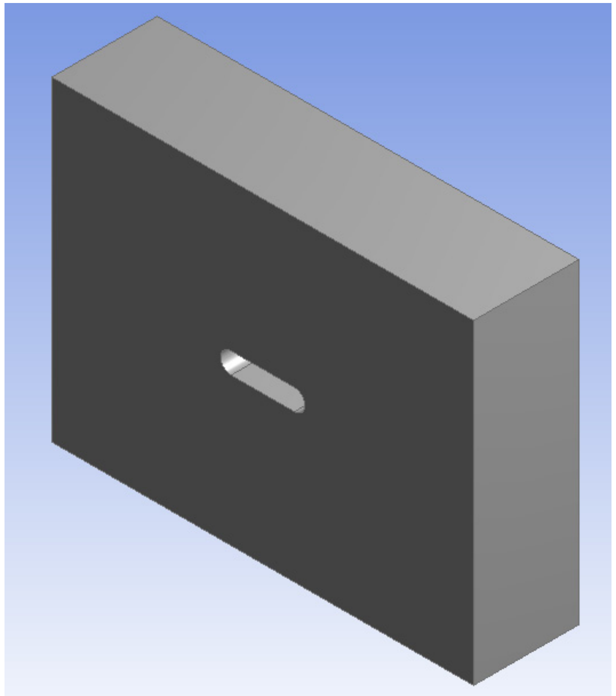
Path on the geometry



## Inglis notch-like, *plane strain*

### *ANSYS Wb*

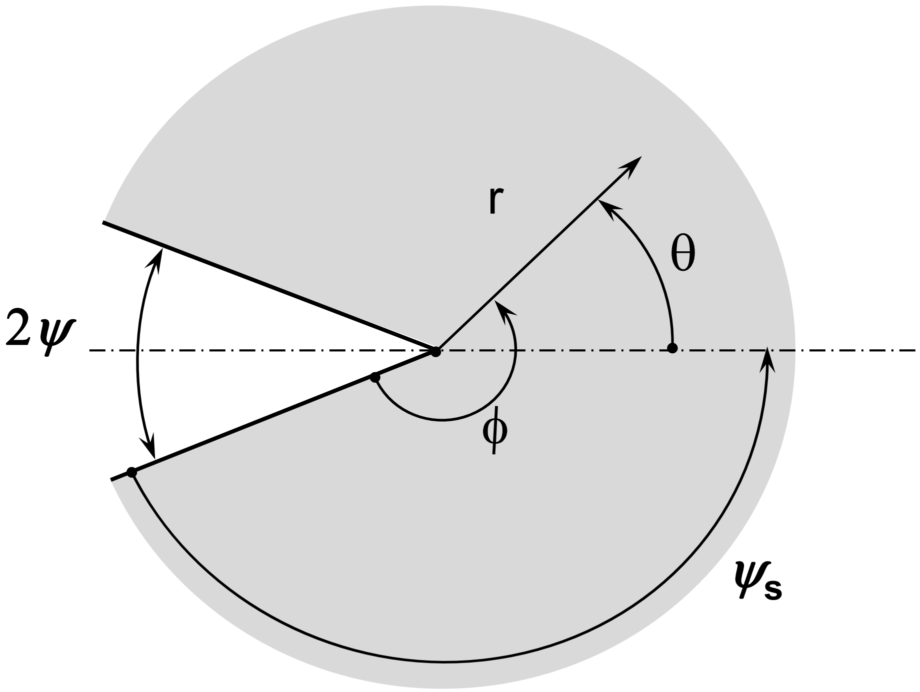


Exercise:

Calculate the Stress components, with ANSYS Workbench, at the notch tip for the large thickness geometry, and then verify the plain strain assumption

Repeat same calculation with imposed (exactly) plain strain constraint

## The Williams problem




- local geometry : $\rho = 0$
- governing parameters: $\psi_s = \pi - \psi$
- local polar coordinates: $r, \theta$
- useful angular variable: $\phi = \theta + \psi_s$

Airy function: $\Phi(x,y)$ $\quad \sigma_{xx} = \frac{\partial^2\Phi}{\partial x^2} \quad \sigma_{yy} = \frac{\partial^2\Phi}{\partial y^2} \quad \sigma_{xy} = \frac{\partial^2\Phi}{\partial x\partial y}$

Governing equation: $\nabla^2\nabla^2\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi = 0$

$$\frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4} = 0$$

Polar coordinates:

$$\nabla^2\nabla^2\Phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\Phi = 0$$

$$\frac{\partial^4\Phi}{\partial r^4} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r^4}\frac{\partial^4\Phi}{\partial\theta^4} + 2\frac{1}{r}\frac{\partial^3\Phi}{\partial r^3} + 2\frac{1}{r^2}\frac{\partial^4\Phi}{\partial r^2\partial\theta^2} + 2\frac{1}{r^3}\frac{\partial^3\Phi}{\partial r\partial\theta^2} = 0$$

$$\sigma_{rr} = \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} \; ; \quad \sigma_{\theta\theta} = \frac{\partial^2\Phi}{\partial r^2} \; ; \quad \sigma_{r\theta} = -\frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\Phi}{\partial\theta}$$

Stress components

- Williams hypothesis for the Airy function: $\Phi = r^{\lambda+1} F_\lambda(\phi)$

$$\Phi = r^{\lambda+1}\left[c_1 \sin(\lambda+1)\phi + c_2 \cos(\lambda+1)\phi + c_3 \sin(\lambda-1)\phi + c_4 \cos(\lambda-1)\phi\right]$$

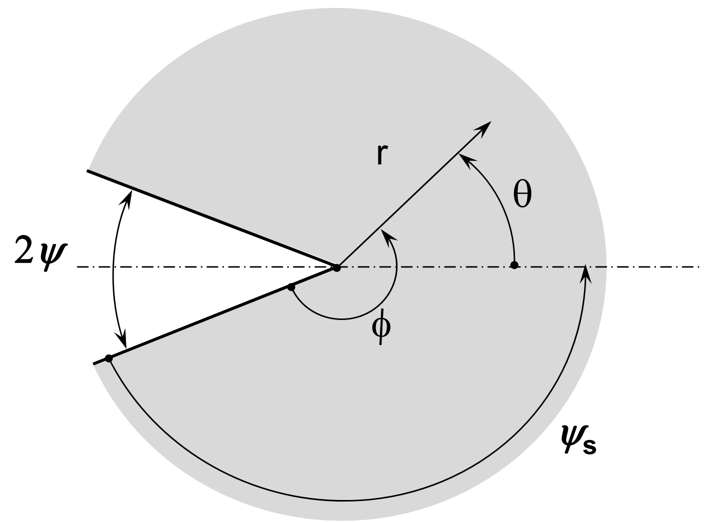
1. General parameters: $c_1, c_2, c_3, c_4$ and exponent $\lambda$ (a dimensionless real number)
2. Airy equation fulfilled in the domain for any combination of $c_1, c_2, c_3, c_4$ and $\lambda$

- Corresponding stress field:

$$\sigma_{rr} = r^{\lambda-1}\left[F_\lambda''(\phi) + (\lambda+1)\cdot F_\lambda(\phi)\right]$$

$$\sigma_{\theta\theta} = r^{\lambda-1}\left[\lambda(\lambda+1)\cdot F_\lambda(\phi)\right]$$

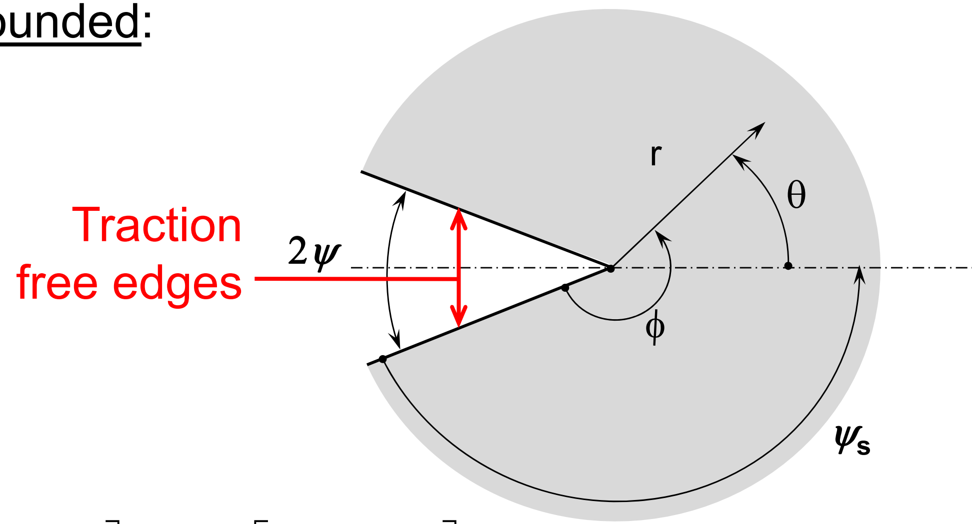
$$\sigma_{r\theta} = r^{\lambda-1}\left[-\lambda\cdot F_\lambda(\phi)\right]$$




- Strain and displacement:

$$\varepsilon_{ij} \propto r^{\lambda-1} \qquad u_i \propto r^{\lambda}$$

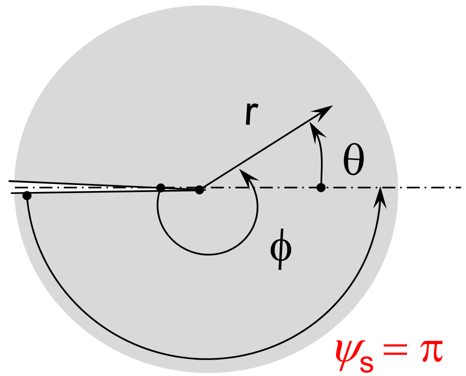
- In order to keep the displacements <u>bounded</u>:
  $u_i \propto r^{\lambda} \rightarrow \lambda > 0$
- Local boundary conditions:
  $\sigma_{\theta\theta}(0) = \sigma_{\theta\theta}(2\psi_s) = 0$
  $\sigma_{r\theta}(0) = \sigma_{r\theta}(2\psi_s) = 0$
- Boundary conditions in explicit form:




$$\begin{cases} c_2 + c_4 = 0 \\ c_1(\lambda+1) + c_3(\lambda-1) = 0 \\ c_1 \sin[2\psi_s(\lambda+1)] + c_2 \cos[2\psi_s(\lambda+1)] + c_3 \sin[2\psi_s(\lambda-1)] + c_4 \cos[2\psi_s(\lambda-1)] = 0 \\ c_1(\lambda+1)\cos[2\psi_s(\lambda+1)] - c_2(\lambda+1)\sin[2\psi_s(\lambda+1)] + c_3(\lambda-1)\cos[2\psi_s(\lambda-1)] - c_4(\lambda-1)\cos[2\psi_s(\lambda-1)] = 0 \end{cases}$$

- Homogeneous linear system with unknowns: $c_1$, $c_2$, $c_3$, $c_4$ and the parameter $\lambda$
- Typical outcome of several problems: instability, free vibrations, etc.
- We are interested in **not trivial solutions** (eigenvalue problem)
- Let's put the determinant of the system matrix to zero
- Characteristic equation with $\lambda$ as unknown (infinite solutions)

**Crack as the special case with $\psi = 0$**




- For this case the eigensolutions are

$$\lambda_n = \frac{n}{2} \quad \text{where } n = 1, 2, 3, ......$$

- and the corresponding Airy's function becomes:

$$\Phi = \sum_{n=1}^{\infty} r^{\frac{n}{2}+1} \left\{ c_{3n} \left[ \sin\left( \left( \frac{n}{2} - 1 \right) \phi \right) - \frac{n-2}{n+2} \sin\left( \left( \frac{n}{2} + 1 \right) \phi \right) \right] + c_{4n} \left[ \cos\left( \left( \frac{n}{2} - 1 \right) \phi \right) - \cos\left( \left( \frac{n}{2} + 1 \right) \phi \right) \right] \right\}$$

- The infinite couples $c_{3n}$, $c_{4n}$ are determined by the other boundary conditions (remote geometry of the body, applied loads, constraints)

- Final general expression for the stress components:

$$\sigma_{ij} = \sum_{n=1}^{\infty} r^{\frac{n}{2}-1} \cdot \mathrm{H}_{ij} \left( \phi, n, c_{3n}, c_{4n} \right) = A_{ij}\left( \phi \right) r^{-\frac{1}{2}} + B_{ij}\left( \phi \right) + C_{ij}\left( \phi \right) r^{\frac{1}{2}} + ....$$

Square root singular term !

## General conclusions of the Williams analysis

- Among the (usually) infinite terms of the stress expansion at the notch tip, only the first is unbounded (it goes to infinite as $r$ approaches zero)

- The other terms are bounded or tends to zero approaching the notch tip

- The power of the singular term is a function of the angle $2\psi$ of the notch

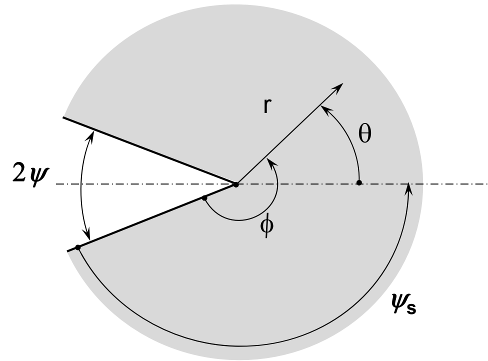
- The strength of the singularity is the highest when $\psi = 0$: the crack is the most severe notch

- The **power** of the leading singular term is **universal** (the same for any crack), the asymptotic terms of the elastic fields at the tip are:

$$\sigma_{ij}, \varepsilon_{ij} \propto \frac{1}{\sqrt{r}} \quad \text{and} \quad u_i \propto \sqrt{r}$$

Exercise – MATLAB:

Implement a parametric calculation for the Willams problem and find the λ solution in the range of angles $\psi = 0° - 89°$




$$\begin{cases} c_2 + c_4 = 0 \\ c_1(\lambda+1) + c_3(\lambda-1) = 0 \\ c_1 \sin[2\psi_s(\lambda+1)] + c_2\cos[2\psi_s(\lambda+1)] + c_3\sin[2\psi_s(\lambda-1)] + c_4\cos[2\psi_s(\lambda-1)] = 0 \\ c_1(\lambda+1)\cos[2\psi_s(\lambda+1)] - c_2(\lambda+1)\sin[2\psi_s(\lambda+1)] + c_3(\lambda-1)\cos[2\psi_s(\lambda-1)] - c_4(\lambda-1)\cos[2\psi_s(\lambda-1)] = 0 \end{cases}$$

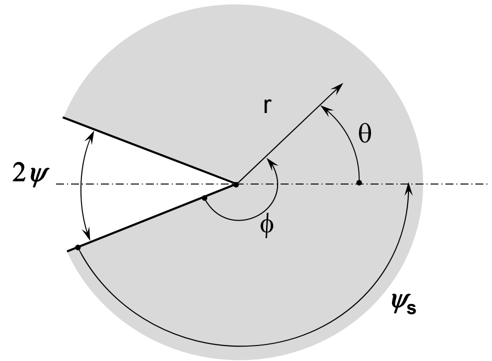
Then the system can be put in matrix form:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ (\lambda+1) & 0 & (\lambda-1) & 0 \\ \sin[2\psi_s(\lambda+1)] & \cos[2\psi_s(\lambda+1)] & \sin[2\psi_s(\lambda-1)] & \cos[2\psi_s(\lambda-1)] \\ (\lambda+1)\cos[2\psi_s(\lambda+1)] & -(\lambda+1)\sin[2\psi_s(\lambda+1)] & (\lambda-1)\cos[2\psi_s(\lambda-1)] & -(\lambda-1)\cos[2\psi_s(\lambda-1)] \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

# *Williams 1957*

Exercise – MATLAB:

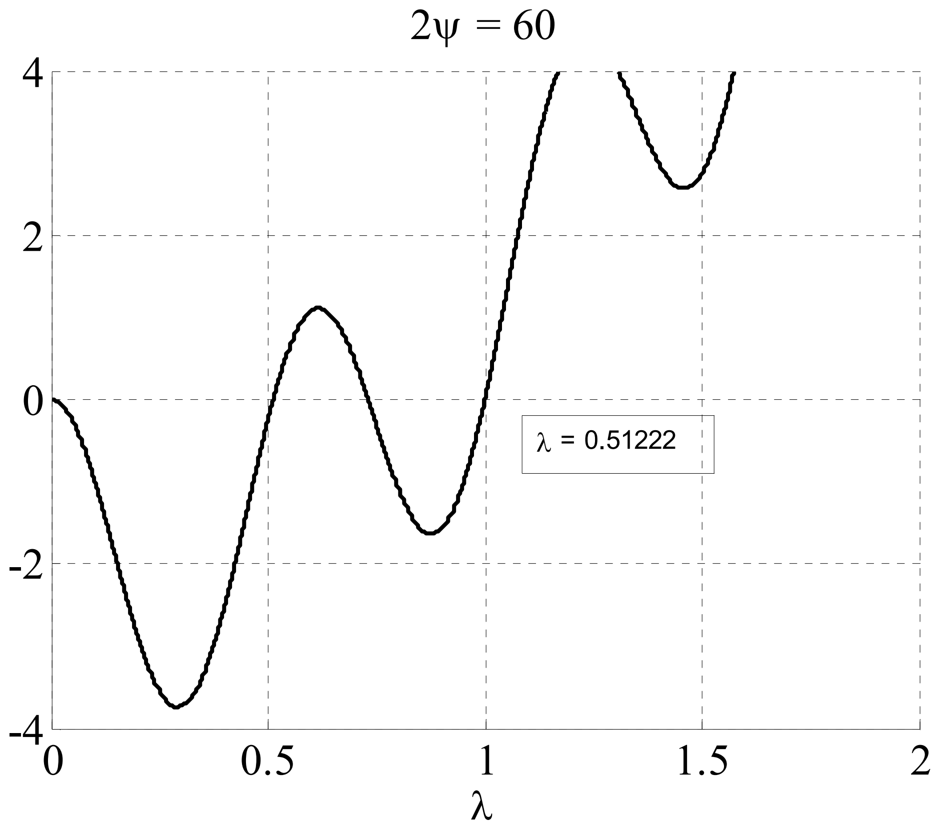
Write the determinant of the matrix, impose it to zero and solve to find $\lambda$

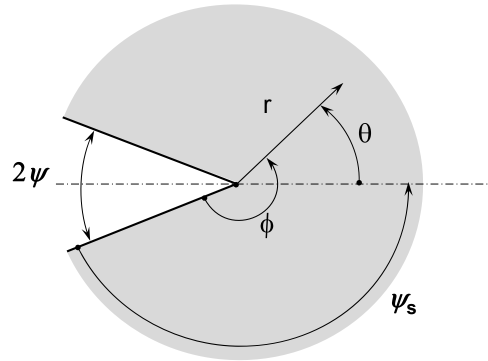



$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ (\lambda+1) & 0 & (\lambda-1) & 0 \\ \sin\left[2\psi_s(\lambda+1)\right] & \cos\left[2\psi_s(\lambda+1)\right] & \sin\left[2\psi_s(\lambda-1)\right] & \cos\left[2\psi_s(\lambda-1)\right] \\ (\lambda+1)\cos\left[2\psi_s(\lambda+1)\right] & -(\lambda+1)\sin\left[2\psi_s(\lambda+1)\right] & (\lambda-1)\cos\left[2\psi_s(\lambda-1)\right] & -(\lambda-1)\cos\left[2\psi_s(\lambda-1)\right] \end{vmatrix} = 0$$

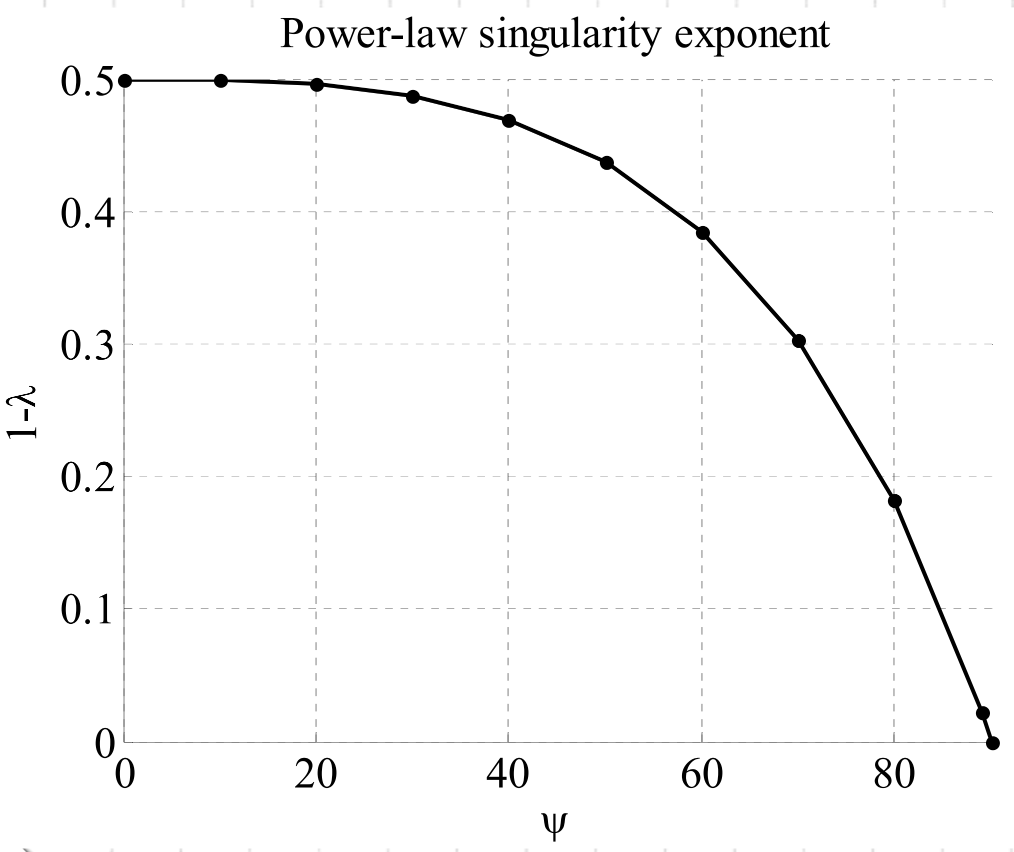
$\lambda = \ldots$

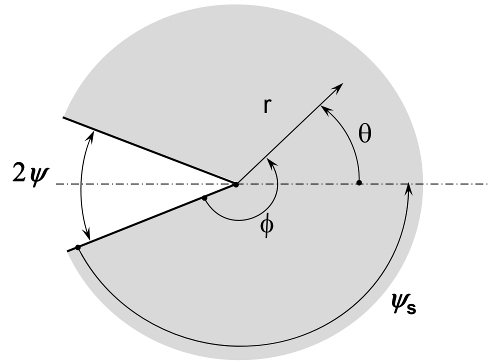
Exercise – MATLAB:



Exercise – MATLAB:




$$\sigma_{ij} \propto r^{\lambda-1} = \frac{1}{r^{1-\lambda}}$$